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An equivalence criterion for 3-manifolds*.

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Abstract

Within geometric topology of 3-manifolds (with or without boundary), a representation theory exists, which makes use of 4-coloured graphs. Aim of this paper is to translate the homeomorphism problem for the represented manifolds into an equivalence problem for 4-coloured graphs, by means of a finite number of graph-moves, called *dipole moves*. Moreover, interesting consequences are obtained, which are related with the same problem in the n-dimensional setting.

1 Introduction and basic notations

Within every topological-combinatorial representation theory of PL-manifolds, great importance has been attached to the problem of deciding whether two different "objects" do represent the same manifold: recall, for instance, [R] and [S] for Heegaard diagrams of 3-manifolds, [K] for framed links, [M] for generalized Heegaard diagrams, [Pi] for simple 3-coverings of $\mathbb{S}^3$ branched over links. From this view-point, our attention is fixed upon 4-coloured graphs representing 3-dimensional manifolds, with or without boundary: as pointed out later on in the present

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paragraph, edge-coloured graphs are a "discrete way" to handle pseudosimplicial complexes triangulating manifolds (i.e. a suitable generalization of simplicial triangulations, where two (curvilinear) simplices may intersect in more than one face: see [HW]).

This paper completes the effort of translating the homeomorphism problem for the represented 3-manifolds into an equivalence problem for 4-coloured graphs, by means of a finite set of graph-moves, called *dipole moves*; moreover, interesting generalizations are obtained, which are related with the same problem in the n-dimensional setting. Actually, the same n-dimensional problem was already faced - and solved - in [FG], but in the closed case only; here, the whole class of PL n-manifolds is considered, and the deeply different approach to the problem involves - among other - up-to-date knowledges related to stellar and bistellar operations (which are reviewed in the third paragraph).

The manifold representation theory via edge-coloured graphs was firstly introduced by M.Pezzana and his school (see [FGG] and its references); further, it has been developed by other researchers, too (see [BM], [LM], [V], [CV]). We shall repeat here the terminology and the basic notions useful for this paper, in order to make it essentially self-contained. For the graph theory involved, we refer to [W]; as far as piecewise-linear (PL) topology is concerned, see [RS]. For sake of completeness, definitions and results will be given - where it is possible - in the n-dimensional piecewise-linear (PL) setting, even if our attention is mostly fixed upon the 3-dimensional case (where no difference arises among topological manifolds, PL-manifolds and differential manifolds: see [Mo]).

An *(n+1)-coloured graph* (Γ,γ) is a multigraph $\Gamma = (V(\Gamma), E(\Gamma))$ endowed with a proper edge-coloration $\gamma : E(\Gamma) \to \Delta_n = \{0,1,\ldots,n\}$ (i.e. $\gamma(e) \neq \gamma(f)$ for any two adjacent edges $e, f \in E(\Gamma)$).

For every $\mathcal{B} \subset \Delta_n$, we set $\Gamma_{\mathcal{B}} = (V(\Gamma), \gamma^{-1}(\mathcal{B}))$; the connected components of $\Gamma_{\mathcal{B}}$ are said to be $\mathcal{B} - residues$, or $\hat{c} - residues$ if $\mathcal{B} = \Delta_n - \{c\}$, $c \in \Delta_n$. Moreover, $\Gamma_{\mathcal{B}}(x)$ denotes the $\mathcal{B}$-residue containing the vertex $x \in V(\Gamma)$.

We say that an (n+1)-coloured graph (Γ,γ) is *regular with respect to the colour* $c \in \Delta_n$ if every $\hat{c}$-residue is a regular graph of degree n. The symbol $\mathcal{G}_n$ denotes the class of all (n+1)-coloured graphs regular with respect to the "last" colour n; note that the degree of a vertex v

in a graph $(\Gamma, \gamma) \in \mathcal{G}_n$ is either $n+1$ (and, in this case, v is said to be an *internal* vertex) or n (and, in this case, v is said to be a *boundary* vertex).

A theorem of Pezzana ([Pe]), together with its subsequent improvements and generalizations (see [FG], [CvG], [G_1])) states that every PL n-manifold M^n (with or without boundary) may be represented by an element of $\mathcal{G}_n$: this means that (at least) a graph $(\Gamma, \gamma) \in \mathcal{G}_n$ exists, from which a suitable pseudosimplicial triangulation $K(\Gamma)$ of M^n - having as many simplices as the vertices of $V(\Gamma)$ - is obtained.

The process leading from (Γ, γ) to $K(\Gamma)$ - and viceversa - is largely exposed in [FGG]; for short, we only remember that the edge-coloration of (Γ, γ) induces a vertex-labelling of $K(\Gamma)$ by means of Δ_n, which is injective on every n-simplex, and that two n-simplices of $K(\Gamma)$ share the (n-1)-dimensional face opposite to the c-labelled vertex ($c \in \Delta_n$) iff the corresponding vertices of $V(\Gamma)$ are joined by a c-coloured edge. Moreover, the underlying multigraph Γ of the (n+1)-coloured graph (Γ, γ) associated to a pseudocomplex K, is nothing but the 1-skeleton of the ball-complex dual to K.

Obviously, not every $(\Gamma, \gamma) \in \mathcal{G}_n$ represents a pseudocomplex $K(\Gamma)$ triangulating an n-manifold M^n: the necessary and sufficient condition is that each $\hat{c}$-residue, $c \in \Delta_n$, represents either a (n-1)-ball or a (n-1)-sphere. In order to understand why this holds, some definitions are needed.

Definition 1. *If σ is a simplex of an n-dimensional pseudocomplex K, the disjoint star of σ in K, $std(\sigma; K)$, is the subcomplex of K consisting of the disjoint union of the n-simplices containing σ and of their proper faces, with re-identification of the faces containing σ and of their faces.*

Definition 2. *If σ is a simplex of an n-dimensional pseudocomplex K, the disjoint link of σ in K, $lkd(\sigma; K)$, is the subcomplex of $std(\sigma; K)$ consisting of simplices disjoint from σ.*

By construction, if $K = K(\Gamma)$ is associated to an (n+1)-coloured graph (Γ, γ), then every m-simplex σ, whose vertices are labelled by $\{c_1, c_2, \ldots, c_{m+1}\} \subset \Delta_n$, corresponds to a $(\Delta_n - \{c_1, c_2, \ldots, c_{m+1}\})$-residue Ξ of (Γ, γ), and the associated (n-m-1)-pseudocomplex $K(\Xi)$ is exactly $lkd(\sigma; K)$; thus, in particular, the previous characterization follows from the fact that every $\hat{c}$-residue ($c \in \Delta_n$) represents $lkd(v_c; K(\Gamma))$, for some c-labelled vertex v_c of $K(\Gamma)$. Moreover, if $(\Gamma,\gamma) \in \mathcal{G}_n$ is assumed, every $\hat{n}$-residue is a regular graph of degree n; this implies that every n-labelled vertex belongs to the interior of the pseudo-triangulation $K = K(\Gamma)$ of M^n, while the induced pseudo-triangulation ∂K of the - possibly void - boundary ∂M^n inherits a vertex-labelling by means of Δ_{n-1}. Actually, ∂K results to be represented by the so called *boundary graph* $(\partial\Gamma, {}^{\partial}\gamma)$, which may be easily recovered from (Γ,γ) by the following construction (see [CvG]):

- $V(\partial\Gamma)$ is the set of boundary vertices of Γ;
- two vertices $v, w \in V(\partial\Gamma)$ are joined by a c-coloured edge $e \in E(\partial\Gamma)$ (with $c \in \Delta_{n-1}$) if and only if v and w belong to the same $\{c, n\}$-residue of Γ.

2 Moves on edge-coloured graphs

As already stated, "dipole moves" constitute the main set of graph-moves we are interested in; their first introduction is due to [FG], but the present paper gives them the extended meaning of "(proper) dipoles", as defined in [G_2] for graphs with boundary.

Definition 3. *Let $(\Gamma,\gamma) \in \mathcal{G}_n$ be an (n+1)-coloured graph with $\#V(\Gamma) > 2$. An h-dipole ($1 \leq h \leq n$) of Γ is a subgraph Θ consisting of two vertices $v, w \in V(\Gamma)$ joined by h edges coloured by $c_1, c_2, \ldots, c_h \in \Delta_n$, and satisfying the following conditions:*

a) $\Gamma_{\mathcal{B}}(v) \neq \Gamma_{\mathcal{B}}(w)$, with $\mathcal{B} = \Delta_n - \{c_1, c_2, \ldots, c_h\}$;

b) if either v or w is an internal vertex, then either $\Gamma_{\mathcal{B}}(v)$ or $\Gamma_{\mathcal{B}}(w)$ is a regular $(n+1-h)$-coloured graph.

The colours $c_1, c_2, \ldots, c_h$ are said to be *involved* in the dipole $\Theta = \{v, w\}$. Moreover, $\Theta = \{v, w\}$ is said to be an *internal* dipole if either v or w is an internal vertex.

Note that, if (Γ,γ) is a regular (n+1)-coloured graph and/or colour n is involved in the dipole Θ and/or $h = n$, then condition b) is always satisfied.

Definition 4. *The elimination (or cancellation) of the h-dipole Θ in (Γ,γ) consists of:*

a) deleting Θ from (Γ, γ);

b) welding the "hanging" pairs of edges of the same colour $c \in \Delta_n - \{c_1, c_2, \ldots, c_h\}$.

The *insertion* of an h-dipole is the inverse process; by a *dipole move* (resp. *internal dipole move*) we mean either the elimination or the insertion of an h-dipole (resp. internal h-dipole), for some $1 \leq h \leq n$. By abuse of language, if Γ' is obtained from Γ by means of a dipole move, we will sometimes indicate as a "dipole move" also the process induced on the associated pseudocomplexes, yielding $K(\Gamma')$ from $K(\Gamma)$.

In [G_2, Prop. 5.3] it is proved that, if $K = K(\Gamma)$ triangulates a manifold M^n, then dipole move yields a new pseudocomplex $K' = K(\Gamma')$ triangulating M^n: in fact, a dipole insertion is nothing but a suitable re-triangulation of an n-ball $\mathbb{D}^n \subset |K(\Gamma)|$. On the other hand, it is easy to check that internal dipole moves do not affect the boundary triangulations; so, the following summarizing result holds.

Proposition 1. *If K is a pseudocomplex triangulating an n-manifold M^n, and K' is obtained from K by a dipole move, then K' triangulates M^n, too; further, if the dipole move is an internal one, then $\partial K = \partial K'$.*

■

Our main result states that in dimension three, if the pseudocomplexes triangulating 3-manifolds are associated to 4-coloured graphs, the first part of the statement may be reversed: $K = K(\Gamma)$ and $K' = K(\Gamma')$, both triangulating the 3-manifold M^3, are always *equal up to dipoles* (i.e. a finite number of dipole moves exists, yielding K' (resp. K) from K (resp. K')).

Up to now, we don't know whether the whole converse of Proposition 1 is true, i.e. whether an equivariant version of our Main Theorem holds.

We are now going to define a second set of graph-moves (called *wound moves*), which is surely a useful tool for our proofs, but is not really a "new" set of moves: in fact, in [G_2] - where the original definition appeared - it is proved that every wound move on graphs representing manifolds may be performed by means of a finite sequence of dipole moves (see [G_2: Prop. 5.7 and Corollary 5.8]).

Definition 5. *Let* $(\Gamma, \gamma) \in \mathcal{G}_n$ *with* $\#V(\partial\Gamma) > 2$, *and let* x *and* y *be two boundary-vertices of* Γ. $W = \{x, y\}$ *is said to be a wound of type* h *(or, simply, an* h*-wound) involving colours* $c_1, ..., c_h$ $(c_i \in \Delta_{n-1}, \ \forall i = 1, \ldots, h)$ *iff:*

i) $\Gamma_{\{c_i,n\}}(x) = \Gamma_{\{c_i,n\}}(y)$ for every $i \in \{1, 2, \ldots, h\}$;

ii) $\Gamma_{\mathcal{B}}(x) \neq \Gamma_{\mathcal{B}}(y)$, with $\mathcal{B} = \Delta_n - \{c_1, c_2, ..., c_h\}$.

Definition 6. *Let* $W = \{x, y\}$ *an* h*-wound in* $\Gamma \in \mathcal{G}_n$ *and let* $\tilde{\Gamma}$ *be the (n+1)-coloured graph obtained by adding a new* n*-coloured edge between the vertices* x *and* y *of* Γ. *Then,* $\tilde{\Gamma}$ *(resp.* Γ*) is said to be obtained from* Γ *(resp.* $\tilde{\Gamma}$*) by suturing (resp. by opening) the wound* W*; finally, both the process from* Γ *to* $\tilde{\Gamma}$ *and the process from* $\tilde{\Gamma}$ *to* Γ *are called wound moves.*

The third set of graph-moves we are going to define was originally introduced in [G2] with the aim of translating the well-known Alexander's stellar operations (see [A] or [Gl], for example) into coloured graph setting.

Definition 7. *Let* $(\Gamma, \gamma) \in \mathcal{G}_n$, *and let* Ξ *be an* $\hat{a}$*-residue* $(a \in \Delta_n)$ *of* Γ. *The bisection of type* (a, b) *on* Ξ *(with* $b \in \Delta_{n-1} - \{a\}$*) is the process yielding the following new graph* $(\beta\Gamma, {}^{\beta}\gamma) \in \mathcal{G}_n$*:*

1) if $\Xi' = (V(\Xi'), E(\Xi'))$ is a copy of the graph $\Xi = (V(\Xi), E(\Xi))$, then
$$V(\beta\Gamma) = V(\Gamma) \cup V(\Xi');$$

2) if $\bar{E}$ is a set of edges connecting every vertex $v \in V(\Xi)$ with its corresponding vertex $v' \in V(\Xi')$, then
$$E(\beta\Gamma) = (E(\Gamma) - E(\Xi_{\{b\}})) \cup E(\Xi') \cup \bar{E};$$

3) ${}^{\beta}\gamma : E(\beta\Gamma) \to \Delta_n$ is the edge colouring defined by
$${}^{\beta}\gamma(e) = \begin{cases} \gamma(e) & \text{if } e \in E(\Gamma) \\ \gamma(\bar{e}) & \text{if } e \in E(\Xi') \text{ is the corresponding} \\ & \qquad \text{of } \bar{e} \in E(\Xi) - E(\Xi_{\{b\}}) \\ a & \text{if } e \in E(\Xi') \text{ is the corresponding} \\ & \qquad \text{of } \bar{e} \in E(\Xi_{\{b\}}) \\ b & \text{if } e \in \bar{E} \end{cases}$$

It is not difficult to check that, if $(\Gamma, \gamma) \in \mathcal{G}_n$ represents the n-manifold M^n, then $(\beta\Gamma, {}^{\beta}\gamma)$ represents M^n, too. In fact - as pointed out in [G$_2$: paragraph 7] - if w is the a-labelled vertex of $K(\Gamma)$ such that $K(\Xi) = lkd(w, K(\Gamma))$, the pseudocomplex $K(\beta\Gamma)$ associated to $\beta\Gamma$ is simply obtained from $K(\Gamma)$ by performing a direct stellar operation on every 1-simplex of $K(\Xi)$ having as end-points w (which is an a-labelled vertex) and a b-labelled vertex; moreover, the vertex-labelling of $K(\beta\Gamma)$ agrees with the one of $K(\Gamma)$ on every "old" vertex, but w (which results to be b-labelled in $K(\beta\Gamma)$ [1]), and assignes label a to every "new" vertex. In this situation, we shall sometimes say that $K(\beta\Gamma)$ itself is obtained from $K(\Gamma)$ by *bisection of type (a,b) around the vertex w.*

Figure 1(a) shows the effect of a bisection on a particular 3-residue Ξ of a 4-coloured graph $(\Gamma, \gamma) \in \mathcal{G}_3$ (note that (Γ, γ) and $(\beta\Gamma, {}^{\beta}\gamma)$ agree in the not depicted parts), while Figure 1(b) shows the effect of the same bisection on a single 3-simplex of $K(\Gamma)$ containing the vertex w, with $K(\Xi) = lkd(w, K(\Gamma))$.

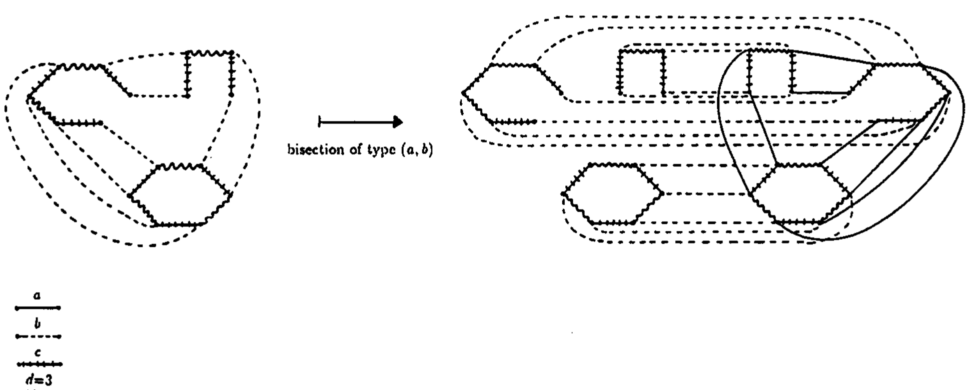

Fig. 1(a)

[1]In fact, $lkd(w; K(\beta\Gamma))$ is represented by the $\hat{b}$-residue Ξ' of $\beta\Gamma$.

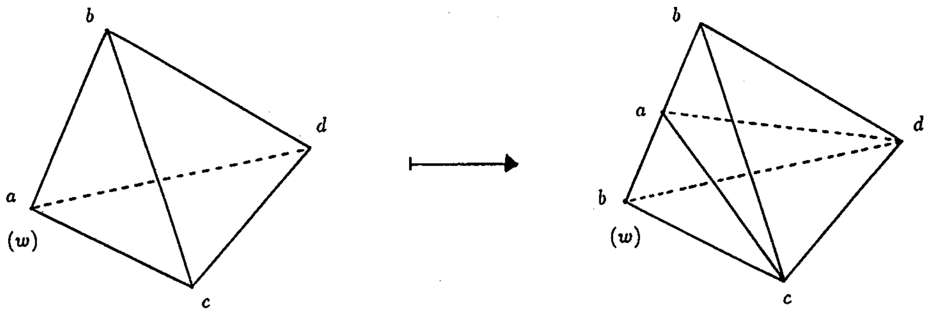

Fig. 1(b)

Let now $(\Gamma, \gamma) \in \mathcal{G}_n$ represent the n-manifold M^n. Obviously, the first baricentric subdivision $\tilde{K}$ of $K(\Gamma)$ is a simplicial triangulation of M^n; moreover, a canonical vertex-labelling on $\tilde{K}$ may be easily obtained, by assigning to every vertex v of $\tilde{K}$ the dimension of the simplex $\sigma(v)$ whose baricenter is v.

Definition 8. *With the above notations, the (n+1)-coloured graph* $(\tilde{\Gamma}, \tilde{\gamma}) \in \mathcal{G}_n$ *associated to* $\tilde{K}$ *with the canonical vertex-labelling is said to be the first baricentric subdivision of* $(\Gamma, \gamma) \in \mathcal{G}_n$.

The following result shows the strict connection between baricentric subdivisions and bisections.

Proposition 2. *Let* $(\Gamma, \gamma) \in \mathcal{G}_n$ *represent the n-manifold* M^n, *and let* $(\tilde{\Gamma}, \tilde{\gamma})$ *be its baricentric subdivision. Then,* (Γ, γ) *and* $(\tilde{\Gamma}, \tilde{\gamma})$ *are equal up to bisections, i.e.* $(\tilde{\Gamma}, \tilde{\gamma})$ *may be obtained from* (Γ, γ) *by means of a finite sequence of bisections.*

Proof. Let $\mathcal{V}_j$ $(j \in \Delta_n)$ be the set of j-labelled vertices of the pseudocomplex $K(\Gamma)$; obviously, in $\tilde{K}$, every vertex $v \in \mathcal{V}_j$ results to be 0-labelled, for each $j \in \Delta_n$. Thus, it is not difficult to check - by remembering the effect of a bisection on the pseudocomplex associated to the coloured graph - that $(\tilde{\Gamma}, \tilde{\gamma})$ may be directly obtained from (Γ, γ) by performing the following sequence of bisections:
for increasing $j = 1, 2, \ldots, n$, do the j bisections of type $(k, k-1)$, with decresing $k = j, j-1, \ldots, 1$, on every $\hat{k}$-residue representing the disjoint link of a vertex $v_j \in \mathcal{V}_j$.

■

Figure 2 illustrates the proof of Proposition 2 in the bidimensional case. In fact, the baricentric subdivision of a 2-symplex (with the canonical vertex-labelling) is performed by the depicted sequence of bisections:

- bisection of type (1,0) around the vertex v_1;
- bisection of type (2,1) around the vertex v_2;
- bisection of type (1,0) around the vertex v_2.

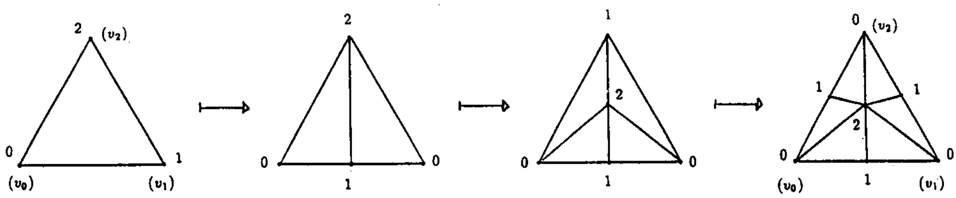

Fig. 2

3 Moves on simplicial triangulations of manifolds

In the present paragraph, K will denote a simplicial triangulation of a PL n-manifold M^n (with possibly void boundary, if not otherwise stated); as usual, ∂K and $Int\ K = K - \partial K$ will denote the (possibly void) *boundary complex* and the *interior* of K respectively.

Moreover, if $A \in K$ is an arbitrary simplex, we set: $st(A;K) := \{C \in K \ / \ \exists B \in K,\ C \subseteq B,\ B \supseteq A\}$; $lk(A;K) := \{C \in K \ / \ C \in st(A;K),\ C \cap A = \emptyset\}$.

Let now introduce the notion of bistellar operation, which was originally defined by U.Pachner in [P], with slightly different notations.

Definition 9. **[P]** *Let $A \neq \emptyset$ be a k-simplex ($0 \leq k \leq n$) of K, such that $lk(A;K)$ is the boundary complex ∂B of an $(n-k)$-simplex B not contained in K. Then, bistellar k-operation $\chi_{(A,B)}$ on K is the process yielding the new simplicial triangulation of M^n*

$$\chi_{(A,B)}K := (K - A * \partial B) \ \cup \ \partial A * B$$

where $L_1 * L_2$ denotes the join of the two simplicial complexes L_1 and L_2.

Note that $\chi^{-1}_{(A,B)} = \chi_{(B,A)}$ is a bistellar $(n-k)$-operation; thus, we can say that two simplicial triangulations K, K' are *bistellar equivalent* if they can be obtained from each other by a finite sequence of bistellar operations.

As far as closed n-manifolds is concerned, the above move is sufficient to solve the equivalence problem; in fact, in 1986, U. Pachner proved the following result, which is analogous to Alexander's one about stellar subdivisions.

Proposition 3. - [P]. *K, K' are simplicial triangulations of the same closed n-manifold M^n if and only if they are bistellar equivalent.*

Recently, the same universal property for bistellar operations has been extended to triangulations of the same n-manifold coinciding on their (non-void) boundary complexes.

Proposition 4. - [C]. *Let K, K' be simplicial triangulations of n-manifolds, with $\partial K = \partial K'$. Then, $|K|, |K'|$ are PL-homeomorphic if and only if K, K' are bistellar equivalent.*

4 Main results

The present paragraph is entirely devoted to prove the equivalence criterion for 4-coloured graphs representing manifolds.

Main Theorem *Let $(\Gamma, \gamma) \in \mathcal{G}_3$ (resp. $(\Gamma', \gamma') \in \mathcal{G}_3$) represent the 3-manifold $M^3 = |K(\Gamma)|$ (resp. $N^3 = |K(\Gamma')|$). Then, M^3 and N^3 are PL-homeomorphic manifolds if and only if (Γ, γ) and (Γ', γ') are equal up to dipoles.*

For, we need some preliminary results; the first one has the purpose of linking together the subjects of the previous two paragraphs, i.e. moves on simplicial triangulations and moves on coloured graphs.

Lemma 5. *Let K, K' be bistellar-equivalent n-dimensional simplicial complexes. If $\Gamma(\tilde{K})$ (resp. $\Gamma(\tilde{K}')$) denotes the (n+1)-coloured graph associated to the (canonically labelled) baricentric subdivision $\tilde{K}$ (resp. $\tilde{K}'$) of K (resp. K'), then $\Gamma(\tilde{K})$ and $\Gamma(\tilde{K}')$ are equal up to (internal) dipoles.*

Proof. Let us assume K' being obtained by means of a bistellar k-operation $\chi_{(A,B)}$ on K (with $0 \le k \le n$). In other words (see the third paragraph), K (resp. K') contains a k-simplex A (resp. an (n-k)-simplex B), such that $st(A;K) = A * \partial B$ (resp. $st(B;K') = B * \partial A$). Since $\partial(st(A;K)) = \partial(st(B;K'))$ and $K - st(A;K) = K' - st(B;K')$, we are going to prove the statement by showing that the baricentric subdivision of $st(A;K)$ may be transformed - by means of dipole moves in the associated coloured graph - into the cone on its boundary.

First of all note that, if H is a (pseudo-)simplicial triangulation of an n-manifold M^n, then the (n+1)-coloured graph $\tilde{\Gamma} = \Gamma(\tilde{H})$ associated to the canonically labelled baricentric subdivision $\tilde{H}$ of H has the following combinatorial properties (which are direct consequences of the particular geometrical structure of $\tilde{H}$):

1) $\#V(\tilde{\Gamma}) = p\ (n+1)!$, where p is the number of n-simplices of H;

2) every $\{i,j\}$-residue of $\tilde{\Gamma}$ has length four, $\forall i,j \in \{0,1,\ldots,n-1\}$, with $j \neq i \pm 1$;

3) every $\{i,n\}$-residue of $\tilde{\Gamma}$ has length four, $\forall i \in \{0,1,\ldots,n-2\}$;

4) every $\{i,i+1\}$-residue of $\tilde{\Gamma}$ has length six, $\forall i \in \{0,1,\ldots,n-2\}$;

5) every $\{n-1,n\}$-residue R of $\tilde{\Gamma}$ corresponds to a 1-simplex $\sigma(R)$ of H and has length $2m$, where m is the number of n-simplices constituting $std(\sigma(R);H)$.

In particular, let us assume $H = st(A;K) = A * \partial B$; since every n-simplex of $st(A;K)$ shares exactly one (n-1)-faces with eachone of the other n-simplices of $st(A;K)$, and since every three n-simplices of $st(A;K)$ constitute $std(\sigma;H)$, for some internal 1-simplex σ of $H = st(A;K)$, then property 5) turns to the following:

5') every $\{n-1,n\}$-residue of $\tilde{\Gamma}$ has length six.

Now, if $\Xi_1, \Xi_2, \ldots, \Xi_{n-k+1}$ are the $\hat{n}$-residues of $\tilde{\Gamma}$ (corresponding to the n-simplices of H), and if e_i (with $1 \leq i \leq n-k$) is an (arbitrarily chosen) n-coloured edge of $\tilde{\Gamma}$ connecting a vertex x_i of Ξ_i with a vertex x'_i of Ξ_{i+1}, it is not difficult to check that the sequence of 1-dipole eliminations $\{x_i, x'_i\}$, for $i = 1,2,\ldots,n-k$, induces the creation of an h-dipole ($h \geq 2$) $\{w,w'\}$ for every pair of n-adjacent vertices w, w' in $\tilde{\Gamma}$. Thus, by performing every such dipole eliminations, an (n+1)-coloured graph $(\bar{\Gamma},\bar{\gamma})$ is obtained, which contains no n-coloured edge, and has the same boundary as $\tilde{\Gamma}$ [2] : this obviously implies that the associated complex $\bar{H} = K(\bar{\Gamma})$ is exactly the cone from an inner n-labelled vertex on the canonically labelled boundary of $\tilde{H}$ (first baricentric subdivision of $H = st(A;K)$).

The statement is now a direct consequence of the fact that $st(A;K)$ and $st(B;K')$ have the same boundary, and that bistellar operation merely "exchanges" $st(A;K)$ with $st(B;K')$.

∎

From now on, we restrict our attention to dimension $n \leq 3$; in this setting, the following result shows how to "factorize" bisections by means of dipole moves.

Lemma 6. *Let $(\Gamma,\gamma) \in \mathcal{G}_n$ represent an n-manifold M^n, with $n \leq 3$, and let $(\beta\Gamma,{}^{\beta}\gamma)$ be obtained by bisection of type (a,b) on a $\hat{a}$-residue Ξ of Γ (with $a \in \Delta_n$ and $b \in \Delta_{n-1} - \{a\}$).*
We have:

- *i) if Ξ is a regular graph, $\beta\Gamma$ may be obtained from Γ by means of a finite sequence of dipole insertions;*
- *ii) if Ξ has not empty boundary, $\beta\Gamma$ may be obtained from Γ by means of a finite sequence of dipole insertions, followed by a finite sequence of wound openings.*

Proof. For sake of notational simplicity, we will consider only the case $n = 3$; on the other hand, the (simpler) bidimensional case has already been handled in [CP: Prop. 5].

Let $R_1, R_2, \ldots, R_s$ be the $(\Delta_3 - \{a,b\})$-residues of the $\hat{a}$-residue Ξ of Γ; by construction, they exactly represent the 1-simplices of $K(\Gamma)$ on which a direct stellar operation is performed, in order to obtain $K(\beta\Gamma)$. Since $b \neq 3$ is assumed, it is obvious that Ξ is a regular graph of degree three if and only if every R_i $(1 \leq i \leq s)$ is a regular graph of degree two. Moreover, since (Γ,γ) represents a 3-manifold M^3, the 3-residue Ξ represents either the 2-sphere $\mathbb{S}^2$ (in case Ξ being a regular graph) or the

[2] In fact, every performed dipole has involved colour n and is, obviously, an internal dipole.

2-ball $\mathbb{D}^2$ (in case Ξ being not regular); thus, $\partial\Xi$ - if not empty - is a connected 2-coloured regular graph, representing the 1-sphere $\mathbb{S}^1$ and containing all the boundary vertices of Ξ. In other words, if $R_1, R_2, \ldots, R_t$ $(t \leq s)$ (resp. $R_{t+1}, R_{t+2}, \ldots, R_s$) are the regular (resp. not regular) $(\Delta_3 - \{a,b\})$-residues of Ξ, and if x_j, y_j are the two boundary vertices of the open path R_{t+j} (for $j = 1, \ldots, s-t$), then $\partial\Xi$ - if not empty - consists of exactly one $\{b,c\}$-coloured cycle (with $\{c\} = \Delta_2 - \{a,b\}$), having $\{x_j, y_j \ / \ 1 \leq j \leq s-t\}$ as vertex-set, with x_j c-adjacent to y_j, for every $j \in \{1, 2, \ldots, s-t\}$, and y_j b-adjacent to $x_{p(j)}$, p being a suitable cyclic permutation of $\{1, 2, \ldots, s-t\}$.

Now, if $(\beta\Gamma, {}^{\beta}\gamma) \in \mathcal{G}_3$ is obtained by bisection of type (a,b) on Ξ, it is easy to check - by making use of Definition 7 - that $\{y'_j, x'_{p(j)}\}$ is a 1-wound in $\beta\Gamma$ involving colour a, for every $j \in \{1, 2, \ldots, s-t-1\}$; moreover, the suture of those 1-wounds yields a 2-wound $\{y'_{t-s}, x'_{p(t-s)}\}$ involving colours a, c. Thus, after suturing $s - t \geq 0$ wounds, a new 4-coloured graph $(\bar{\Gamma}, \bar{\gamma})$ is obtained, so that the (possibly boundary) vertex w of $K(\Gamma)$ originally represented by Ξ is represented in $K(\bar{\Gamma})$ by a regular $\hat{b}$-residue $\bar{\Xi}$, whose $(\Delta_3 - \{a,b\})$-residues are $R'_1, R'_2, \ldots, R'_t$ (copies of $R_1, R_2, \ldots, R_t$) and $\bar{R}$ (obtained by connecting through n-coloured edges the copies of $R_{t+1}, R_{t+2}, \ldots, R_s$).

It is not difficult to prove that, since $\bar{\Xi}$ is a regular and planar graph (representing $\mathbb{S}^2$), a finite sequence $\{v'_1, w'_1\}, \ldots, \{v'_l, w'_l\}$ (with $l = (\#V(\Xi) - 2)/2)$ of dipoles, all involving colour a, exists, whose eliminations transforms $\bar{\Xi}$ into the standard 3-coloured graph consisting of two vertices (v'_0, w'_0, say). [3]

Further, the particular structure of $\bar{\Gamma}$ ensures that $\{v'_1, w'_1\}, \ldots,$ $\{v'_l, w'_l\}$ constitute subsequent dipoles in $\bar{\Gamma}$, too: for, note that, if $\bar{v}, \bar{w}$ are boundary vertices of Ξ, then $\bar{v}, \bar{w}$ may be a-adjacent if and only if they are also 3-adjacent in $\bar{\Gamma}$. Finally, it is very easy to check that, after elimination of the sequence of dipoles $\{v'_1, w'_1\}, \ldots, \{v'_l, w'_l\}$ and of the 3-dipole $\{v'_0, w'_0\}$, the $\hat{b}$-residue $\bar{\Xi}$ disappears, and the starting graph (Γ, γ) is re-obtained.

Hence, the statement results to be proved, by simply inverting the

[3]Note that, precisely in this step of our proof, the dimensional assumption is essential: in fact, for $n \geq 4$, it is known the existence of n-coloured graphs representing S^{n-1}, which can not be "reduced" by a sequence of dipole eliminations.

whole process from (Γ, γ) to $(\beta\Gamma, {}^{\beta}\gamma)$.

∎

Corollary 7. *Let $(\Gamma, \gamma) \in \mathcal{G}_n$ represent an n-manifold M^n, with $n \leq 3$. If $\tilde{\Gamma}$ is the baricentric subdivision of Γ, then Γ and $\tilde{\Gamma}$ are equal up to dipoles.*

Proof. By Proposition 2, we know that, if $(\Gamma, \gamma) \in \mathcal{G}_n$ represents an n-manifold M^n, then its baricentric subdivision $(\tilde{\Gamma}, \tilde{\gamma})$ may be obtained from (Γ, γ) by means of a finite sequence of bisections of type (a, b), with $b \neq n$; on the other hand, Lemma 6 ensures that, in the particular case of dimension $n \leq 3$, every such bisection is equivalent to a finite sequence of dipole insertions and/or wound openings. Finally, the statement follows by remembering that wound moves on graphs representing manifolds are nothing but compositions of finite sequences of dipole moves (see [G2: Prop. 5.7]).

∎

We are now able to prove that our Main Theorem holds for 4-coloured graphs having the same boundary; this will be the key-stone to prove it in the general situation.

Proposition 8. *Let $(\Gamma, \gamma), (\Gamma', \gamma') \in \mathcal{G}_3$ represent the same 3-manifold M^3. If $\partial\Gamma = \partial\Gamma'$, then (Γ, γ) and (Γ', γ') are equal up to dipoles.*

Proof. Let us consider the baricentric subdivision $(\tilde{\Gamma}, \tilde{\gamma}) \in \mathcal{G}_3$ (resp. $(\tilde{\Gamma}', \tilde{\gamma}') \in \mathcal{G}_3$) of (Γ, γ) (resp. (Γ', γ')); since $\tilde{K} = K(\tilde{\Gamma})$ and $\tilde{K}' = K(\tilde{\Gamma}')$ are simplicial triangulations of the same 3-manifold M^3, with $\partial\tilde{K} = \partial\tilde{K}'$, Proposition 4 ensures that $\tilde{K}$ and $\tilde{K}'$ are bistellar equivalent. Thus, if $(\tilde{\tilde{\Gamma}}, \tilde{\tilde{\gamma}}) \in \mathcal{G}_3$ (resp. $(\tilde{\tilde{\Gamma}}', \tilde{\tilde{\gamma}}') \in \mathcal{G}_3$) denotes the baricentric subdivision of $(\tilde{\Gamma}, \tilde{\gamma})$ (resp. $(\tilde{\Gamma}', \tilde{\gamma}')$), then Lemma 5 states that $(\tilde{\tilde{\Gamma}}, \tilde{\tilde{\gamma}})$ and $(\tilde{\tilde{\Gamma}}', \tilde{\tilde{\gamma}}')$ are equal up to (internal) dipoles. Now, a double use of Corollary 7 completes the proof: (Γ, γ) (resp. (Γ', γ')), $(\tilde{\Gamma}, \tilde{\gamma})$ (resp. $(\tilde{\Gamma}', \tilde{\gamma}')$) and $(\tilde{\tilde{\Gamma}}, \tilde{\tilde{\gamma}})$ (resp. $(\tilde{\tilde{\Gamma}}', \tilde{\tilde{\gamma}}')$) are equal up to dipoles, and so (Γ, γ) and (Γ', γ') are.

∎

Remark 1. The proof of Proposition 8, without the aid of an n-dimensional result similar to Lemma 6 (which probably does not hold!), yields quickly to the following statement:

Let $(\Gamma,\gamma),(\Gamma',\gamma') \in \mathcal{G}_n$ represent the same n-manifold M^n. If $\partial\Gamma = \partial\Gamma'$, then (Γ,γ) and (Γ',γ') are equal up to (internal) dipoles and bisections.

Since both internal dipole moves and bistellar operations do not affect the boundary triangulations, in the general situation it is useful to know how to induce boundary moves on a manifold triangulation; actually, an indirect proof of the following result is already contained in [CG, Lemma A and B].

Lemma 9. *Let (Λ,λ), $(\Lambda',\lambda') \in \mathcal{G}_{n-1}$ be equal up to dipoles. Then, for every $(\Gamma,\gamma) \in \mathcal{G}_n$ with $(\partial\Gamma, {}^{\partial}\gamma) = (\Lambda,\lambda)$, there exists $(\Gamma',\gamma') \in \mathcal{G}_n$ with $(\partial\Gamma', {}^{\partial}\gamma') = (\Lambda',\lambda')$ such that (Γ,γ), (Γ',γ') are equal up to dipoles.*

Proof. Obviously, it is sufficient to prove the statement in the following two cases:
case a) (Λ',λ') is obtained from (Λ,λ) by a dipole elimination;
case b) (Λ',λ') is obtained from (Λ,λ) by a dipole insertion.

Case a) Let (Λ',λ') be obtained from (Λ,λ) by eliminating the h-dipole $\theta = \{\bar{x},\bar{y}\}$ $(1 \le h \le n-1)$ and let x (resp. y) be the boundary vertex of (Γ,γ) corresponding to $\bar{x}$ (resp. $\bar{y}$). Then, the required $(\Gamma',\gamma') \in \mathcal{G}_n$ is simply obtained from (Γ,γ) by adding a new n-coloured edge between x and y (i.e. by *suturing the wound* $\{x,y\}$, according with Definition 6).

For, let $(\bar{\Gamma},\bar{\gamma})$ be the (n+1)-coloured graph obtained from (Γ,γ) by adding four new vertices x',y',x'',y'' and the following $2n+2$ new edges: $e'_0,\dots,e'_{h-1}$ between x' and y'; $e''_0,\dots,e''_{h-1}$ between x'' and y''; $e_h^{(x)},\dots,e_{n-1}^{(x)}$ between x' and x''; $e_h^{(y)},\dots,e_{n-1}^{(y)}$ between y' and y'', $e_n^{(x)}$ between x and x'; $e_n^{(y)}$ between y and y', with colouring:

$$\begin{aligned} \bar{\gamma}^{(h)}(e'_i) &= \bar{\gamma}^{(h)}(e''_i) = i & \qquad 0 \le i \le h-1 \\ \bar{\gamma}^{(h)}(e_j^{(x)}) &= \bar{\gamma}^{(h)}(e_j^{(y)}) = j & \qquad h \le j \le n \end{aligned}$$

It is now easy to check that (Γ,γ) (resp. (Γ',γ')) may be obtained from $(\bar{\Gamma},\bar{\gamma})$ by eliminating the dipole $\{x',x''\}$ of type $n-h$ (resp. $\{x'',y''\}$ of type h) and the resulting dipole $\{y',y''\}$ of type n (resp. $\{x',y'\}$ of type n). Since $(\partial\Gamma', {}^{\partial}\gamma') = (\Lambda', \lambda')$ is obviously verified, in case a) the thesis follows.

Case b) Let (Λ', λ') be obtained from (Λ, λ) by adding the h-dipole $\theta = \{\bar{x}, \bar{y}\}$ involving colours $c_1, c_2, \ldots, c_h \in \Delta_{n-1}$ within the $(\Delta_{n-1} - \{c_1, c_2, \ldots, c_h\})$-residue Z of (Λ, λ). Further, let $(\bar{\Gamma}, \bar{\gamma})$ be the (n+1)-coloured graph obtained by taking two copies Z', Z'' of Z, and by adding them to (Γ, γ) in the following way: if z (resp. z', z'') is the vertex of (Γ, γ) (resp. Z') (resp. Z'') corresponding to the vertex $\bar{z} \in V(Z)$, then join each vertex $z' \in V(Z')$ with the (boundary) vertex $z \in V(\Gamma)$ by an n-coloured edge and with the vertex $z'' \in V(Z'')$ by n-h multiple edges coloured by $\Delta_{n-1} - \{c_1, c_2, \ldots, c_h\}$. Since, for every $\bar{z} \in V(Z)$, $\{z', z''\}$ is an $(n-h)$-dipole in $(\bar{\Gamma}, \bar{\gamma})$, it is easy to check that (Γ, γ) may be obtained from $(\bar{\Gamma}, \bar{\gamma})$ by p dipole eliminations (of type m, with $1 \leq m \leq n$), where $p = \#V(Z)$.

On the other hand, $(\partial\bar{\Gamma}, {}^{\partial}\bar{\gamma}) = (\partial\Gamma, {}^{\partial}\gamma) = (\Lambda, \lambda)$, and a dipole θ'' isomorphic with θ may be added to $(\bar{\Gamma}, \bar{\gamma})$ within the $(\Delta_{n-1} - \{c_1, c_2, \ldots, c_h\})$-residue Z'', giving rise to an $(n+1)$-coloured graph (Γ', γ') with the required properties: $(\partial\Gamma', {}^{\partial}\gamma') = (\Lambda', \lambda')$ and (Γ, γ), (Γ', γ') equal up to dipoles.

■

Proof of the Main Theorem. Since it is known that dipole moves do not affect the homeomorphism class of the represented manifold (see [G2] or Proposition 1), one only implication has to be proved.

For, let us assume $K(\Gamma)$ and $K(\Gamma')$ to be (different) pseudosimplicial triangulations of the same manifold M^n. If $\partial\Gamma = \partial\Gamma'$ (for example, if Γ and Γ' are both graphs with void boundary), then Proposition 8 yields the thesis. Otherwise, we will prove the statement (which is trivial in dimension 1) by induction on dimension n (with $n \leq 3$). In fact, inductive hypothesis (in the closed case) ensures the existence of a finite sequence of dipole moves connecting $\partial\Gamma$ and $\partial\Gamma'$; by Lemma 9, a finite sequence of dipole moves on (Γ', γ') exists, yielding a new graph (Γ'', γ'') with $\partial\Gamma'' = \partial\Gamma$. The thesis now directly follows from Proposition 8, applied to (Γ, γ) and (Γ'', γ'').

■

In order to derive a partial generalization of the Main Theorem to dimension n, the following result is of use.

Lemma 10. *Let (Λ, λ), $(\Lambda', \lambda') \in \mathcal{G}_{n-1}$, with Λ' obtained from Λ by bisections. Then, for every $(\Gamma, \gamma) \in \mathcal{G}_n$ with $(\partial\Gamma, {}^{\partial}\gamma) = (\Lambda, \lambda)$, there exists $(\Gamma', \gamma') \in \mathcal{G}_n$ with $(\partial\Gamma', {}^{\partial}\gamma') = (\Lambda', \lambda')$ such that Γ' is obtained from Γ by bisections.*

Proof. Without loss of generality, we may assume (Λ', λ') to be obtained from (Λ, λ) by a single bisection of type (a, b) on a $\hat{a}$-residue Ξ of Λ (with $a, b \in \Delta_{n-1}$, $a \neq b$). Obviously, since $(\partial\Gamma, {}^{\partial}\gamma) = (\Lambda, \lambda)$, a $\hat{a}$-residue $\tilde{\Xi}$ of Γ exists, such that $\partial\tilde{\Xi} = \Xi$; hence, the required $(\Gamma', \gamma') \in \mathcal{G}_n$ is simply obtained from (Γ, γ) by bisection of type (a, b) on $\tilde{\Xi}$. In fact, if w is the a-labelled vertex of $K(\Gamma)$ such that $K(\tilde{\Xi}) = lkd(w, K(\Gamma))$, we know that the pseudocomplex $K(\Gamma')$ associated to Γ' is obtained from $K(\Gamma)$ by performing a direct stellar operation on the set $E_{(a,b)}(w)$ of 1-simplices of $K(\tilde{\Xi})$ having as end-points w and a b-labelled vertex; on the other hand, since w is also the a-labelled vertex of $K(\Lambda) = \partial K(\Gamma)$ such that $K(\Xi) = lkd(w; K(\Lambda)) = lkd(w; \partial K(\Gamma))$, the 1-simplices of $K(\Xi)$ on which a stellar operation is performed in order to obtain $K(\Lambda')$ are nothing but the boundary ones belonging to $E_{(a,b)}(w)$.

The thesis now follows by finite iteration on the number of bisections involved in the process.

■

Remark 2. A proof similar to that of the Main Theorem, with the aid of Remark 1 and Lemma 10, yields quickly to the following statement:

Let $(\Gamma, \gamma) \in \mathcal{G}_n$ (resp. $(\Gamma', \gamma') \in \mathcal{G}_n$) represent the n-manifold $M^n = |K(\Gamma)|$ (resp. $N^n = |K(\Gamma')|$). Then, M^n and N^n are PL-homeomorphic manifolds if and only if (Γ, γ) and (Γ', γ') are equal up to dipoles and bisections.

References

[A] J. W. Alexander, *The combinatorial theory of complexes.* Annals of Math. (2) **31** (1930), 294-322.

[BM] J. Bracho - L. Montejano, *The combinatorics of colored triangulations of manifolds.* Geom. Dedicata **22** (1987), 303-328.

[C] M. R. Casali, *A note about bistellar operations on PL-manifolds with boundary.* Geom. Dedicata **56** (1995), 257-262.

[CG] M. R. Casali - C. Gagliardi, *A combinatorial proof of Rohlin theorem.* Geom. Dedicata, to appear.

[CP] M. R. Casali - R. Panisi, *Equivalenze combinatorie di varietà bidimensionali,* Atti Sem. Mat. Fis. Univ. Modena **44** (1996), 271-294.

[CvG] A. Cavicchioli - C. Gagliardi, *Crystallizations of PL-manifolds with connected boundary,* Boll. Un. Mat. Ital.**17-B** (1980), 902-917.

[CV] A. Costa - P. del Val-Melus, *Crystallizations and permutations representing 3-manifolds,* Boll. Un. Mat. Ital. **1-B** (1987), 477-490.

[FG] M. Ferri - C. Gagliardi, *Crystallization moves,* Pacific J.Math. **100** (1982), 233-246.

[FGG] M. Ferri - C. Gagliardi - L. Grasselli, *A graph-theoretical representation of PL-manifolds. A survey on crystallizations,* Aequationes Math.**31** (1986), 121-141.

[G_1] C. Gagliardi, *Cobordant crystallizations,* Discrete Math. **45** (1983), 61-73.

[G_2] C. Gagliardi, *On a class of 3-dimensional polyhedra,* Ann. Univ. Ferrara **33** (1987), 51-88.

[Gl] L. C. Glaser, *Geometrical combinatorial Topology,* Van Nostrand Reinhold (1970).

[HW] P. J. Hilton - S. Wylie, *An introduction to algebraic topology - Homology theory,* Cambridge Univ. Press (1960).

[K] R. Kirby, *A calculus for framed links in S^3,* Inv. Math. **45** (1978), 35-56.

[LM] S. Lins - A. Mandel, *Graph-encoded 3-manifolds,* Discrete Math. **57** (1985), 261-284.

[M] J. M. Montesinos, *Heegaard diagrams for closed 4-manifolds*, Geometric topology - Proc. Georgia Topol.Conference, J.C.Cantrell Ed., Academic Press (1979), 219-237.

[Mo] E. E. Moise, *Affine structures in 3-manifolds - V - The triangulation theorem and Haupvermutung*, Annals of Math. **56** (1952), 96-114.

[P] U. Pachner, *Konstruktionsmethoden und das kombinatorische Homoomorphieproblem fur Triangulationen kompakter semilinearer Mannigfaltigkeiten*, Abh. Math. Sem. Hamb. **57** (1986), 69-85.

[Pe] M. Pezzana, *Sulla struttura topologica delle varietà compatte*, Atti Sem. Mat. Fis. Univ. Modena **23** (1974), 269-277.

[Pi] R. Piergallini, *Covering moves*, Trans. Amer. Math. Soc. **325 n.2** (1991), 903-920.

[R] K. Reidemeister, *Heegaarddiagramme und invarianten von Mannigfaltigkeiten*, Hamburg Abh. **10** (1933), 109-118.

[RS] C. Rourke - B. Sanderson, *Introduction to Piecewise-linear topology*, Springer Verlag (1972).

[S] J. Singer, *Three-dimensional manifolds and their Heegaard diagrams*, Trans. Amer. Math. Soc. **35** (1933), 88-111.

[V] A. Vince, *Combinatorial maps*, J. Combin. Theory Ser. B **34** (1983), 1-21.

[W] A. T. White, *Graphs, groups and surfaces*, North Holland, Amsterdam - London (1973).

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