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A remark on the blow-up of the solutions of the equation $u_t + f(x)\,a(u)\,u_x = h(x,u)$

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Abstract

We consider the Cauchy problem for the equation $u_t + f(x)a(u)u_x = h(x,u)$ where f, a and h are real C^1 functions, $f \geq \theta > 0, a' > 0$, $h_u \geq 0$ and $h_x \leq 0$. Following the ideas of Lax [4] and Klainerman-Majda [3], we prove a blow-up result for the solutions with special data corresponding, in certain cases, to the development of a singularity in u_x.

1 Introduction and statement of the result

Let us consider the scalar conservation law

$$u_t + a(u)\,u_x = 0, \quad a \in C^1(\mathbf{R}),\ (x,t) \in \mathbf{R}^2 \tag{1.1}$$

The study of the development of singularities for the solution of the Cauchy problem for the equation (1.1) has been treated by Lax [4] and Majda [6] by proving the appearence of shocks if we impose some conditions to the function a and to the initial data u_0. In this paper we extend some of these results to the equation

$$u_t + f(x)\,a(u)\,u_x = h(x,u), \quad f, a \in C^1(\mathbf{R}),\ h \in C^1\left(\mathbf{R}^2\right) \tag{1.2}$$

by using a method similar to the one employed by Klainerman - Majda [3] for a system of conservation laws.

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For the special cases of the equation (1.2) related to the generalised Burgers equation, Natalini - Tesei [8] gave some conditions for the initial data in order to obtain blow-up results for the L^∞ norm of the solution. In the last section we give some applications to a class of equations arising in physics.

We assume

$$\begin{array}{l} f(\xi) \geq \theta > 0, \quad a'(\xi) \geq \rho > 0, \ \forall \xi \in \mathbf{R}, \quad h_u \geq 0, \ h_x \leq 0 \\ \text{and} \quad f \in W^{1,\infty}(\mathbf{R}), \quad h(.,\xi) \in W^{1,\infty}(\mathbf{R}) \quad \text{for each } \xi \in \mathbf{R}. \end{array} \tag{1.3}$$

Following Douglis [1] and Li-Yu [5], ch.1, if we take the initial data $u_0 \in C^1(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R})$ then there exists a unique local solution

$$u \in C^1(\mathbf{R} \times [0, T_0]) \cap C^1([0, T_0]; L^\infty(\mathbf{R})) \cap C([0, T_0]; W^{1,\infty}(\mathbf{R})) \tag{1.4}$$

of the equation (1.2) such that $u(\alpha, 0) = u_0(\alpha), \forall \alpha \in \mathbf{R}$. We will denote by $[0, T'[$ the corresponding maximal interval of existence where the sharp continuation principle (cf. [6], 2.3), can be applied.

For such a solution let us consider the equation of the characteristics

$$\frac{dx}{dt}(t) = f(x(t))\ a(u(x(t), t)) \quad \text{with} \quad x(0) = \alpha, \ \alpha \in \mathbf{R}. \tag{1.5}$$

Along this characteristic curve the solution u satisfies the differential equation

$$\frac{d}{dt} u\,(x(t), t) = h\,(x(t), u(x(t), t)) \quad \text{with} \quad u\,(x(0), 0) = u\,(\alpha, 0) = u_0(\alpha). \tag{1.6}$$

We can now state our result which extends previous results of Lax [4] and Majda [6] for conservation laws.

Theorem 1 *Under the above assumption* (1.3) *consider the unique local solution u of* (1.2) *for the initial data $u_0 \in C^1(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R})$ and assume that $u \in L^\infty(\mathbf{R} \times [0, T'[)$ where $[0, T'[$ is the corresponding maximal interval of existence. Let $\alpha_0 \in \mathbf{R}$ be such that $u_0'(\alpha_0) < 0$ and let $x(t) = x(t; \alpha_0)$ be the corresponding characteristic curve starting in $x(0) = \alpha_0$. Then, or* $\limsup_{t \to T'} \left(\|u_x(., t)\|_{L^\infty} + \|u_t(., t)\|_{L^\infty} \right) = +\infty$

$$or \quad \liminf_{t \to T'} \int_0^t a'(u(x(\tau), \tau))\ u_x(x(\tau), \tau)\, d\tau = -\infty$$

and hence $\liminf_{t\to T'} u_x(x(t),t) = -\infty$.
Moreover $T' \le T^* = (-\rho f(\alpha_0)\, u_0'(\alpha_0))^{-1}$.

Remark. Since we suppose $u \in L^\infty(\mathbf{R}\times[0,T'[)$, it is enough to assume $a' > 0$ to prove the blow-up result.

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2. Proof of Theorem 1.

Following an idea of Klainerman - Majda [3] we have, along the characteristic curve defined by (1.5),

$$\frac{d}{dt}\left(\frac{\partial x}{\partial\alpha}\right) = \frac{\partial}{\partial\alpha}\left(\frac{dx}{dt}\right) = f'\frac{\partial x}{\partial\alpha}a(u) + fa'(u)\frac{\partial u}{\partial\alpha}$$

Hence, by the theory of linear ordinary differential equations, we have

$$\frac{\partial x}{\partial\alpha}(t) = \left(\exp\int_0^t f'a(u)d\tau\right)\left\{1+\int_0^t fa'(u)\frac{\partial u}{\partial\alpha}\left[\exp\left(-\int_0^s f'a(u)d\tau\right)\right]ds\right\} \tag{2.1}$$

On the other hand, by the results on the derivative of the solution of an ordinary differential equation in order to the initial data (cf. Petrovski [9], for example) we obtain, from (1.5),

$$\frac{\partial x}{\partial\alpha}(t;\alpha) = \exp\int_0^t \frac{\partial}{\partial x}\left(fa(u)\right)(x(\tau),\tau)\,d\tau \tag{2.2}$$

Also, we have, from (1.6), if $u_0'(\alpha) \neq 0$,

$$\frac{d}{dt}\left(\frac{\partial u}{\partial u_0}\right) = h_u\frac{\partial u}{\partial u_0} + h_x\frac{\partial x}{\partial u_0} = h_u\frac{\partial u}{\partial u_0} + h_x\frac{\partial x}{\partial\alpha}\left(u_0'(\alpha)\right)^{-1} \tag{2.3}$$

Hence, in a neighborhood of $\alpha_0 \in \mathbf{R}$ such that $u_0'(\alpha_0) < 0$ we have (since $h_u \ge 0$, $h_x \le 0$ and $\frac{\partial x}{\partial\alpha} \ge 0$), by (2.3),

$$\frac{\partial u}{\partial u_0}(t;u_0) \ge \frac{\partial u}{\partial u_0}(0;u_0) = 1$$

and so

$$\frac{\partial u}{\partial \alpha}(t;\alpha) = \frac{\partial u}{\partial u_0}(t;u_0)\, u_0'(\alpha) \le u_0'(\alpha) < 0 \tag{2.4}$$

Furthermore, since $f > 0$, we obtain from (1.5), along the characteristic curve

$$f'a(u) = \frac{f'}{f}\frac{dx}{dt} = \frac{d}{dt}\log f$$

and so

$$\exp\left(\int_0^t f'a(u)\,d\tau\right) = \frac{f(x(t))}{f(\alpha)}. \tag{2.5}$$

Introducing (2.4) and (2.5) in (2.1) we obtain

$$\frac{\partial x}{\partial \alpha}(t;\alpha) = \frac{f(x(t))}{f(\alpha)}\left[1 + f(\alpha)\int_0^t a'(u(x(s),s))\frac{\partial u}{\partial u_0}(s;u_0)\,u_0'(\alpha)\,ds\right] \tag{2.6}$$

Hence, since by (1.3) $a' \ge \rho > 0$, we obtain if $u_0'(\alpha_0) < 0$, by applying (2.4) and (2.6),

$$\frac{\partial x}{\partial \alpha}(t;\alpha_0) \le \frac{f(x(t))}{f(\alpha_0)}(1 + \rho\, f(\alpha_0)\, u_0'(\alpha_0)t) \tag{2.7}$$

and the right hand side is equal to zero for $t = T^* = (-\rho f(\alpha_0)\, u_0'(\alpha_0))^{-1}$. Introducing (2.5) in (2.2) we derive

$$\frac{\partial x}{\partial \alpha}(t;\alpha) = \frac{f(x(t))}{f(\alpha)}\exp\left(\int_0^t (fa'(u)u_x)\,(x(\tau),\tau)\,d\tau\right) \tag{2.8}$$

and so, by (2.7) and (2.8), there exists a $T \le T^*$ such that, or

$$\limsup_{t\to T'} (\,\|u_x(.,t)\|_{L^\infty} + \|u_t(.,t)\|_{L^\infty}) = +\infty$$

or

$$\liminf_{t\to T}\frac{\partial x}{\partial \alpha}(t;\alpha_0) = \liminf_{t\to T}\frac{f(x(t))}{f(\alpha_0)}\exp\left(\int_0^t (fa'(u)u_x)\,(x(\tau),\tau)\,d\tau\right) = 0,$$

that is, since $f(x) \ge \theta > 0$,

$$\liminf_{t\to T}\int_0^t (a'(u)u_x)\,(x(\tau),\tau)\,d\tau = -\infty$$

and the theorem is proved.

3. Examples.

Our first example of application of theorem 1 is a semi-linear perturbation of the Burgers equation which can not be reduced by a suitable transformation to the Burgers equation in the framework of [2]:

$$u_t + u\,u_x = \lambda u^p, \quad \text{for odd} \quad p > 1 \text{ and } \lambda > 0 \tag{3.1}$$

For this equation we obtain the following blow-up result under the assumptions of theorem 1:

$$\liminf_{t \to T} \int_0^t u_x(x(\tau), \tau)\, d\tau = -\infty$$

for a certain $T \leq T^* = (-u_0'(\alpha_0))^{-1}$ if $u_0'(\alpha_0) < 0$ and where $x(t)$ is the characteristic curve corresponding to $x(0) = \alpha_0$.
Other kind of results, concerning the blow-up of the L^∞ space norm of the solutions of (3.1) for suitable initial data can be found in [8].

Now, consider the more general equation

$$u_t + a(u)\,u_x + \lambda h(u) = 0, \tag{3.2}$$

with $\lambda < 0$, $a'(\xi) \geq \rho > 0$, $\forall \xi \in \mathbf{R}$ and $h' \geq 0$. These equations are introduced in [7] (with the suplementary condition $h'(\xi) > 0$ for $\xi > 0$) and appear in the study of the so called Gunn effect in semiconductors. The situation described in theorem 1 corresponds to the appearence of shocks pointed out in section 2.4 of [7] in the case of the existence of a negative dissipation term.

Finally, consider the equation (3.2) in the special case

$$u_t + u^k u_x - u^p = 0, \quad 0 < p < 1, \; k > 0, \tag{3.3}$$

for positive solutions (see [7] for the positive dissipation case). For this equation the theorem 1 can not be applied without modification. Take a smooth strictly positive initial data u_0. Along the characteristic curve $x(t;\alpha)$ defined by (1.5) we easily obtain

$$u(x(t;\alpha), t) = \left[(1-p)\,t + u_0(\alpha)^{1-p}\right]^{\frac{1}{1-p}}, \; t \geq 0.$$

Following the proof of theorem 1 we derive

$$\frac{\partial x}{\partial \alpha}(t;\alpha) = 1 + k\, u_0'(\alpha)\, u_0^{-p}(\alpha) \int_0^t \Big[(1-p)s + u_0(\alpha)^{1-p}\Big]^q\, ds, \qquad (3.4)$$

where $q = \dfrac{k-1+p}{1-p} > -1$.

Hence, for α_0 such that $u_0'(\alpha_0) < 0$, the right hand side of (3.4) attains zero for a certain $T^* < +\infty$. Therefore we obtain, as in the proof of theorem 1,

$$\liminf_{t\to T^*} \int_0^t (u^{k-1}u_x)\,(x(\tau;\alpha_0),\tau)\, d\tau = -\infty$$

and hence $\liminf_{t\to T^*} u_x(x(t;\alpha_0),t) = -\infty$.

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