

## Denting point in the space of operator-valued continuous maps

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### Abstract

In the paper<sup>1</sup> we describe the geometric properties of the space of continuous functions with values in the space of operators acting on a Hilbert space. In particular we show that

$$\text{dent } B(\mathcal{L}(H)) = \begin{cases} \text{ext } B(\mathcal{L}(H)) & \text{if } \dim H < \infty \text{ and } \text{card } K < \infty \\ \emptyset & \text{if } \dim H = \infty \text{ or } \text{card } K = \infty \end{cases}$$

and  $x\text{-ext } C(K, \mathcal{L}(H)) = \text{ext } C(K, \mathcal{L}(H))$ .

In this paper we consider the geometric properties of the space of continuous functions from a compact Hausdorff space  $K$  with values in the space of operators acting on a Hilbert space  $H$ . Namely, we deal with the unit ball. We consider such points of the unit ball as strongly extreme, and denting points.

For a Banach space  $E$  we denote by  $\mathbf{B}(E)$  and  $\mathbf{S}(E)$  respectively the unit ball and the unit sphere of  $E$ .

Recall that a point  $q$  of a convex set  $Q \subset E$  is *strongly extreme* ( $q \in s\text{-ext } Q$ ) if  $\| \frac{x_n + y_n}{2} - q \| \rightarrow 0$  for  $x_n, y_n \in Q$  implies  $\| x_n - q \| \rightarrow 0$  (or equivalently  $\| x_n - y_n \| \rightarrow 0$ , since  $x_n - q = \frac{x_n - y_n}{2} + \left( \frac{x_n + y_n}{2} - q \right)$ ); *exposed* ( $q \in \text{exp } Q$ ) if there exists  $\xi \in Q^*$  such that  $\xi(q) = \sup \xi(Q) >$

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$\xi(x)$  for all  $x \in Q \setminus \{q\}$ ; *strongly exposed* ( $q \in s\text{-exp } Q$ ) if it is exposed and if  $\xi(x_n) \rightarrow \xi(q)$  for  $x_n \in Q$  then  $\|x_n - q\| \rightarrow 0$ ; and *denting* ( $q \in \text{dent } Q$ ) if  $q \notin \overline{\text{conv}}(Q \setminus \{q + \varepsilon \mathbf{B}(E)\})$  for all  $\varepsilon > 0$ . Point out that  $s\text{-exp } Q \subset \text{dent } Q \subset s\text{-ext } Q \subset \text{ext } Q$  and  $s\text{-exp } Q \subset \text{exp } Q \subset \text{ext } Q$ . Moreover, if  $Q$  is compact then  $\text{dent } Q = s\text{-ext } Q = \text{ext } Q$  and  $s\text{-exp } Q = \text{exp } Q$ . Note that if  $q \in \text{ext } Q$  is a point of continuity for  $Q(x_n \rightarrow q \text{ weakly}, x_n \in Q)$ , implies  $x_n \rightarrow q$  in norm) then  $q \in \text{dent } Q$  ([10]).

For an operator  $T : E \rightarrow E$  we denote by

$\text{IsDom } T = \{x \in E : \|Tx\| = \|x\|\}$  its isometric domain.

Let  $H$  be a (real or complex) Hilbert space equipped with the inner product  $\langle \cdot, \cdot \rangle$ . Obviously  $\text{ext } \mathbf{B}(H) = s\text{-exp } \mathbf{B}(H) = \text{smooth } \mathbf{B}(H) = \mathbf{S}(H)$ . By  $\mathcal{L}(H)$  we denote the space of bounded operators from  $H$  into  $H$ . The space  $\mathcal{L}(H)$  is equipped with the standard operator norm  $\|T\| = \sup\{\|Tx\| : x \in \mathbf{B}(H)\}$ . Note that  $\text{IsDom } T$  is a closed linear subspace for all  $T \in \mathbf{BL}(H)$ . Moreover,  $T(\{x\}^\perp) \subset (Tx)^\perp$  for  $x \in \text{IsDom } T$  and  $T((\text{IsDom } T)^\perp) \perp T(\text{IsDom } T)$ ,  $T \in \mathbf{B}(\mathcal{L}(H))$ . An operator  $T \in \mathcal{L}(H)$  is called *isometry* if  $\text{IsDom } T = H$  and  $T \in \mathcal{L}(H)$  is a *coisometry* if its adjoint  $T^* \in \mathcal{L}(H)$  is an isometry.

For  $y, z \in H$  we denote by  $y \otimes z$  the one dimensional operator defined by  $(y \otimes z)(x) = y \langle x, z \rangle$ ,  $x \in H$ .

Let  $C(K, E)$  denote the Banach space for all continuous functions from a compact Hausdorff space  $K$  into a Banach space  $E$  equipped with the supremum norm  $\|f\| = \sup_{k \in K} \|f(k)\|_E$ .

It is easy to see that for a convex set  $Q \subset E$  if  $f(K) \subset \text{ext } Q$  then  $f \in \text{ext } \{f \in C(K, E) : f(K) \subset Q\}$ . There is a natural question for which classes of convex sets  $Q$  the inverse implication is a characterization of extreme points. Negative example can be easy given for  $Q \subset \mathbb{R}^3$  with non-closed  $\text{ext } Q$ . More complicated negative example of closed symmetric subset  $Q$  of  $\mathbb{R}^4$  was presented in [1]. In fact it is presented an example of  $f \in \text{ext } \mathbf{B}(C(K, E))$  with  $f(k) \notin \text{ext } \mathbf{B}(E)$  for all  $k \in K$ .

Using Michael's selection theorem ([11]) one can prove that  $\text{ext } \{f \in C(K, E) : f(K) \subset Q\} = \{f \in C(K, E) : f(K) \subset \text{ext } Q\}$  for any stable convex subset  $Q$  of  $E$ . Recall that a convex set  $Q \subset E$  is said to be *stable* if the barycenter map  $Q \times Q \ni (x, y) \rightarrow \frac{x+y}{2} \in Q$  in open. Point out that in finite dimensional space a set is stable (see [12]) if and only if all  $m$ -skeletons ( $m = 0, 1, \dots, n$ ) of  $Q$  are closed (a  $m$ -skeleton of  $Q$  is a set of all  $x \in Q$  such that the face generated by  $x$  in  $Q$  has

dimension less than or equal to  $m$ ).

For the space of operators on Hilbert space we have

$$\text{ext } \mathbf{B}(\mathcal{L}(H)) = \{T \in \mathcal{L}(H) : T \text{ is isometry or coisometry}\} \quad ([8], [4])$$

$$\text{exp } \mathbf{B}(\mathcal{L}(H)) = \begin{cases} \text{ext } \mathbf{B}(\mathcal{L}(H)) & \text{if } H \text{ is separable} \\ \emptyset & \text{if } H \text{ is not separable} \end{cases} ,$$

$$s - \text{exp } \mathbf{B}(\mathcal{L}(H)) = \text{dent } \mathbf{B}(\mathcal{L}(H)) = \begin{cases} \text{ext } \mathbf{B}(\mathcal{L}(H)) & \text{if } \dim H < \infty \\ \emptyset & \text{if } \dim H = \infty \end{cases} ,$$

$$s - \text{ext } \mathbf{B}(\mathcal{L}(H)) = \text{ext } \mathbf{B}(\mathcal{L}(H)) \quad ([5], [7]).$$

$$\text{smooth } \mathbf{B}(\mathcal{L}(H)) = \{T \in \mathbf{S}(\mathcal{K}(H)) : \dim \text{IsDom } T = 1$$

and  $\text{dist}(T, \mathcal{K}(H)) < 1\}$  ([9]).

For the space of continuous functions with values in  $\mathcal{L}(H)$  we have ([6]) the following results

$$\text{ext } \mathbf{B}(C(K, \mathcal{L}(H))) = \{f \in C(K, \mathcal{L}(H)) : f(K) \subset \text{ext } \mathbf{B}(\mathcal{L}(H))\} ,$$

$$\text{exp } \mathbf{B}(C(K, \mathcal{L}(H))) = \begin{cases} \text{ext } \mathbf{B}(C(K, \mathcal{L}(H))) & \text{if } H \text{ is separable} \\ \emptyset & \text{if } H \text{ is not separable} \end{cases} ,$$

$$s - \text{exp } \mathbf{B}(C(K, \mathcal{L}(H))) = \begin{cases} \text{ext } \mathbf{B}(C(K, \mathcal{L}(H))) & \text{if } H \text{ is separable} \\ \emptyset & \text{if } H \text{ is not separable} \end{cases} ,$$

$$s - \text{ext } \mathbf{B}(C(K, \mathcal{L}(H))) = \begin{cases} \text{ext } \mathbf{B}(C(K, \mathcal{L}(H))) & \text{if } \dim H < \infty \\ \emptyset & \text{if } \dim H = \infty \end{cases} .$$

The aim of this paper is to continue investigation giving the characterizations of strongly extreme and denting points of the unit ball of  $C(K, \mathcal{L}(H))$ . Namely we show that strongly extreme points of  $\mathbf{B}(C(K, \mathcal{L}(H)))$  coincides and

$$\text{dent } \mathbf{B}(C(K, \mathcal{L}(H))) = \begin{cases} \text{ext } \mathbf{B}(C(K, \mathcal{L}(H))) & \text{if } \dim H < \infty \text{ and } \text{card } K < \infty \\ \emptyset & \text{if otherwise} \end{cases} .$$

**Theorem 1** For any Hilbert space  $H$  we have

$$s - \text{ext } \mathbf{B}(C(K, \mathcal{L}(H))) = \text{ext } \mathbf{B}(C(K, \mathcal{L}(H)))$$

**Proof.** Let  $f \in \text{ext } \mathbf{B}(C(K, \mathcal{L}(H)))$ . Fix  $\varepsilon > 0$ . We need to show that there exists  $\delta > 0$  such that  $\| \frac{g_n + h_n}{2} - f \| < \delta, x, y \in \mathbf{B}(H)$  implies  $\| g_n - h_n \| < \varepsilon$ .

From the uniform convexity of  $H$  ([2]) there exists  $\delta_\varepsilon$  such that  $\| \frac{x+y}{2} \| > 1 - \delta_\varepsilon$  implies  $\| x - y \| < \varepsilon$ .

Put  $K_1 = \{k \in K : f(k) \text{ is an isometry}\}$  and  $K_2 = \{k \in K : f(k) \text{ is a coisometry}\}$ .

Suppose that  $k \in K_1$ . For  $x \in \mathbf{S}(H)$  we have  $\| \frac{g_n(k)x + h_n(k)x}{2} \| \geq \| f_n(k)x \| - \| \frac{g_n(k)x + h_n(k)x}{2} - f_n(k)x \| \geq 1 - \delta_\varepsilon$ .

Thus  $\| g_n(k)x - h_n(k)x \| \leq \varepsilon$ . Hence  $\| g_n(k) - h_n(k) \| \leq \varepsilon$  for all  $k \in K_1$ .

Now suppose that  $k \in K_2$ . Obviously  $\| f^* \| = \| f \|$  and  $f^*(k)$  is an isometry if and only if  $f(k)$  is a coisometry. Now we consider  $f^*$  instead of  $f$  and we get  $\| g_n^*(k) - h_n^*(k) \| = \| g_n(k) - h_n(k) \| \leq \varepsilon$  for all  $k \in K_2$ .

Therefore  $\| g_n - h_n \| \leq \varepsilon$ .  $\square$

**Proposition 1** *If  $\dim H = \infty$  or  $\text{card } K = \infty$  then*

$$\text{dent } \mathbf{B}(C(K, \mathcal{L}(H))) = \emptyset.$$

**Proof.** At first consider the case  $\dim H = \infty$ .

Fix  $f \in \text{ext } \mathbf{B}(C(K, \mathcal{L}(H)))$ . Obviously  $\dim \text{IsDom } f(k) = \infty$  for all  $k \in K$ . Fix  $k_0 \in K$ . Let  $\{e_i\}_{i=1}^\infty$  be orthonormal system in  $\text{IsDom } f(k_0)$ . Let  $P_j$  be an orthogonal projection on  $\{e_j\}^\perp$ . Put  $f_j = P_j f$ . Obviously  $\| f_j - f \| \geq \| f_j(k_0) - f(k_0) \| = \| (f(k_0)e_j) \otimes e_j \| = 1$ . We have  $\| I - \frac{1}{n} \sum_{i=1}^n P_i \| = \| \frac{1}{n} \sum_{i=1}^n e_i \otimes e_i \| = \frac{1}{n}$  and  $\| f - \frac{1}{n} \sum_{i=1}^n f_i \| \leq \| I - \frac{1}{n} \sum_{i=1}^n P_i \| = \frac{1}{n}$ , i.e.  $f \notin \text{dent } \mathbf{B}(C(K, \mathcal{L}(H)))$ .

Now consider the case when  $\text{card } K = \infty$ . Choose a sequence  $\{k_n\}$  of distinct points of  $K$  such that  $\lim_n k_n = k_0$ . We choose the sequence of continuous functions  $\gamma_n : K \rightarrow [0, 1]$  such that  $\gamma_n(k_n) = 1$  and  $\text{supp } \gamma_{n_1} \cap \text{supp } \gamma_{n_2} = \emptyset$  if  $n_1 \neq n_2$ . Put  $f_j = (1 - \gamma_j)f_0 \in \mathbf{B}(C(K, \mathcal{L}(H)))$ . It is easy to see that  $\| f_j - f \| \geq \| f_j(k_j) - f(k_j) \| = 1$ . We have  $\| f - \frac{1}{n} \sum_{i=1}^n f_i \| \leq \| \frac{1}{n} \sum_{i=1}^n h_i \| = \frac{1}{n}$ , i.e.  $f \notin \text{dent } \mathbf{B}(C(K, \mathcal{L}(H)))$ . ■

As a consequence of Proposition 1 we get the following characterization of denting points.

**Theorem 2** *For any Hilbert space  $H$  we have*

$$\text{dent } \mathbf{B}(C(K, \mathcal{L}(H))) = \begin{cases} \text{ext } \mathbf{B}(C(K, \mathcal{L}(H))) & \text{if } \dim H < \infty \text{ and } \text{card } K < \infty \\ \emptyset & \text{if } \dim H = \infty \text{ or } \text{card } K = \infty \end{cases}$$

**Proof.** If  $\dim H < \infty$  and  $\text{card } K < \infty$  then  $C(K, \mathcal{L}(H))$  is finite dimensional, so  $\mathbf{B}(C(K, \mathcal{L}(H)))$  is compact. Hence exposed and strongly exposed are coincide. In view of Theorem 2 and 3 in [6] we get  $s\text{-exp } \mathbf{B}(C(K, \mathcal{L}(H))) = \text{ext } \mathbf{B}(C(K, \mathcal{L}(H)))$ , which finish the proof. ■

### Question.

In [3] it is shown that the unit ball of  $\mathcal{L}(H)$  is stable if  $\dim H < \infty$ . Is  $\mathbf{B}(\mathcal{L}(H))$  ( and  $\mathbf{B}(\mathcal{K}(H))$ ) stable for infinite dimensional  $H$ ?

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