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Mittag-Leffler Methods in Analysis

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Dedicated to the memory of my teacher, Leopoldo Nachbin (1922-1993)

ABSTRACT. In this survey we present two Mittag-Leffler lemmas and several applications to topics as varied as the $\overline{\partial}$-equation, Fréchet algebras, inductive limits of Banach spaces and quasi-normable Fréchet spaces.

1. INTRODUCTION

The classical Mittag-Leffler theorem asserts the existence of meromorphic functions with prescribed poles and singular parts. If the prescribed poles form a finite set, then it is clear that the sum of the corresponding singular parts is a function with the desired properties. But in the general case, that is when the prescribed poles form a sequence without accumulation points, then the corresponding series of singular parts is not necessarily convergent. But then, by means of suitable corrections of the terms of the series, so as to make it convergent, one obtains a function with the desired properties.

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Such a procedure has been so widely used in analysis, that has become known as *Mittag-Leffler procedure.* Several general results whose proofs follow such a procedure are often called *Mittag-Leffler lemmas* or *Mittag-Leffler abstract theorems.* In this survey we present two such lemmas and several applications to topics as varied as the $\overline{\partial}$ equation, Fréchet algebras, inductive limits of Banach spaces and quasi-normable Fréchet spaces.

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1. MITTAG-LEFFLER LEMMAS

Since we will be dealing with projective limits, we recall the definitions. Let $(X_i)_{i\in I}$ be a family of nonvoid sets, indexed by a directed set I. Suppose that for each pair of indices i, j with $i \leq j$ there is a mapping $\xi_{ij} : X_j \to X_i$ such that ξ_{ii} is the identity mapping on X_i for every i and $\xi_{ij} \circ \xi_{jk} = \xi_{ik}$ whenever $i \leq j \leq k$. Then the collection (X_i, ξ_{ij}) is said to be a *projective system*, and the set

$$X = \{(x_i) \in \prod_{i\in I} X_i : \ \xi_{ij}(x_j) = x_i \text{ whenever } i \leq j\}$$

is called the *projective limit* of the sets X_i and is denoted by proj X_i. The canonical mapping $X \to X_i$ is denoted by ξ_i. If each X_i is a topological space (resp. a group, a vector space, etc.) and each ξ_{ij} is continuous (resp. a homomorphism, linear, etc.), then X is a topological space (resp. a group, a vector space, etc.) as a subset of the product $\prod_{i\in I} X_i$ and each ξ_i is continuous (resp. a homomorphism, linear, etc.).

The following lemma sharpens results of Arens [1, Theorem 2.4] and Esterle [6, Theorem 2.1].

1.1. Lemma. *Let $X = proj(X_m, d_m)$ be the projective limit of a sequence of complete metric spaces such that*

$$d_m(\xi_{m,m+1}(x), \xi_{m,m+1}(y)) \leq d_{m+1}(x,y) \ \textit{for all } x, y \in X_{m+1}. \quad (1.1)$$

Suppose that each X_m contains a nonvoid set T_m such that

$$d_m(t, \xi_{m,m+1}(T_{m+1})) < \varepsilon_m \text{ for all } t \in T_m, \tag{1.2}$$

where $\varepsilon_m > 0$ for every $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} \varepsilon_m < \infty$. Then the set

$$B_m = \bigcap_{n=m}^{\infty} \bigcup_{t \in T_n} \xi_n^{-1}\left(B_{X_n}\left(t, \sum_{k=n}^{\infty} \varepsilon_k\right)\right) \tag{1.3}$$

is nonvoid for every $m \in \mathbb{N}$ and

$$d_m(t, \xi_m(B_m)) \le \sum_{k=m}^{\infty} \varepsilon_k \text{ for all } t \in T_m. \tag{1.4}$$

Proof. Fix $m \in \mathbb{N}$ and $t_m \in T_m$. By repeated applications of (1.2) we can find $t_n \in T_n$ for every $n > m$ such that

$$d_n(t_n, \xi_{n,n+1}(t_{n+1})) < \xi_n \text{ for every } n \ge m.$$

We claim that $(\xi_{np}(t_p))_{p=n}^{\infty}$ is a Cauchy sequence in X_n for every $n \ge m$. Indeed for $q \ge p \ge n \ge m$ we have that

$$\begin{aligned} d_n(\xi_{np}(t_p), \xi_{nq}(t_q)) &\le \sum_{k=p}^{q-1} d_n(\xi_{nk}(t_k), \xi_{n,k+1}(t_{k+1})) \\ &\le \sum_{k=p}^{q-1} d_k(t_k, \xi_{k,k+1}(t_{k+1})) < \sum_{k=p}^{\infty} \varepsilon_k. \end{aligned} \tag{1.5}$$

Let $x_n = \lim_{p\to\infty} \xi_{np}(t_p) \in X_n$ for every $n \ge m$. Since $\xi_{np} \circ \xi_{pq}(tq) = \xi_{nq}(t_q)$ for $q \ge p \ge n \ge m$, we see that $\xi_{np}(x_p) = x_n$ for $p \ge n \ge m$. If we define $x_n = \xi_{nm}(x_m)$ for every $n < m$, we see that $x = (x_n)_{n=1}^{\infty}$ belongs to X. By taking $p = n$ and $q \to \infty$ in (1.5) we get that

$$d_n(t_n, x_n) \leq \sum_{k=n}^{\infty} \varepsilon_k \text{ for every } n \geq m.$$

This shows that $x \in B_m$ and $d_m(t_m, \xi_m(x)) \leq \sum_{k=m}^{\infty} \varepsilon_k$, as we wanted.

1.2 Corollary *(Esterle [6]). Let $X = proj\ (X_m, d_m)$ be the projective limit of a sequence of complete metric spaces such that*

$$d_m(\xi_{m,m+1}(x), \xi_{m,m+1}(y)) \leq d_{m+1}(x,y) \text{ for all } x, y \in X_{m+1}.$$

Suppose that

$$d_m(x, \xi_{m,m+1}(X_{m+1})) < \varepsilon_m \text{ for all } x \in X_m \text{ and } m \in \mathbb{N},$$

where $\varepsilon_m > 0$ for every $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} \varepsilon_m < \infty$. Then X is nonvoid and

$$d_m(x, \xi_m(X)) \leq \sum_{k=m}^{\infty} \varepsilon_k \text{ for all } x \in X_m \text{ and } m \in \mathbb{N}.$$

1.3 Corollary. *Let $X = proj\ X_m$ be the projective limit of a sequence of complete metric spaces. Suppose that each X_m contains a nonvoid set T_m such that $T_m \subset \overline{\xi_{m,m+1}(T_{m+1})}$ for every $m \in \mathbb{N}$. Then X is nonvoid and $T_m \subset \overline{\xi_m(X)}$ for every $m \in \mathbb{N}$.*

Proof. Let δ_m denote the metric on X_m for every $m \in \mathbb{N}$ and define

$$d_m(x,y) = \max_{1 \leq n \leq m} \delta_n(\xi_{nm}(x), \xi_{nm}(y)) \text{ for all } x, y \in X_m.$$

Since each ξ_{nm} is continuous, d_m and δ_m define the same topology on X_m. And since (X_m, δ_m) is complete, one can readily verify that (X_m, d_m) is complete. Since d_m satisfies condition (1.1) in Lemma 1.1, the desired conclusion follows.

The next result, due to Arens [1, Theorem 2.4], is probably the first Mittag-Leffler lemma.

1.4 Corollary *(Arens [1]). Let $X = proj\ X_m$ be the projective limit of a sequence of complete metric spaces. If each $\xi_{m,m+1} : X_{m+1} \to X_m$ has a dense range, then X is nonvoid and each $\xi_m : X \to X_m$ has a dense range.*

The next lemma summarizes results of Palamodov [19, p. 215, Proposition 11], Komatsu [12, Lemma 1.3], Bierstedt et al. [3, Lemma 2.8] and Galbis [8]. We shall derive this lemma from Corollary 1.3.

1.5 Lemma. *Let $X = proj\ X_m$, $Y = proj\ Y_m$ and $Z = proj\ Z_m$ be the projective limits of three sequences of abelian groups. Suppose that for every $m \in \mathbb{N}$ there are homomorphisms $\varphi_m : X_m \to Y_m$ and $\psi_m : Y_m \to Z_m$ such that the following diagram is commutative and each row is exact.*

$$\begin{array}{ccccc} X_m & \xrightarrow{\varphi_m} & Y_m & \xrightarrow{\psi_m} & Z_m \\ \uparrow \xi_{m,m+1} & & \uparrow \eta_{m,m+1} & & \uparrow \zeta_{m,m+1} \\ X_{m+1} & \xrightarrow{\varphi_{m+1}} & Y_{m+1} & \xrightarrow{\psi_{m+1}} & Z_{m+1} \end{array}$$

Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be the unique homomorphisms such that the following diagram is commutative and each row is exact.

$$\begin{array}{ccccc} X_m & \xrightarrow{\varphi_m} & Y_m & \xrightarrow{\psi_m} & Z_m \\ \uparrow \xi_m & & \uparrow \eta_m & & \uparrow \zeta_m \\ X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & Z \end{array}$$

(a) If each φ_m is injective, then φ is injective.

(b) Suppose that $X = proj\ X_m$ is actually a projective limit of complete, abelian, metric groups. Suppose in addition that

$$\xi_{m,m+1}(X_{m+1}) \subset \overline{\xi_{m,m+2}(X_{m+2})}\ for\ every\ m \in \mathbb{N} \qquad (1.6)$$

Under these conditions, if $\zeta_m(Z) \subset \psi_m(Y_m)$ for every $m \in \mathbb{N}$, then ψ is surjective.

Proof. If $\varphi(x) = (\varphi_m(x_m))$ for every $x = (x_m) \in X$, and $\psi(y) = (\psi_m(y_m))$ for every $y = (y_m) \in Y$, then the only nontrivial assertion in the lemma is the surjectivity of ψ. Let $z = (z_m) \in Z$. Since $\zeta_m(Z) \subset \psi_m(Y_m)$, for each $m \in \mathbb{N}$ there exists $y_m \in Y_m$ such that $\psi_m(y_m) = z_m$. Then $(y_m) \in \Pi Y_m$, but there is no guarantee that $(y_m) \in Y$. The idea is to find a sequence of corrections x_m, with $x_m \in X_m$ for every $m \in \mathbb{N}$, such that $(y_m - \varphi_m(x_m)) \in Y$. Since

$$\psi_m(y_m - \varphi_m(x_m)) = \psi_m(y_m) = z_m$$

for every $m \in \mathbb{N}$, this will complete the proof. Thus we want to find $(x_m) \in \Pi X_m$ such that

$$\eta_{m,m+1}(y_{m+1} - \varphi_{m+1}(x_{m+1})) = y_m - \varphi_m(x_m)$$

that is

$$\begin{aligned}\eta_{m,m+1}(y_{m+1}) - y_m &= \eta_{m,m+1} \circ \varphi_{m+1}(x_{m+1}) - \varphi_m(x_m) = \\ &= \varphi_m(\xi_{m,m+1}(x_{m+1}) - x_m)\end{aligned} \tag{1.7}$$

for every $m \in \mathbb{N}$. Now since $(z_m) \in Z$ we have that

$$\begin{aligned}\psi_m(\eta_{m,m+1}(y_{m+1}) - y_m) &= \zeta_{m,m+1} \circ \psi_{m+1}(y_{m+1}) - \psi_m(y_m) = \\ &= \zeta_{m,m+1}(z_{m+1}) - z_m = 0\end{aligned}$$

and therefore $\eta_{m,m+1}(y_{m+1}) - y_m \in \psi_m^{-1}(0) = \varphi_m(X_m)$. Thus for each $m \in \mathbb{N}$ there exists $a_m \in X_m$ such that

$$\varphi_m(a_m) = \eta_{m,m+1}(y_{m+1}) - y_m. \tag{1.8}$$

Comparing (1.7) and (1.8) we see that to complete the proof it suffices to find $(x_m) \in \Pi X_m$ such that

$$\varphi_m(a_m) = \varphi_m(\xi_{m,m+1}(x_{m+1}) - x_m) \tag{1.9}$$

for every $m \in \mathbb{N}$. Now let $\tilde{\xi}_{m,m+1} : X_{m+1} \to X_m$ de defined by

$$\tilde{\xi}_{m,m+1}(x) = \xi_{m,m+1}(x) - a_m$$

for every $x \in X_{m+1}$, and let $\tilde{X}$ denote the projective limit of the complete metric spaces X_m with respect to the linking mappings $\tilde{\xi}_{m,m+1} : X_{m+1} \to X_m$. If we use (1.6), then a straightforward verification shows that

$$\tilde{\xi}_{m,m+1}(X_{m+1}) \subset \overline{\tilde{\xi}_{m,m+2}(X_{m+2})} \text{ for every } m \in \mathbb{N}.$$

If we set $T_m = \tilde{\xi}_{m,m+1}(X_{m+1})$ for every $m \in \mathbb{N}$, then Corollary 1.3 applies and guarantees that $\tilde{X}$ is nonvoid. If $(x_m) \in \tilde{X}$, then $\tilde{\xi}_{m,m+1}(x_{m+1}) = x_m$, that is $\xi_{m,m+1}(x_{m+1}) - a_m = x_m$ for every $m \in \mathbb{N}$. This achieves (1.9) and completes the proof.

Let us mention that Petzsche [20] derives the classical Mittag-Leffler theorem from Lemma 1.5, whereas Esterle [6] derives the Baire category theorem and the classical Mittag-Leffler theorem from Corollary 1.4.

1.6. Remark. In all of the preceding lemmas we required that $X = proj\ X_m$ be the projective limit of a sequence of complete metric spaces. An examination of the proofs shows that the conclusions remain true if $X = proj\ X_m$ is the projective limit of a sequence of pseudometric spaces with the property that whenever (a_p) is a Cauchy sequence in X_{m+1}, then the sequence $(\xi_{m,m+1}(a_p))$ has a unique limit in X_m.

2. THE $\bar{\partial}$ EQUATION ON POLYNOMIALLY CONVEX DOMAINS

If U is an open set in $\mathbb{C}^n$, then $C^\infty_{pq}(U)$ denotes the vector space of all C^∞ differential forms of type (p,q) on U. If K is a compact set in $\mathbb{C}^n$, then $C^\infty_{pq}(K)$ denotes the vector space of all germs of C^∞ differential forms of type (p,q) around K.

The solution of the $\bar{\partial}$ equation around polynomially convex compact sets is essentially due to Oka [18] (see Hörmander [11, Lemma 2.7.4 and Theorem 2.7.6]). And then a standard Mittag-Leffler procedure extends the solution to the case of polinomially convex open sets (see Hörmander [11, Theorem 2.7.8]). Let us see how this result follows from Lemma 1.5.

2.1. Theorem *(Oka [18], Hörmander [11]). Let U be a polynomially convex open set in $\mathbb{C}^n$. Then for each $g \in C^\infty_{p,q+1}(U)$ with $\bar{\partial} g = 0$, there exists $f \in C^\infty_{pq}(U)$ such that $\bar{\partial} f = g$.*

Proof. Let (K_m) be a sequence of polynomially convex compact sets such that $U = \cup_{m=1}^\infty K_m$ and $K_m \subset int\ K_{m+1}$ for every $m \in \mathbb{N}$.

(a) Consider first the case $q \geq 1$. If $\mathcal{F}^\infty_{pq}(U)$ (resp. $\mathcal{F}^\infty_{pq}(K)$) denotes the subspace of all $f \in C^\infty_{pq}(U)$ (resp. $C^\infty_{pq}(K)$) with $\bar{\partial} f = 0$, then we have the following commutative diagram of vector spaces and linear mappings.

$$\begin{array}{ccccc} C^\infty_{p,q-1}(K_m) & \xrightarrow{\bar{\partial}} & C^\infty_{pq}(K_m) & \xrightarrow{\bar{\partial}} & \mathcal{F}^\infty_{p,q+1}(K_m) \\ \uparrow & & \uparrow & & \uparrow \\ C^\infty_{p,q-1}(U) & \xrightarrow{\bar{\partial}} & C^\infty_{pq}(U) & \xrightarrow{\bar{\partial}} & \mathcal{F}^\infty_{p,q+1}(U) \end{array}$$

Then the mappings $\bar{\partial} : C^\infty_{pq}(K_m) \to \mathcal{F}^\infty_{p,q+1}(K_m)$ and $C^\infty_{p,q-1}(U) \to C^\infty_{p,q-1}(K_m)$ are surjective. If we endow each of the spaces $C^\infty_{p,q-1}(K_m)$ with the discrete metric, then Lemma 1.5 applies and guarantees that the mapping $\bar{\partial} : C^\infty_{pq}(U) \to \mathcal{F}^\infty_{p,q+1}(U)$ is surjective.

(b) Consider next the case $q = 0$. Recall that if $f \in C^\infty_{p0}(U)$, then $\bar{\partial} f = 0$ if and only if f is a holomorphic mapping on U with values in $\mathcal{L}^a({}^p\mathbb{C}^n)$, the Banach space of all alternating p-linear forms on $\mathbb{C}^n$ (see Mujica [16, Proposition 21.6]). Then we have the following commutative diagram of vector spaces and linear mappings.

$$\begin{array}{ccccc}
\mathcal{H}(K_m;\mathcal{L}^a({}^{P}\mathbb{C}^n)) & \hookrightarrow & C^\infty_{p0}(K_m) & \xrightarrow{\bar\partial} & \mathcal{F}^\infty_{p1}(K_m) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{H}(U;\mathcal{L}^a({}^{P}\mathbb{C}^n)) & \hookrightarrow & C^\infty_{p0}(U) & \xrightarrow{\bar\partial} & \mathcal{F}^\infty_{p1}(U)
\end{array}$$

Then the mapping $\bar\partial : C^\infty_{p0}(K_m) \to \mathcal{F}^\infty_{p1}(K_m)$ is surjective for every $m \in \mathbb{N}$. If $\mathcal{H}(K_m; \mathcal{L}^a({}^{P}\mathbb{C}^n))$ is endowed with the seminorm of the supremum on K_m, then a theorem of Oka (see Hörmander [11, Theorem 2.7.7]) implies that the mapping $\mathcal{H}(U;\mathcal{L}^a({}^{P}\mathbb{C}^n)) \to \mathcal{H}(K_m;\mathcal{L}^a({}^{P}\mathbb{C}^n))$ has a dense range. By Lemma 1.5 and Remark 1.6 the mapping $\bar\partial : C^\infty_{p0}(U) \to \mathcal{F}^\infty_{p1}(U)$ is surjective.

3. FRECHET ALGEBRAS

If A is a commutative Fréchet algebra, then $S(A)$ denotes the spectrum of A, that is the set of all continuous nonzero homomorphisms $\varphi : A \to \mathbb{C}$. All Fréchet algebras are assumed to have an identity.

3.1. Theorem. *(Brooks [5]). Let A be a commutative Fréchet algebra. Let (a_j) be a sequence in A such that*

$$\bigcap_{j=1}^{\infty}\{\varphi \in S(A) : \varphi(a_j) = 0\} = \phi \tag{3.1}$$

Then there is a sequence (x_j) in A such that $\sum_{j=1}^{\infty} a_j x_j = 1$.

Theorem 3.1 is due to Brooks [5, Theorem 2.2] in the case of infinite sequences, and to Arens [1, Theorem 4.2] in the case of finite sequences. Arens obtained the result with the aid of Corollary 1.4 and the following lemma.

3.2. Lemma. *(Arens [1]). Let A and B be two commutative Banach algebras and let $\pi : A \to B$ be a homomorphism with a dense* *range. Let* $a_1, \ldots, a_p \in A$ *such that* $A = a_1 A + \cdots + a_p A$*. Then, given* $\varepsilon > 0$ *and* $y_1, \ldots, y_p \in B$ *such that* $\pi(a_1)y_1 + \cdots + \pi(a_p)y_p = 1$*, there are* $x_1, \ldots, x_p \in A$ *such that* $a_1 x_1 + \cdots + a_p x_p = 1$ *and* $\|\pi(x_j) - y_j\| \leq \varepsilon$ *for* $j = 1, \ldots, p$*.*

In the case of infinite sequences, Corollary 1.4 is not strong enough to prove Theorem 3.1, and Brooks gave a direct, rather cumbersome proof (see also Goldmann [9, p. 136]). We now prove Theorem 3.1 with the aid of Corollary 1.3.

Proof of Theorem 3.1. By a result of Michael [14, Theorem 5.1], A can be represented as the projective limit of a sequence of commutative Banach algebras A_m, where each homomorphism $\pi_m : A \to A_m$ has a dense range. It follows from (3.1) that

$$\bigcap_{j=1}^{\infty} \{\varphi \in S(A_m) : \varphi \circ \pi_m(a_j) = 0\} = \phi$$

for every $m \in \mathbb{N}$. Since the sets $S(A_m)$ are all compact, we can find an increasing sequence (p_m) in $\mathbb{N}$ such that

$$\bigcap_{j=1}^{p_m} \{\varphi \in S(A_m) : \varphi \circ \pi_m(a_j) = 0\} = \phi$$

for every $m \in \mathbb{N}$. Since each A_m is a Banach algebra, we can find $y_1, \ldots, y_{p_m} \in A_m$ such that $\sum_{j=1}^{p_m} \pi_m(a_j) y_j = 1$. We will now show the existence of a sequence (x_j) in A such that $\sum_{j=1}^{\infty} \|\pi_m(a_j)\|_m \|\pi_m(x_j)\|_m < \infty$ for every $m \in \mathbb{N}$ and $\sum_{j=1}^{\infty} a_j x_j = 1$. This will complete the proof. For each $m \in \mathbb{N}$ consider the Banach space

$$E_m = \{(x_j) \in A_m^{\mathbb{N}} : \|(x_j)\|_m := \sum_{j=1}^{\infty} \|\pi_m(a_j)\|_m \|x_j\|_m < \infty\}$$

and its subsets

$$X_m = \{(x_j) \in E_m : \sum_{j=1}^{\infty} \pi_m(a_j)x_j = 1\}$$

and

$$T_m = \{(x_j) \in X_m : x_j = 0 \text{ for every } j > p_m\}.$$

Thus X_m is a complete metric space, as a closed subset of E_m. Consider also the Fréchet space

$$E = \{(x_j) \in A^{\mathbf{N}} : ||(x_j)||_m :=$$

$$= \sum_{j=1}^{\infty} ||\pi_m(a_j)||_n||\pi_m(x_j)||_m < \infty \text{ for all } m \in \mathbb{N}\}$$

and its subset

$$X = \{(x_j) \in E : \sum_{j=1}^{\infty} a_j x_j = 1\}$$

Then $E = \text{proj } E_m$ and $X = \text{proj } X_m$. If $\xi_{m,m+1} : X_{m+1} \to X_m$ denotes the natural mapping, then it follows from Lemma 3.2 that $T_m \subset \overline{\xi_{m,m+1}(T_{m+1})}$ for every $m \in \mathbb{N}$. By Corollary 1.3 X is nonvoid.

In a similar manner we can use Corollary 1.3 to prove another result of Brooks, namely [5, Theorem 2.4].

4. INDUCTIVE LIMITS OF A BANACH SPACES

We recall that if F is a Fréchet space, then the *inductive dual* F'_i of F is the inductive limit of the Banach spaces $(F')_{V_m^\circ}$, where (V_m) is any basis of convex, balanced, 0-neighborhoods in F. It follows from the work of Grothendieck (see [10, Théoréme 6]) that F'_i is always complete. We refer to Bierstedt's survey [2] for information on the inductive dual.

Let $E = \text{ind } E_m$ be the inductive limit of an increasing sequence of Banach spaces. In [15, Theorem 1] we proved that if there is a Hausdorff locally convex topology τ on E such that the closed unit ball of each E_m is τ-compact, then E is topologically isomorphic to F'_i for a suitable Fréchet space F. In particular E is complete.

When trying to apply this theorem in concrete situations, it is natural to seek for a Hausdorff locally convex topology τ_m on each E_m such that:

(a) the inclusion mapping $(E_m, \tau_m) \hookrightarrow (E_{m+1}, \tau_{m+1})$ is continuous;

(b) the closed unit ball of E_m is τ_m-compact.

If the inductive limit $(E, \tau) := \text{ind } (E_m, \tau_m)$ is Hausdorff, then the preceding theorem directly applies. But in certain situations it may be difficult to prove that (E, τ) is Hausdorff. Hence the following variant of the preceding theorem is sometimes more useful.

4.1. Theorem. *Let $E = \text{ind } E_m$ be the inductive limit of an increasing sequence of Banach spaces. Suppose that for each $m \in \mathbb{N}$ there is a Hausdorff locally convex topology τ_m on E_m such that:*

(a) the inclusion mapping $(E_m, \tau_m) \hookrightarrow (E_{m+1}, \tau_{m+1})$ is continuous;
(b) the closed unit ball B_m of E_m is τ_m-compact.

If we set

$$F = \{\varphi \in E' : \varphi|B_m \text{ is } \tau_m - \text{continuous for every } m \in \mathbb{N}\},$$

then F is a Fréchet space for the topology of uniform convergence on each B_m, and E is topologically isomorphic to F'_i. In particular E is Hausdorff, regular and complete.

Proof. If we set

$$F_m = \{\varphi \in E'_m : \varphi|B_m \text{ is } \tau_m - \text{continuous}\},$$

then F_m is a Banach space for the norm $||\varphi|| = \sup\limits_{B_m} |\varphi|$, and F can be canonically identified with proj F_m. Let $R_m : F \to F_m$ and $R_{mn} : F_n \to F_m (m \leq n)$ denote the restriction mappings. Let $J : E \to F'$ and $J_m : E_m \to F'_m$ denote the evaluation mappings. By a result of Waelbroeck [21] and Ng [17], J_m is an isometric isomorphism for every $m \in \mathbb{N}$. Since $F = \text{proj } F_m$, it is clear that J is surjective, but it is far from clear that J is injective. We shall prove that F is indeed injective.

Since $F'_i = \text{ind } F'_m$, it will follow that $J: E \to F'_i$ is a topological isomorphism. Now since the following diagram is commutative,

$$\begin{array}{ccccc} & E_m & \hookrightarrow & E_{m+1} & \\ J_m & \downarrow & & \downarrow & J_{m+1} \\ & F'_m & \stackrel{R'_{m,m+1}}{\longrightarrow} & F'_{m+1} & \end{array}$$

we see that the dual mapping $R'_{m,m+1}: F'_m \to F'_{m+1}$ is injective. By the Hahn-Banach theorem the mapping $R_{m,m+1}: F_{m+1} \to F_m$ has a dense range for every $m \in \mathbb{N}$. By Corollary 1.4 the mapping $R_m: F \to F_m$ has a dense range as well, and hence the dual mapping $R'_m: F'_m \to F'$ is injective. Since the following diagram is commutative,

$$\begin{array}{ccccc} & E_m & \hookrightarrow & E & \\ J_m & \downarrow & & \downarrow & J \\ & F'_m & \stackrel{R'_m}{\longrightarrow} & F' & \end{array}$$

we see that the mapping $J: E \to F'$ is injective, as we wanted.

The following corollary improves a result of Floret [7, Corollary 2].

4.2. Corollary. *Let $E = \text{ind } G'_m$ be the inductive limit of an increasing sequence of duals G'_m of Banach spaces G_m such that the inclusion mappings $G'_m \hookrightarrow G'_{m+1}$ are dual mappings. If $G := \text{proj } G_m$, then E is topologically isomorphic to G'_i. In particular E is Hausdorff, regular and complete.*

Proof. Apply Theorem 4.1 with $\tau_m = \sigma(G'_m, G_m)$ for every $m \in \mathbb{N}$. It follows from Grothendieck's characterization of the completion that each F_m coincides with G_m, and hence F coincides with G.

I obtained these results in 1986, but did not publish them. I communicated the results to Klaus Bierstedt, who quoted them without proofs in his survey on inductive limits (see [2, Theorem 3.15]).

5. QUASI-NORMABLE FRECHET SPACES

Quasi-normable spaces were introduced by Grothendieck [10]. A Fréchet space E is said to be *quasi-normable* if it has a decreasing basis of closed, convex, balanced 0-neighborhoods U_m such that for each $m \in \mathbb{N}$ and each $\varepsilon > 0$ there is a bounded set $B \subset E$ such that

$$U_{m+1} \subset \varepsilon U_m + B \tag{5.1}$$

The following theorem is part of a result of Bonet [4].

5.1. Theorem. *(Bonet [4]). A Fréchet space E is quasi-normable if and only if has a decreasing basis of closed, convex, balanced 0-neighborhoods U_m such that for each $m \in \mathbb{N}$, each $\varepsilon > 0$ and each $n > m$ there exists $\lambda > 0$ such that*

$$U_{m+1} \subset \varepsilon U_m + \lambda U_n. \tag{5.2}$$

Proof. If B is a bounded subset of E, then $B \subset \bigcap_{n=1}^{\infty} \lambda_n U_n$ for suitable $\lambda_n > 0$, and hence it is clear that (5.1) implies (5.2). Bonet [4] gave two proofs of the reverse implication. One proof is based on a result of Meise and Vogt [13, Theorem 7], and the other one is based on a Mittag-Leffler procedure. We now derive this implication from Lemma 1.1. Fix $m \in \mathbb{N}$ and $\varepsilon > 0$. By using (5.2) and induction we can easily find a sequence $(\mu_n)_{n=m+1}^{\infty}$, with $\mu_{m+1} = 1$ and $\mu_n > 0$ for all $n > m$ such that

$$\mu_{n+1} U_{n+1} \subset \frac{\varepsilon}{2^n} U_n + \mu_{n+2} U_{n+2} \text{ for all } n \geq m. \tag{5.3}$$

Without loss of generality we way assume that $E = \text{proj } E_n$ is the projective limit of a sequence of Banach spaces, and $U_n = \xi_n^{-1}(V_n)$ is the inverse image of the closed unit ball of E_n. Set $T_n = \xi_n(\mu_{n+1} U_{n+1})$ for every $n \geq m$. Since we may assume that the natural mapping $\xi_{n,n+1} : E_{n+1} \to E_n$ has norm not greater than one, condition (1.1) in Lemma

1.1 is satisfied. And (5.3) guarantees that condition (1.2) is satisfied too with $\varepsilon_n = \varepsilon/2^n$. Then Lemma 1.1 implies that

$$T_m \subset \xi_m(B_m) + \varepsilon V_m,$$

where

$$B_m \subset \bigcap_{n=m}^{\infty} (\mu_{n+1} U_{n+1} + \varepsilon U_n) \subset \bigcap_{n=m}^{\infty} (\mu_{n+1} + \varepsilon) U_n.$$

Thus B_m is a bounded subset of E and $U_{m+1} \subset B_m + \varepsilon U_m$, as we wanted.

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