

A Continuous Surjection from the Unit Interval onto the Unit Square

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ABSTRACT. We show that there exists a continuous surjection $\varphi: I \rightarrow I^2$ which admits an averaging operator in the sense of Pełczyński and which has the additional property that the map $\varphi^\circ: f \rightarrow f \circ \varphi$ is an isomorphism from $L_p(I^2)$ onto a subspace of $L_p(I)$, where $1 \leq p < \infty$.

1. INTRODUCTION

In [T] the author proved that for a wide class of pairs of compact metric spaces (K, K_1) there exists a continuous surjection $\psi: K \rightarrow K_1$ admitting an averaging operator in the sense of Pełczyński, [P]. The results of [T] contain the important special case that there exists a continuous surjection $\varphi: I \rightarrow I^2$, where $I = [0, 1] \subset \mathbb{R}$, having a regular averaging operator (for the terminology, see below). The aim of this paper is to show that the definition of φ can be modified such that $\varphi^\circ: f \rightarrow f \circ \varphi$ in addition becomes an isomorphism from $L_p(I^2)$ onto a subspace of $L_p(I)$, where $1 \leq p < \infty$ (Corollary 7 and Theorem 8). So, we get an operator $\varphi^\circ: C(I^2) \rightarrow C(I)$ which has good properties *simultaneously* with respect to the sup- and L_p -norms. This result, while being of interest in itself, is connected with the study of some Fréchet function spaces, see Section 4.

We introduce the notations and definitions used in this paper. If K is a compact metric space, we denote by $C(K)$ the Banach space of continuous, real or complex valued mappings, endowed with the sup-norm. If K_1 and K_2 are compact metric spaces and $\varphi: K_1 \rightarrow K_2$ is a continuous surjection, we denote by φ° the linear isometry from $C(K_2)$ into $C(K_1)$ given by $\varphi^\circ f = f \circ \varphi$. If $\varphi^\circ(C(K_2))$ is 1-complemented in $C(K_1)$, i.e., if there exists a contractive projection from $C(K_1)$ onto $\varphi^\circ(C(K_2))$, we say that φ admits a regular averaging operator. For more details we recommend the reference [LT], Sections II.4.h.i; see also [P].

Let $\Delta \subset I$ be the "middle thirds"-Cantor set; see for example [R], p. 179. Using the homeomorphism

$$(\varepsilon_m)_{m=1}^\infty \rightarrow \sum_{m=1}^\infty 2\varepsilon_m 3^{-m},$$

where $\varepsilon_m = 0$ or 1 for all $m \in \mathbb{N}$, we identify the topological product

$$\prod_{m=1}^\infty \{0,1\} \tag{1}$$

with Δ . By $\psi: \Delta \rightarrow [0,1]$ we denote the continuous surjection

$$\psi((\varepsilon_m)_{m=1}^\infty) = \sum_{m=1}^\infty \varepsilon_m 2^{-m}. \tag{2}$$

Each dyadic point of the form

$$\sum_{m=0}^n \varepsilon_m 2^{-m} \in I,$$

where $\varepsilon_n = 1$, $n \geq 1$, has two inverse images, $(\varepsilon_1, \dots, \varepsilon_n, 0, 0, 0, \dots)$ and $(\varepsilon_1, \dots, \varepsilon_{n-1}, 0, 1, 1, 1, \dots)$. The other points of I have only one inverse image. We define the discontinuous right inverse $\varrho: I \rightarrow \Delta$ of ψ by

$$\varrho(x) = \min\{y \in \Delta \mid \psi(y) = x\}, \tag{3}$$

where "min" is taken with respect to the usual order of $I \supset \Delta$. The

mapping q° is an isometry from $C(\Delta)$ onto

$$D(I), \tag{4}$$

which is the subspace of $l_\infty(I)$ (the Banach space of bounded scalar valued functions on I endowed with the sup-norm) spanned by continuous functions and the characteristic functions of intervals with dyadic endpoints. It is easy to check that such characteristic functions are contained in $q^\circ(C(\Delta))$, and that the other details of this statement also hold.

The elements of Δ^4 are considered as $4 \times \infty$ -matrices consisting of numbers 0 or 1 (see (1.2)). We denote 4×1 -matrices, i.e., the columns of elements of Δ^4 , by $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)^T$. By $\bar{0}$ (resp. $\bar{1}$) we denote a matrix which consists of numbers 0 (resp. 1) only; the dimension of such a matrix will be clear from context. If $A = (\epsilon_{ij})$ is a matrix with $\epsilon_{ij} = 0$ or 1 for all i and j , we denote by A^- the matrix (ϵ_{ij}^-) , where

$$\epsilon_{ij}^- = \begin{cases} 0, & \text{if } \epsilon_{ij} = 1 \\ 1, & \text{if } \epsilon_{ij} = 0. \end{cases} \tag{5}$$

The space of $4 \times m$ -matrices, consisting of numbers 0 and 1, is denoted by Δ_m^4 .

We denote by m_n the n -dimensional Lebesgue measure. We define the σ -algebra \mathcal{M} of subsets of Δ by

$$\mathcal{M} = \{ \psi^{-1}(\mathcal{A}) \mid \mathcal{A} \subset I \text{ is Lebesgue measurable} \},$$

and we define the measure μ_1 on (Δ, \mathcal{M}) by $\mu_1(\mathcal{A}) = m_1(\psi(\mathcal{A}))$, where $\mathcal{A} \in \mathcal{M}$. Note that μ_1 is additive and even σ -additive in spite of the fact that ψ is not an injection: if $\mathcal{A} \subset \Delta$ and $\mathcal{B} \subset \Delta$ are disjoint, then $\psi(\mathcal{A}) \cap \psi(\mathcal{B})$ is contained in the subset of the dyadic points of I ; this set has Lebesgue measure 0. We denote by μ_n the n -fold product of the measure μ_1 .

We define the homeomorphism $\eta: \Delta \rightarrow \Delta^4$,

$$\eta: (\varepsilon_m)_{m=1}^{\infty} \rightarrow ((\varepsilon_{4m-3})_{m=1}^{\infty}, (\varepsilon_{4m-2})_{m=1}^{\infty}, (\varepsilon_{4m-1})_{m=1}^{\infty}, (\varepsilon_{4m})_{m=1}^{\infty})^T. \quad (6)$$

By

$$\sigma: I \times I \rightarrow I \quad (7)$$

we mean the continuous surjection which assigns to $(x, y) \in I^2$ the unique number $t \in I$ such that (x, y) belongs to the line segment joining $(0, t)$ with $(1, t^2)$. This map is used in the proof of the so called Milutin's lemma, see [LT], II.4.21.

2. CONSTRUCTION OF THE MAP φ

We first define the continuous surjection $\gamma: \Delta^4 \rightarrow \Delta^4$ as follows.

Let $A \in \Delta^4$; we write $A = (A_1, A_2, A_3, \dots)$ where each $A_m = (B_m, C_m)$ is a 4×2 -matrix consisting of $(2m-1)$:th and $2m$:th columns B_m and C_m of A . We first define for all $m \in \mathbb{N}$ the 4×1 -matrices D_m inductively as follows. Let $D_1 = C_1$. Let $m \in \mathbb{N}$, $m > 1$, and assume that D_k is defined for $k < m$. We first define $\Gamma_A^m: \{0, 1\} \rightarrow \Delta_1^4$ by

$$\Gamma_A^m(0) = \begin{cases} D_{m-1}, & \text{if } A_{m-1} = \bar{0} \\ D_{m-1}^-, & \text{if } A_{m-1} \neq \bar{0}, \end{cases}$$

$$\Gamma_A^m(1) = \begin{cases} D_{m-1}, & \text{if } A_{m-1} = \bar{1} \\ D_{m-1}^-, & \text{if } A_{m-1} \neq \bar{1} \end{cases}$$

To define D_m we distinguish between several cases.

1^o. If $B_m \neq C_m$ and $B_m \neq C_m^-$, we set $D_m = C_m$.

2^o. Assume that $A_m = \bar{0}$ or $\bar{1}$. If $A_m = A_{m-1}$, we set $D_m = D_{m-1}$, and if $A_m \neq A_{m-1}$, we set $D_m = D_{m-1}^-$. Remark. If $A_m = \bar{0}$, we have $D_m = \Gamma_A^m(0)$, and if $A_m = \bar{1}$, we have $D_m = \Gamma_A^m(1)$.

3^o. a) Assume that $B_m = C_m$, $A_m \neq \bar{0}$ and $A_m \neq \bar{1}$. If $\Gamma_A^m(0) = C_m$, we set $D_m = (0,0,0,0)^T$. If $\Gamma_A^m(0) \neq C_m$, we set $D_m = C_m$.

b) Assume $B_m = C_m^-$, $C_m \neq (0,0,0,0)^T$ and $C_m \neq (1,1,1,1)^T$. If $\Gamma_A^m(1) = C_m$, we set $D_m = (1,1,1,1)^T$ and $\Gamma_A^m(1) \neq C_m$, we define $D_m = C_m$.

4^o. Consider the case $B_m = C_m^-$, and $C_m = (0,0,0,0)^T$ or $C_m = (1,1,1,1)^T$.

a) If $\Gamma_A^m(0) = (1,1,1,1)^T$ and $\Gamma_A^m(1) \neq (0,0,0,0)^T$, we set $D_m = (0,0,0,0)^T$.

b) If $\Gamma_A^m(1) = (0,0,0,0)^T$ and $\Gamma_A^m(0) \neq (1,1,1,1)^T$, we define $D_m = (1,1,1,1)^T$.

c) In the other cases we set $D_m = C_m$.

We define the element $D \in \Delta^4$ by $D = (D_1, D_2, D_3, \dots)$. We define also for all m the mapping $\gamma_m: \Delta_{2m}^4 \rightarrow \Delta_m^4$ by

$$\gamma_m((A_1, \dots, A_m)) = (D_1, \dots, D_m).$$

and we set

$$\gamma(A) = D.$$

In the following we denote $\bar{\psi} := (\psi, \psi, \psi, \psi): \Delta^4 \rightarrow I^4$ and $\bar{\sigma} := (\sigma, \sigma): I^4 \rightarrow I^2$.

Lemma 1. *The map $\gamma: \Delta^4 \rightarrow \Delta^4$ is a continuous surjection which admits a continuous right inverse and for which the map*

$$\varphi = \bar{\sigma} \circ \bar{\psi} \circ \gamma \circ \eta \circ \varrho: I \rightarrow I^2 \tag{8}$$

is a continuous surjection.

Proof. The continuity of γ , the existence of a continuous right inverse of γ and the surjectivity of φ can be proved exactly as in Lemma 3.2 of [T]. Also the idea for the proof of the continuity of φ is the same as in [T], but because of the details it is necessary to give the proof here. In

view of Lemma 3.1 of [T] it is enough to show that

$$J := \bar{\psi} \circ \gamma \circ \eta : \Delta \rightarrow I^4 \quad (9)$$

maps the elements, $a, \bar{a} \in \Delta$ of the form

$$\begin{aligned} a &= (b, 1, 0, 0, 0, \dots) \\ \bar{a} &= (b, 0, 1, 1, 1, \dots), \end{aligned} \quad (10)$$

where b is a finite sequence consisting of numbers 0 or 1, to the same element of I^4 .

We denote $\eta(a) = A = (A_1, A_2, \dots) \in \Delta^4$ and $\eta(\bar{a}) = (\tilde{A}_1, \tilde{A}_2, \dots) \in \Delta^4$, where A_m (respectively, \tilde{A}_m) consists of the $2m-1$:th and $2m$:th columns B_m and C_m of A (resp. $\tilde{B}_m, \tilde{C}_m, \tilde{A}$). In view of (10) and the definition of η , (6), there exists a unique number m such that $A_k = \tilde{A}_k$ for $k < m$ (if $m > 1$), $A_m \neq \tilde{A}_m$ and $A_k = \bar{0}$, $\tilde{A}_k = \bar{1}$ for $k > m$.

We now consider $\gamma(A) = D = (D_1, D_2, \dots)$ and $\gamma(\tilde{A}) = \tilde{D} = (\tilde{D}_1, \tilde{D}_2, \dots)$.

It is clear from the definition of γ that $D_k = \tilde{D}_k$ for $k < m$. Moreover, the matrix A_m contains an element equal to 1 and, similarly, \tilde{A}_m contains an element 0. Hence, we have $A_m \neq A_{m+1}$ and $\tilde{A}_m \neq \tilde{A}_{m+1}$. By 2^o we get $D_{m+1} = D_m^-$ and $\tilde{D}_{m+1} = \tilde{D}_m^-$, and, moreover, $D_k = D_{m+1}$ and $\tilde{D}_k = \tilde{D}_{m+1}$ for $k > m+1$. So, we have

$$\gamma \circ \eta(a) = D = (D_1, D_2, \dots, D_{m-1}, D_m, D_m^-, D_m^-, D_m^-, \dots),$$

$$\gamma \circ \eta(\bar{a}) = \tilde{D} = (D_1, D_2, \dots, D_{m-1}, \tilde{D}_m, \tilde{D}_m^-, \tilde{D}_m^-, \tilde{D}_m^-, \dots).$$

Let us consider the i :th ($1 \leq i \leq 4$) rows $(d_1^{(i)}, d_2^{(i)}, \dots) \in \Delta$ and $(\tilde{d}_1^{(i)}, \tilde{d}_2^{(i)}, \dots) \in \Delta$ of D and \tilde{D} , respectively. We have $d_j^{(i)} = \tilde{d}_j^{(i)}$ for $1 \leq j < m$. Moreover, $d_k^{(i)} = d_m^{(i)-}$ and $\tilde{d}_k^{(i)} = \tilde{d}_m^{(i)-}$ for all $k > m$. This shows that

$$\psi((d_1^{(i)}, d_2^{(i)}, \dots)) = \psi((\tilde{d}_1^{(i)}, \tilde{d}_2^{(i)}, \dots))$$

and, hence, $J(a) = \bar{\psi}(D) = \bar{\psi}(\tilde{D}) = J(\bar{a})$.

Theorem 2. *The map $\varphi: I \rightarrow I^2$ (see (8)) has a regular averaging operator.*

The proof is the same as that of Theorem 3.3 of [T].

3. MAIN RESULT

We now show that φ also has the additional property that φ° defines an isomorphism from $L_p(I^2)$ into $L_p(I)$ for $1 \leq p < \infty$.

Lemma 3. *Let $m > 1$ and $A \in \Delta_{2^{m-2}}^4$ and $D \in \Delta_1^4$. There exist exactly 16 different matrices $A_m \in \Delta_2^4$ such that*

$$\gamma_m((A, A_m)) = (\gamma_{m-1}(A), D). \tag{11}$$

Proof. The proof of this lemma consists of a straightforward but elaborate verification of the different cases in the definition of γ_m . The numbers 1^o-4^o refer there.

i) We first assume that $D \neq (0,0,0,0)^T$ and $D \neq (1,1,1,1)^T$. There exist 14 vectors $B \in \Delta_1^4$ such that $B \neq D$ and $B \neq \bar{D}$. By 1^o we see that $A_m = (B, D)$ satisfies (11) for all such B .

Next we check if the cases $A_m = \bar{0}, \bar{1}, (D, D)$ or (\bar{D}, D) could satisfy (11). First, if $\Gamma_A^m(0) = D$, then, by 2^o and 3^o a), $A_m = \bar{0}$ satisfies (11) and $A_m = (D, D)$ does not (use the remark in 2^o). If $\Gamma_A^m(0) \neq D$, then $A_m = (D, D)$ satisfies (11) and $A_m = \bar{0}$ does not. Hence, in every case exactly one of the matrices $\bar{0}$ and (D, D) satisfies (11). In the same way we see that exactly one the matrices $\bar{1}$ and (\bar{D}, D) satisfies (11).

Since $D \neq (0,0,0,0)^T, (1,1,1,1)^T$ we see that 4^o cannot produce other matrices satisfying (11). Finally, by 1^o-4^o, a matrix A_m of the form $A_m = (B, C)$, where $C \neq D$ and, moreover, either B or C is different from $(0,0,0,0)^T$ and $(1,1,1,1)^T$, cannot satisfy (11).

Summing up, we see that (11) holds for exactly 16 different matrices $A \in \Delta_2^4$.

ii) We assume $D = (0,0,0,0)^T$. Again there exist 14 vectors $B \in \Delta_1^4$, $B \neq D$ and $B \neq D^-$. By 1^o, (11) holds for $A_m = (B,D)$.

It follows immediately from 1^o-4^o that a matrix $A_m = (B_m, C_m)$, where $C_m \neq D$, can satisfy (11) only if $C_m = D^-$ and $B_m = D$ or D^- , or if $B_m = C_m = \Gamma_A^m(0)$ (see 3^o a)). Hence, we need only to consider such cases and the cases $A_m = \bar{0}$ and $A_m = (D^-, D)$. We should find exactly two matrices of these types satisfying (11).

a) We assume $\Gamma_A^m(0) = (0,0,0,0)^T$. Then $A_m = \bar{0}$ satisfies (11). Moreover, if $\Gamma_A^m(1) = (0,0,0,0)^T$, then also $A_m = \bar{1}$ works, by 2^o, and, by 4^ob), the cases $A_m = (D^-, D)$, $A_m = (D, D^-)$ do not work. If $\Gamma_A^m(1) \neq (0,0,0,0)^T$, then $A_m = \bar{1}$ (see 2^o) and $A_m = (D, D^-)$ (see 4^o c)) do not work but $A_m = (D^-, D)$ does, by 4^o c). So we get altogether two positive cases.

b) We assume $\Gamma_A^m(0) = (1,1,1,1)^T$ so that $A_m = \bar{0}$ does not work. If $\Gamma_A^m(1) = (0,0,0,0)^T$, then by 2^o, $A_m = \bar{1}$ satisfies (11). Moreover, by 4^o c), (D^-, D) satisfies (11) and (D, D^-) does not. If $\Gamma_A^m(1) \neq (0,0,0,0)^T$, then $A_m = \bar{1}$ does not satisfy (11), but by 4^o a), (D^-, D) and (D, D^-) do.

c) Assume $\Gamma_A^m(0) \neq D$ and $\Gamma_A^m(0) \neq D^-$. The case $A_m = \bar{0}$ does not work. By 3^o a), $A_m = (\Gamma_A^m(0), \Gamma_A^m(0))$ satisfies (11). If $\Gamma_A^m(1) = (0,0,0,0)^T$, then $A_m = \bar{1}$ works, and by 4^o b), $A_m = (D, D^-)$ and $A_m = (D^-, D)$ do not. If $\Gamma_A^m(1) \neq (0,0,0,0)^T$, then $A_m = \bar{1}$ does not work, and by 4^o c), $A_m = (D^-, D)$ satisfies (11) and $A_m = (D, D^-)$ does not.

iii) The case $D = (1,1,1,1)^T$ is analogous to ii). But since the point in this kind of proofs is a careful verification of all the details, we want to give the proof also in this case.

By 1^o, there exist 14 vectors $B \in \Delta_1^4$, $B \neq D$, $B \neq D^-$ such that (11) holds for $A_m = (B,D)$. From now on we need only to consider the cases $A_m = \bar{1}, \bar{0}, (D^-, D), (D, D^-)$ and $(\Gamma_A^m(1), \Gamma_A^m(1))$.

a) We assume $\Gamma_A^m(1) = D$. Now $A_m = \bar{1}$ works. If $\Gamma_A^m(0) = D$, then also $\bar{0}$ works but (D^-,D) and (D,D^-) do not. If $\Gamma_A^m(0) \neq D$, then $\bar{0}$ and (D,D^-) do not work but (D^-,D) does.

b) Assume $\Gamma_A^m(1) = D^-$. If $\Gamma_A^m(0) = D$, then $\bar{0}$ and (D^-,D) satisfy (11) but $\bar{1}$ and (D,D^-) do not. If $\Gamma_A^m(0) \neq D$, then the cases $A_m = \bar{0}$ and $A_m = \bar{1}$ are negative and the cases (D^-,D) and (D,D^-) are positive.

c) Assume $\Gamma_A^m(1) \neq D, D^-$. By 3^a b), $A_m = (\Gamma_A^m(1), \Gamma_A^m(1))$ satisfies (11). If $\Gamma_A^m(0) = D$, then also $\bar{0}$ works but $\bar{1}$, (D^-,D) and (D,D^-) do not. If $\Gamma_A^m(0) \neq D$, then $\bar{0}, \bar{1}$ and (D,D^-) do not work but (D^-,D) does.

We have now gone through all the cases.

Corollary 4. Given $m \in \mathbb{N}$ and $D \in \Delta_m^4$ there exist exactly 2^{4m} different matrices $A \in \Delta_{2m}^4$ such that $\gamma_m(A) = D$.

Proof. Let $A = (A_1, \dots, A_m) \in \Delta_{2m}^4$, where $A_m \in \Delta_2^4$, and let $A' = (A_1, \dots, A_{m-1})$. Since the matrix formed by the first $m-1$ columns of $\gamma_m(A)$ is equal to $\gamma_{m-1}(A')$, we can prove Corollary 4 using Lemma 3 and induction with respect to the number of the columns of D . Note that by definition, the 16 matrices (B,C) , where $B \in \Delta_1^4$, are the preimages of C with respect to γ_1 .

Lemma 5. Let $K(i) \in \mathbb{N}$ for all $i = 1, 2, 3, 4$ and let $a_m^{(i)} \in \{0, 1\}$ for all i and for all $m \leq K(i)$. Let us denote by $A \subset \Delta^4$ the set

$$A = \{(x_m^{(i)})_{\substack{m \in \mathbb{N} \\ 1 \leq i \leq 4}} \in \Delta^4 \mid x_m^{(i)} = a_m^{(i)} \text{ for } m \leq K(i)\}. \quad (12)$$

We have

$$\mu_4(\gamma^{-1}(A)) = \prod_{i=1}^4 2^{-K(i)}. \quad (13)$$

Proof. Let $K = \max \{K(i) \mid 1 \leq i \leq 4\}$. Let us introduce the set

$$\mathcal{A}_K = \{(x_m^{(i)})_{1 \leq m \leq K} \in \Delta_K^4 \mid x_m^{(i)} = a_m^{(i)} \text{ for } m \leq K(i)\}.$$

It is a direct consequence of the definition of γ and γ_K that $A \in \gamma^{-1}(\mathcal{A})$ if and only if $A = (A_1, B)$, where $A_1 \in \Delta_{2K}^4$, $B \in \Delta^4$ and A_1 satisfies

$$\gamma_K(A_1) \in \mathcal{A}_K. \quad (14)$$

For a fixed $A_1 \in \gamma_K^{-1}(\mathcal{A}_K) \subset \Delta_{2K}^4$

$$\mu_4(\{(A_1, B) \in \Delta^4 \mid B \in \Delta^4\}) = 2^{-8K}. \quad (15)$$

We thus need only to calculate $\#(\gamma_K^{-1}(\mathcal{A}_K))$. (We denote by $\#(C)$ the cardinality of the set C). But by Corollary 4,

$$\#(\gamma_K^{-1}(\mathcal{A}_K)) = 2^{4K} \#(\mathcal{A}_K). \quad (16)$$

On the other hand it is elementary to see that

$$\#(\mathcal{A}_K) = \prod_{i=1}^4 2^{K-K(i)}.$$

Since the sets $\{(A, B) \mid B \in \Delta^4\}$ and $\{(A_1, B) \mid B \in \Delta^4\}$ are disjoint for $A \neq A_1$, we get by (15) and (16)

$$\mu_4(\gamma^{-1}(\mathcal{A})) = 2^{-8K} \#(\gamma_K^{-1}(\mathcal{A}_K)) = 2^{-4K} \prod_{i=1}^4 2^{K-K(i)} = \prod_{i=1}^4 2^{-K(i)}.$$

Proposition 6. *There exist positive constants C_1 and C_2 such that*

$$C_1 m_1(\varphi^{-1}(\mathcal{A})) \leq m_2(\mathcal{A}) \leq C_2 m_1(\varphi^{-1}(\mathcal{A}))$$

holds for all rectangles $\mathcal{A} \subset I^2$.

Proof. 1°. We first find positive constants d_1 and d_2 such that

$$d_1\mu_4(\gamma^{-1}(\mathcal{A})) \leq \mu_4(\mathcal{A}) \leq d_2\mu_4(\gamma^{-1}(\mathcal{A})) \tag{17}$$

holds for all rectangles $\mathcal{A} \subset \Delta^4$,

$$\mathcal{A} = \prod_{i=1}^4 [a^{(i)}, \bar{a}^{(i)}] \tag{18}$$

where $a^{(i)}, \bar{a}^{(i)} \in \Delta$ for all i . (Intervals $[a^{(i)}, \bar{a}^{(i)}]$ in Δ are defined with respect to the natural order of Δ .)

Let us denote for $x = (x_m)_{m=1}^\infty, y = (y_m)_{m=1}^\infty \in \Delta, x \neq y,$

$$n(x,y) = \min\{m \mid x_m \neq y_m\} \tag{19}$$

and let $m(a,i)$ (resp. $m(\bar{a},i)$) stand for the smallest number m such that $m > n(a^{(i)}, \bar{a}^{(i)})$ and $a_m^{(i)} = 0$ (resp. $\bar{a}_m^{(i)} = 1$). Then (*) $x \in [a^{(i)}, \bar{a}^{(i)}]$ if either $x_m = a_m^{(i)}$ for $m < m(a,i)$ and $x_{m(a,i)} = 1$, or $x_m = \bar{a}_m^{(i)}$ for $m < m(\bar{a},i)$ and $x_{m(\bar{a},i)} = 0$. Moreover, (**) if $x \in [a^{(i)}, \bar{a}^{(i)}]$ then either $x_m = a_m^{(i)}$ for $m < m(a,i)$, or $x_m = \bar{a}_m^{(i)}$ for $m < m(\bar{a},i)$. Denoting $M(i) := \min\{m(a,i), m(\bar{a},i)\}$ for all $1 \leq i \leq 4$ we thus get

$$2^{-M(i)} \leq \mu_1([a, \bar{a}]) \leq 2^{-M(i)+2}$$

so that,

$$\prod_{i=1}^4 2^{-M(i)} \leq \mu_4(\mathcal{A}) \leq 2^8 \prod_{i=1}^4 2^{-M(i)}. \tag{20}$$

Let us denote, for all $i, b_m^{(i)} = a_m^{(i)}$ and $b_{M(i)} = 1$, if $M(i) = m(a,i)$, or $b_m^{(i)} = \bar{a}_m^{(i)}$ and $b_{M(i)} = 0$, if $M(i) = m(\bar{a},i)$. From (*) we see that \mathcal{A} contains the set

$$\mathcal{B} = \{(x_m^{(i)}) \in \Delta^4 \mid x_m^{(i)} = b_m^{(i)} \text{ for } m \leq M(i)\}. \tag{21}$$

By Lemma 5,

$$\mu_4(\gamma^{-1}(\mathcal{A})) \geq \mu_4(\gamma^{-1}(\mathcal{B})) = \prod_{i=1}^4 2^{-M(i)}. \quad (22)$$

Moreover, by (***) \mathcal{A} is contained in the union of the 16 sets

$$\mathcal{B}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \{(x_m^{(i)}) \in \Delta^4 \mid x_m^{(i)} = c_m^{(i)} \text{ for } m < M(i)\} \quad (23)$$

where $\varepsilon_i \in \{0, 1\}$ for $i = 1, \dots, 4$ and $c_m^{(i)} = a_m^{(i)}$ for $m < M(i)$, if $\varepsilon_i = 0$, or $c_m^{(i)} = \bar{a}_m^{(i)}$ for $m < M(i)$, if $\varepsilon_i = 1$. Hence, by Lemma 5

$$\mu_4(\gamma^{-1}(\mathcal{A})) \leq \mu_4\left(\bigcup_{\varepsilon_i \in \{0,1\}} \gamma^{-1}(\mathcal{B}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4))\right) \leq 2^8 \prod_{i=1}^4 2^{-M(i)}. \quad (24)$$

Combining (20), (22) and (24) we see that (17) holds with $d_1 = 2^{-8}$, $d_2 = 2^8$.

2°. We find constants $c_1, c_2 > 0$ such that

$$c_1 \mu_1(\eta^{-1}(\mathcal{A})) \leq \mu_4(\mathcal{A}) \leq c_2 \mu_1(\eta^{-1}(\mathcal{A})) \quad (25)$$

holds for all 4-rectangles $\mathcal{A} \subset \Delta^4$. We define the elements $a^{(i)}$ and $\bar{a}^{(i)}$ and the numbers $m(a, i)$, $m(\bar{a}, i)$, $b_m^{(i)}$ and $M(i)$ as in 1°. Let $M = \max \{M(i) \mid i = 1, \dots, 4\}$. Note that by (20) we again have

$$\prod_{i=1}^4 2^{-M(i)} \leq \mu_4(\mathcal{A}) \leq 2^8 \prod_{i=1}^4 2^{-M(i)}. \quad (26)$$

Let us define the set $\mathcal{B} \subset \mathcal{A}$ as in 1°, (21). From the definition of η , (6), we see that

$$\eta^{-1}(\mathcal{B}) = \{(x_m)_{m=1}^\infty \in \Delta_{4M} \mid x_{4(m-1)+i} = b_m^{(i)} \text{ for all } 1 \leq i \leq 4 \text{ and } m \leq M(i)\}. \quad (27)$$

Let us denote

$$C = \{(x_m)_{m=1}^{4M} \in \Delta_{4M} \mid x_{4(m-1)+i} = b_m^{(i)} \text{ for all } 1 \leq i \leq 4 \text{ and } m \leq M(i)\}. \quad (28)$$

Now $x = (x_m) \in \eta^{-1}(\mathcal{B})$ if and only if $x = (y, z)$, where $y \in C$ and $z \in \Delta$. For a fixed $y \in C$ we have

$$\mu_1(\{(y, z) \mid z \in \Delta\}) = 2^{-4M}. \quad (29)$$

We calculate $\#(C)$. In view of (28), the elements of C are vectors with $4M$ components out of which $\sum_{i=1}^4 M(i)$ are fixed and thus $4M - \sum_{i=1}^4 M(i)$ may be chosen arbitrarily from the set $\{0, 1\}$. So,

$$\#(C) = 2^{4M - \sum_{i=1}^4 M(i)}. \quad (30)$$

Combining this with (29) we get

$$\mu_1(\eta^{-1}(\mathcal{A})) \geq \mu_1(\eta^{-1}(\mathcal{B})) = \prod_{i=1}^4 2^{-M(i)}. \quad (31)$$

To get an upper estimate for $\mu(\eta^{-1}(\mathcal{A}))$ we define the 16 sets $\mathcal{B}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$, where $\epsilon_i \in \{0, 1\}$, as in 1^o, (23). Since these sets are of the same form as \mathcal{B} above, we get by (31)

$$\mu_1(\eta^{-1}(\mathcal{B}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4))) = \prod_{i=1}^4 2^{-M(i)+1}. \quad (32)$$

Since the union of all the sets $\mathcal{B}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ contains \mathcal{A} , we get from (32)

$$\mu_1(\eta^{-1}(\mathcal{A})) \leq 2^8 \prod_{i=1}^4 2^{-M(i)}. \quad (33)$$

Combining (26), (31) and (33) yields (25) with $c_1 = 2^{-8}$, $c_2 = 2^8$.

3^o. We consider the map $\bar{\sigma}$. If $[a, b] \subset I$, the definition of σ implies

$$m_2(\sigma^{-1}([a,b])) = ((b+b^2) - (a+a^2))/2 = (b-a)(1+b+a)/2.$$

Hence, $m_1([a,b])/2 \leq m_2(\sigma^{-1}([a,b])) \leq 2m_1([a,b])$, and so

$$m_4(\bar{\sigma}^{-1}(\mathcal{A}))/4 \leq m_2(\mathcal{A}) \leq 4m_4(\bar{\sigma}^{-1}(\mathcal{A})) \quad (34)$$

for all rectangles $\mathcal{A} \subset I^2$.

4°. Our statement now follows by combining (17), (25) and (34); the maps σ and $\bar{\sigma}$ are measure preserving. Note that if $\mathcal{A} \subset I^2$ is a rectangle, then $\sigma^{-1}(\mathcal{A})$ is not a 4-rectangle but there is no difficulty to approximate it as well as we wish by finite unions of 4-rectangles. Moreover, if $\mathcal{A} \subset \Delta^4$ is a rectangle, then $\gamma^{-1}(\mathcal{A})$ need not be. However, the lower and upper estimates for $\mu_4(\gamma^{-1}(\mathcal{A}))$ are done using 4-rectangles in Δ^4 , see (21) and (23), respectively. Hence, we need also the result of 2° only for rectangles.

Corollary 7. *There exist positive constants C_1 and C_2 such that for all $f \in C(I^2)$*

$$C_1 \int_I |f \circ \varphi(x)| dx \leq \int_{I^2} |f(x)| dx \leq C_2 \int_I |f \circ \varphi(x)| dx. \quad (35)$$

Proof. If $(A_i)_{i=1}^n$ is a sequence of disjoint rectangles in I^2 , we have for all sequences $(a_i)_{i=1}^n$ of scalars

$$\int_{I^2} \left| \sum_{i=1}^n a_i \chi_i(x) \right| dx = \sum_{i=1}^n |a_i| m_2(A_i),$$

$$\int_I \left| \sum_{i=1}^n a_i \chi_i \circ \varphi(x) \right| dx = \sum_{i=1}^n |a_i| m_1(\varphi^{-1}(A_i)),$$

where χ_i is the characteristic function of A_i . So, for simple functions of

this form (35) follows from Proposition 6, and for continuous functions we get the statement by approximation.

We immediately get the following

Theorem 8. *The operator φ° can be extended to an isomorphism from $L_p(I^2)$ into $L_p(I)$, where $1 \leq p < \infty$.*

4. ON THE SPACES $C(\Omega) \cap L_p(\Omega)$

If $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is an open set, we denote by $C(\Omega) \cap L_p(\Omega)$, $1 \leq p < \infty$, the Fréchet space of continuous, L_p -integrable functions from Ω into \mathbb{K} . The topology of this space is determined by the seminorms

$$p_0(f) = \left(\int_{\Omega} |f|^p \right)^{1/p},$$

$$p_k(f) = \sup_{x \in \Omega_k} |f(x)|, \quad k \in \mathbb{N} \tag{36}$$

where $(\Omega_k)_{k=1}^\infty$ is an increasing sequence of compact subsets of Ω , whose union is Ω . For more details on these spaces we refer to [BT].

The isomorphic classification of such spaces is an open problem. Probably the most interesting question in this area is, whether the spaces $C(\mathbb{R}) \cap L_p(\mathbb{R})$ and $C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$ are isomorphic to each other. It is not difficult to see, using a natural imbedding, that $C(\mathbb{R}) \cap L_p(\mathbb{R})$ is isomorphic to a complemented subspace of $C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$. (First, select a continuous cut-off function with compact support $\varphi \in C(\mathbb{R})$ such that $0 \leq \varphi \leq 1$ on \mathbb{R} and $\varphi = 1$ for every $x \in [0,1]$. Put $E := C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$ and $F := C(\mathbb{R}) \cap L_p(\mathbb{R})$ and define $T: F \rightarrow E$ by $Tf(x,y) = f(x)\varphi(y)$ for all $f \in F$, $x,y \in \mathbb{R}$, and $S: E \rightarrow F$ by $Sg(x) := \int_0^1 g(x,y)dy$ for all $g \in E$, $x \in \mathbb{R}$. It is a direct matter to check that T and S are continuous linear maps such that $S \circ T$ is the identity of F . So, $P := T \circ S$ is a continuous projection on E whose image is isomorphic to F .) So, in view of the decomposition

method of Pełczyński, the crucial problem is, whether $C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$ is isomorphic to a complemented subspace of $C(\mathbb{R}) \cap L_p(\mathbb{R})$. Using the results in earlier sections we can prove that the space defined on \mathbb{R}^2 is isomorphic to a subspace of $C(\mathbb{R}) \cap L_p(\mathbb{R})$, but to prove the complementedness we would still need a "measure preserving" continuous surjection $\varphi: I \rightarrow I^2$ and a projection P which is simultaneously bounded $C(I) \rightarrow \varphi^\circ(C(I^2))$ and $L_p(I) \rightarrow \varphi^\circ(L_p(I^2))$, and this result is not (yet?) available.

So, let us prove what we can do.

Lemma 9. *There exists an enumeration $(Q_n)_{n \in \mathbb{Z}}$ of the family of closed squares $(Q_{n,m})_{n,m \in \mathbb{Z}}$, where $Q_{n,m} = \{(x,y) \in \mathbb{R}^2 \mid n \leq x \leq n+1, m \leq y \leq m+1\}$, such that Q_n and Q_{n+1} have a common side for all $n \in \mathbb{Z}$.*

There is no difficulty to make such an enumeration for example according to the following picture:

		-17	-16	15	16	17	18	
	-13	-14	-15	14	13	12	19	
	-12	-3	-2	1	2	11	20	
	-11	-4	-1	0	3	10	21	
	-10	-5	-6	5	4	9	22	
	-9	-8	-7	6	7	8	23	
				27	26	25	24	
				28	29			

Figure 1.

Proposition 10. *For all p , $1 \leq p < \infty$, the space $C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$ is isomorphic to a subspace of $C(\mathbb{R}) \cap L_p(\mathbb{R})$.*

Proof. Let ϕ be the map constructed in Section 2. It can be verified from the definition that $\phi(0) = (0,0)$ and $\phi(1) = (1,1)$. Let $\psi^{(1)}$ be a homeomorphism from I^2 onto itself such that $\psi^{(1)}(0,0) = (0,0)$ and $\psi^{(1)}(1,1) = (1,0)$. It is then clear that also the operator ψ° , where $\psi := \psi^{(1)} \circ \phi$, is an isometry from $C(I^2)$ onto a subspace of $C(I)$ and an isomorphism from $L_p(I^2)$ onto a subspace of $L_p(I)$. Let $(Q_n)_{n=-\infty}^\infty$ be the sequence of closed squares as in Lemma 9. We claim that it is possible to choose a sequence of continuous surjections $\phi^{(n)}: [n, n+1] \rightarrow Q_n$ such that

(i) each $\phi^{(n)}$ is of the form $\tau_n^{(2)} \circ r_n \circ \phi \circ \tau_n$ or $\tau_n^{(2)} \circ r_n \circ \psi \circ \tau_n$, where τ_n is the translation from $[n, n+1]$ onto I , r_n is an isometry from I^2 onto itself, and $\tau_n^{(2)}$ is the translation from I^2 onto Q_n , and

(ii) $\phi^{(n-1)}(n) = \phi^{(n)}(n)$ for all $n \in Z$.

Note that ii) means in particular that

$$\phi^{(n-1)}(n) \in Q_n. \tag{37}$$

To prove this we first choose $\phi^{(0)}$ of the form i) such that $\phi^{(0)}(0) \in Q_{-1}$ and $\phi^{(0)}(1) \in Q_1$. Assume that $n \geq 1$ and that $\phi^{(k)}$ is constructed for $-n+1 \leq k \leq n-1$ such that i) holds for these $\phi^{(k)}$ and such that $\phi^{(k-1)}(k) = \phi^{(k)}(k)$ for $-n+1 < k \leq n-1$ and $\phi^{(n-1)}(n) \in Q_n$ and $\phi^{(-n+1)}(-n+1) \in Q_{-n}$. By Lemma 9 the squares Q_n and Q_{n+1} have one common side S_n . So, it is possible to join one of the endpoints, say s_n , of S_n and $\phi^{(n-1)}(n)$ by one side of Q_n or one diagonal of Q_n . We thus can find a map $\phi^{(n)}$ which is of the form i) and satisfies $\phi^{(n)}(n) = \phi^{(n-1)}(n)$ and $\phi^{(n)}(n+1) = s_n \in Q_{n+1}$. (If s_n and $\phi^{(n-1)}(n)$ are the endpoints of a side of Q_n , we can take a map of the form $\tau_n^{(2)} \circ r_n \circ \psi \circ \tau_n$, and if s_n and $\phi^{(n-1)}(n)$ are in the opposite corners of Q_n , i.e. they are the endpoints of a diagonal of Q_n , we can take a map of the form $\tau_n^{(2)} \circ r_n \circ \phi \circ \tau_n$. In each case r_n is the combination of some rotation and reflection.) The map $\phi^{(-n)}$ is defined analogously.

Defining

$$\phi(t) = \phi^{(n)}(t) \text{ for } t \in [n, n+1], n \in Z, \tag{38}$$

we get a continuous surjection from \mathbb{R} onto \mathbb{R}^2 . We claim that $\phi^\circ: f \rightarrow f \circ \phi$

is the desired isomorphism from $C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$ onto a subspace of $C(\mathbb{R}) \cap L_p(\mathbb{R})$. Since ϕ is a continuous surjection and since $\phi^{-1}(K)$ is compact for all compact $K \subset \mathbb{R}^2$, ϕ° is an isomorphism from $C(\mathbb{R}^2)$ onto a subspace of $C(\mathbb{R})$. It is thus enough to prove the corresponding statement between L_p -spaces. For all $f \in C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$

$$\int_{\mathbb{R}} |f \circ \phi|^p = \sum_{n \in \mathbb{Z}} \int_n^{n+1} |f \circ \phi|^p = \sum_{n \in \mathbb{Z}} \int_0^1 |f \circ \tau_n^{(2)} \circ r_n \circ \phi_n|^p \quad (39)$$

where, for all n , ϕ_n equals ϕ or ψ . According to the definition of ψ we can find positive constants c_1 and c_2 such that

$$c_1 \int_{I^2} |f \circ \tau_n^{(2)} \circ r_n|^p \leq \int_0^1 |f \circ \tau_n^{(2)} \circ r_n \circ \phi_n|^p \leq c_2 \int_{I^2} |f \circ \tau_n^{(2)} \circ r_n|^p \quad (40)$$

for all f and n . Since r_n is an isometry, we can further write

$$\sum_{n \in \mathbb{Z}} \int_{I^2} |f \circ \tau_n^{(2)} \circ r_n|^p = \sum_{n \in \mathbb{Z}} \int_{I^2} |f \circ \tau_n^{(2)}|^p = \sum_{n \in \mathbb{Z}} \int_{Q_n} |f|^p = \int_{\mathbb{R}^2} |f|^p. \quad (41)$$

Combining (39), (40) and (41) we see that ϕ° is also an isomorphism from $L_p(\mathbb{R}^2)$ onto a subspace of $L_p(\mathbb{R})$.

NOTE ADDED IN PROOF

After the paper "A continuous surjection from the unit interval onto the unit square" and the reference [T] in it, "Averaging operators on spaces of continuous functions" were submitted, I realized that some of the results of [T] were already proved by B. Hoffmann in "An injective characterization of Peano spaces", *Topol. and Appl.* 11 (1980), 37-46. This is why [T] does not appear anywhere. In this note we give the missing details of the proofs of Lemma 1 and Theorem 2 of "A continuous surjection from the unit interval onto the unit square".

Proof of Lemma 1: The map $\gamma: \Delta^4 \rightarrow \Delta^4$ is continuous, since the

first m columns of $\gamma(A)$, $A \in \Delta^4$ depend only on the first $2m$ columns of A . We show that γ is a surjection having a continuous right inverse. Let $D=(D_1,D_2,\dots) \in \Delta^4$, where each $D_m \in \Delta_1^4$. We define the element $A \in \Delta^4$, using the same notation as in the definition of γ , as follows. Let $B_1 = C_1 = D_1$. For $m > 1$ we set $C_m = D_m$ and for B_m we choose a matrix which is not equal to anyone of the matrices $D_m, D_m^-, \bar{0}$ or $\bar{1}$. We set $A = (A_1,A_2,\dots)$, where $A_m = (B_m,C_m)$ for all m . It follows now directly from 1^o in the definition of γ that $\gamma(A) = D$. Moreover, the element A depends continuously on D , since the first $2m$ columns of A depend only on the first m columns of D .

We denote by γ^{-1} the continuous right inverse of γ constructed above.

Finally, we show that ϕ is a surjection. Since $\bar{\sigma}$ and $\bar{\psi}$ are surjections, it is enough to prove that $\gamma \circ \eta \circ \rho$ is surjective. Let $D \in \Delta^4$ be arbitrary and let $A = (A_1,A_2,\dots) = \gamma^{-1}(D)$. Each $A_m, m > 1$, contains both numbers 0 and 1. Hence, $\eta^{-1}(A)$ is of the form $(\epsilon_m)_{m=1}^\infty$, where both numbers 0 and 1 occur as ϵ_m for arbitrarily large m . But for such sequences we have

$$\rho\left(\sum_{m=1}^\infty \epsilon_m 2^{-m}\right) = (\epsilon_m)_{m=1}^\infty$$

so that $\gamma \circ \eta \circ \rho$ is surjective.

Proof of Theorem 2. The original proof of Milutin's lemma, which is also presented in [LT], Proposition 2.4.21, shows that the continuous map $\sigma \circ (\psi, \psi): \Delta^2 \rightarrow I$ admits a regular averaging operator. Hence, the same is also true for

$$\bar{\sigma} \circ \bar{\psi}: \Delta^4 \rightarrow I^2.$$

Let $P, \|P\| = 1$, be a projection from $C(\Delta^4)$ onto $(\bar{\sigma} \circ \bar{\psi}) \circ (C(I^2))$.

Let $\gamma: \Delta^4 \rightarrow \Delta^4$ be as above. The operator $f \mapsto \gamma \circ P(\gamma^{-1}) \circ f$ is a contractive projection from $C(\Delta^4)$ onto $(\bar{\sigma} \circ \bar{\psi} \circ \gamma) \circ (C(I^2))$. Hence, also $(\bar{\sigma} \circ \bar{\psi} \circ \gamma \circ \eta) \circ (C(I^2))$ is a 1-complemented subspace of $C(\Delta)$.

Let $\varrho: I \rightarrow \Delta$ be the discontinuous map defined in (3). By (4), ϱ° is an isometry from $C(\Delta)$ onto $D(I)$ so that there exists a contractive projection R from $D(I)$ onto $\varphi^\circ(C(I^2))$, where

$$\varphi := \sigma \circ \psi \circ \gamma \circ \eta \circ \varrho.$$

By Lemma 1, φ is continuous so that $\varphi^\circ(C(I^2))$ is a subspace of $C(I) \subset D(I)$. The restriction of R to $C(I)$ gives the desired projection.

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Recibido: 28 de agosto de 1992
Revisado: 10 de diciembre de 1992