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Embedding of Real Varieties and their Subvarieties into Grassmannians

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ABSTRACT. Given a compact affine nonsingular real algebraic variety X and a nonsingular subvariety $Z \subset X$ belonging to a large class of subvarieties, we show how to embed X in a suitable Grassmannian so that Z becomes the transverse intersection of the zeros of a section of the tautological bundle on the Grassmannian.

In [2] Bochnak and Kucharz prove the following characterization of a compact nonsingular algebraic hypersurface Z in a compact affine nonsingular real algebraic variety X: There is an algebraic embedding $f: X \to RP^n$ (for some n) and a projective hyperplane $H \subset RP^n$ transverse to $f(X)$ such that $H \cap f(X) = f(Z)$. This fact (or rather a closely related statement about strongly algebraic real line bundles) plays a crucial role in their construction of algebraic models Y of a compact, connected, smooth manifold M of dimensions $m \geq 3$ such that

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the algebraic homology elements in $H^1(Y, Z/2) = H^1(M, Z/2)$ form a prescribed subgroup $G \subset H^1(M, Z/2)$. If we wish to extend this result to subgroups of $H^k(M, Z/2)$ for $k > 1$ it seems desirable, as a first step, to extend the above characterization of hypersurfaces to subvarieties of higher codimension.

Let $G_{n,k}(R)$ denote the Grassmannian of k-planes in R^n. Let $\gamma_{n,k}$ denote the universal bundle over $G_{n,k}(R)$. For definitions and results concerning real varieties, strongly algebraic vector bundles etc. see [1].

Theorem 1. *Let X be a compact affine nonsingular real algebraic variety. Let ζ be a strongly algebraic real vector bundle over X of rank k. Let σ be a regular section of ζ transverse to the zero section. Let $Z = \sigma^{-1}(0)$. Then*

(i) There exists a regular embedding $f : X \to G_{n,k}(R)$ for suitable n such that ζ and $f^*(\gamma_{n,k})$ are isomorphic.

(ii) There exists a regular section s of $\gamma_{n,k}$ such that s is transversal to the zero section and $s^{-1}(0) \cap f(X) = f(Z)$ (the intersection $s^{-1}(0) \cap f(X)$ being transverse intersection).

Proof. We can assume that X is a subvariety of real projective q space RP^q for some q. By theorem 12.1.7 of [1] there is a regular map $g : X \to G_{\ell,k}(R)$ (for suitable ℓ) such that $g^*(\gamma_{\ell,k})$ and ζ are isomorphic. Let $G_{\ell,k}(C)$ denote the Grassmannian of complex k-planes in C^ℓ and $\gamma^C_{\ell,k}$ the corresponding universal complex bundle. Let X_C denote the complexification of X in CP^q. Then g extends to a regular map $\tilde{g} : U \to G_{\ell,k}(C)$ where $U \subset X_C$ is a Zariski open set containing X. We can assume U and $\tilde{g}$ are defined over R. By resolution of singularities we can find a complex nonsingular subvariety Y of some complex projective space CP^m with Y defined over R and a regular map (defined over R) $\tau : Y \to X_C$ where τ is the composition of a sequence of blowings-up with real centers outside U such that $\tilde{g} \circ \tau$ extends to a regular map on Y. Denote this extension by h. To simplify notation we identify X with $\tau^{-1}(X)$. Then $h^*(\gamma^C_{\ell,k})$ is a bundle defined over R and $h^*(\gamma^C_{\ell,k})|X$ is isomorphic to $\zeta \otimes C$.

Now, for $E \to M$ a holomorphic vector bundle of rank k over the compact complex manifold M, let $H^0(M, E)$ denote the space of holomorphic sections. Denote the dimension of $H^0(M, E)$ by n. Let

$i_E(x) = \{$ sections vanishing at $x\}$. Assume that each fiber of E is generated by global sections. Then identifying $H^0(M, E)$ with C^n we see that i_E maps M to $G_{n,n-k}(C) \simeq G_{n,k}(C)$. If $F \to M$ is a positive holomorphic line bundle then for p sufficiently large $i_{E\otimes F^p}$ is an embedding of M into $G_{n,k}(C)$ where, now, $n = dim_C H^0(M, E \otimes F^P)$ and $i^*_{E\otimes F^P}(\gamma^C_{n,k})$ is isomorphic to the bundle $E \otimes F^P \to M$. Apply this to $E \to M$ replaced by $h^*(\gamma^C_{\ell,k})$ (so M is replaced by Y) and F replaced by $\gamma^C_{m,1}|Y$. In this case $i_{E\otimes F^P}$ is a regular map defined over R. Abbreviating $i_{E\otimes F^p}$ by i, we can write

$$i^*(\gamma^C_{n,k}) \simeq h^*(\gamma^C_{\ell,k}) \otimes (\gamma^C_{m,1}|Y)^p$$

(as complex bundles). We now restrict both sides to X and obtain

$$(i|X)^*(\gamma_{n,k}) \otimes C \simeq (\zeta \otimes C) \otimes ((\gamma_{m,1}|X) \otimes C)^p$$

and hence

$$(i|X)^*(\gamma_{n,k}) \simeq \zeta \otimes (\gamma_{m,1}|X)^p .$$

We can assume p is even. Then $(\gamma_{m,1}|X)^p$ is topologically trivial. Hence $(i|X)^*(\gamma_{n,k})$ is topologically and hence algebraically isomorphic to ζ. This completes the proof of (i) with $f = i|X$.

To simplify notation we now identify X with $f(X)$ and ζ with $\gamma_{n,k}|X$. Let $s_1, \ldots, s_n$ be sections of $\gamma_{n,k}$ (over $G_{n,k}(R)$) spanning the fiber at each point of $G_{n,k}(R)$. Write $\sigma = \Sigma\lambda_i(s_i|X)$ where λ_i are regular real-valued functions on X. Let $\tilde{\lambda}$ be a regular extension of λ_i to $G_{n,k}(R)$. Let ϕ be a regular real-valued function on $G_{n,k}(R)$ such that $\phi^{-1}(0) = Z(= \sigma^{-1}(0))$. For $t = (t_1, \ldots, t_n)$, define $s_t = \Sigma^n_{i=1}(\tilde{\lambda}_i + t_i\phi^2)s_i$. We can find t (suitably small) so that s_t is transverse to the zero section, $s_t^{-1}(0)$ is transverse to X and $s_t^{-1}(0) \cap X = \sigma^{-1}(0)(= Z)$. This completes the proof of (ii).

References

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