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Conjugacy pinched and cyclically pinched one-relator groups.

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Abstract

Here we consider two classes of torsion-free one-relator groups which have proved quite amenable to study-the **cyclically pinched one-relator groups** and the **conjugacy pinched one-relator groups**. The former is the class of groups which are free products of free groups with cyclic amalgamations while the latter is the class of HNN extensions of free groups with cyclic associated subgroups. Both are generalizations of surface groups. We compare and contrast results in these classes relative to n-freeness, separability properties including conjugacy separability, subgroup separability and residual finiteness, decision theoretic properties including the isomorphism problem and hyperbolicity.

1 Introduction

The theory of one-relator groups has always been of central importance in combinatorial group theory. The motivations for this interest are varied and arise from both topology and complex function theory as well as from algebra. From a topological viewpoint the class of fundamental groups of compact surfaces - the surface groups - falls into the class of one-relator groups. Many general questions concerning one-relator groups are in turn motivated by surface groups. From an algebraic

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viewpoint one-relator groups are of interest because they provide a natural generalization of free groups with which they exhibit many similarities. Good general references on one-relator groups are the article by G. Baumslag [G.B. 1] and the book of Lyndon and Schupp [L-S].

The cornerstone of one-relator group theory is the Freiheitssatz. The proof and its many modifications (see [F-R 1]) have provided several techniques for a general treatment of one-relator groups. However many examples and counterexamples show that the theory is quite complex and in order to proceed further focus must be narrowed and in particular restrictions must be placed on the form of the relator. If $G = < x_1, ..., x_n; R >$ then Karrass, Magnus and Solitar [K-M-S] proved that G is torsion-free unless $R = S^m$ for some $m \geq 2$ and some nontrivial, non-power word S in the free group on $x_1, .., , x_n$, in which case all elements of finite order are conjugate to powers of S. Torsion one-relator groups have proved somewhat more amenable to study than torsion-free due to the Spelling Theorem of B.B. Newman [Ne 1].

In this paper we examine two classes of torsion-free one-relator groups which have proven tractable for study. Both arise originally from the study of surface groups but have additional motivations as well. The first is the class of **cyclically pinched one relator groups**. These are groups which have presentations of the form

$$G = < a_1, ..., a_p, a_{p+1}, ..., a_n; U = V > \qquad (1)$$

where $1 \neq U = U(a_1, ..., a_p)$ is a non-primitive (not part of a free basis) word in the free group F_1 on $a_1, ..., a_p$ and $1 \neq V = V(a_{p+1}, ..., a_n)$ is a non-primitive word in the free group F_2 on $a_{p+1}, ..., a_n$. Structurally such a group is the free product of the free groups on $a_1, ..., a_p$ and $a_{p+1}, ..., a_n$ respectively amalgamated over the cyclic subgroups generated by U and V. The class of cyclically pinched one-relator groups has been extensively studied and it has been shown that in general such a group shares many of the algebraic properties of surface groups. In the next section we will discuss these groups in detail.

The second class is the class of **conjugacy pinched one relator groups**. These are groups which have presentations of the form

$$G = < a_1, ..., a_n, t; tUt^{-1} = V > \qquad (2)$$

where $1 \neq U = U(a_1, ..., a_n)$ and $1 \neq V = V(a_1, ..., a_n)$ are words in the free group on $a_1, ..., a_n$. Structurally such a group is an HNN extension of the free group on $a_1, ..., a_n$ with cyclic associated subgroups generated by U and V. These are closely related to the cyclically pinched case and will be discussed in detail in section 3.

2 Surface Groups and Cyclically Pinched One-Relator Groups

A **surface group** of genus g is the fundamental group of a compact surface of genus g. In the orientable case with genus $g \geq 2$ the resulting group T_g has a presentation

$$T_g = < a_1, b_1, ..., a_g, b_g; [a_1, b_1]...[a_g, b_g] = 1 > \quad (3)$$

If we let $U = [a_1, b_1]...[a_{g-1}, b_{g-1}]$, $V = [a_g, b_g]$ then T_g has the form

$$T_g = < a_1, b_1, ..., a_g, b_g; U = V^{-1} > .$$

and hence T_g is a cyclically pinched one-relator group. In the non-orientable case the resulting group U_g has a presentation

$$U_g = < a_1, ..., a_g; a_1^2...a_g^2 = 1 > .$$

If we let $U = a_1^2...a_{g-1}^2$, $V = a_g^2$ then for $g \geq 2$, U_g is also a cyclically pinched one-relator group.

The question arises as to which properties of surface groups are shared by all cyclically pinched one-relator groups. The generalization in turn of cyclically pinched one-relator groups where torsion is allowed in the generators leads to **groups of F-type** which can be considered as a natural algebraic generalization of finitely generated Fuchsian groups. These latter class was introduced in [F-R 2] and has also been extensively studied (see [F-R 3] and the references there).

An orientable surface group T_g has the property that any $2g - 1$ elements generate a free subgroup. A simple topologically motivated proof of this is as follows. By abelianizing it is clear that the rank (minimum number of necessary generators) of T_g is $2g$. Suppose H is a subgroup of T_g, then from covering space theory $H = \pi_1(S)$ where S is a cover of S_g, the orientable surface of genus g. If $|T_g : H| < \infty$, then S must be another orientable surface of genus $g_1 \geq g$ and hence

$H = T_{g_1}$. If H has infinite index in T_g then homotopically S is a wedge of circles and H is a free group. Now suppose we have $x_1, ..., x_n \in T_g$ with $n \leq 2g - 1$, and let $H = < x_1, ..., x_n >$. If H had finite index in T_g, then the rank of H would be greater than $2g$ which is impossible since H has rank $\leq n$. Therefore H must have infinite index in T_g and hence must be a free group. In general we say that a group G is **n-free** if any set of $\leq n$ elements of G generates a free subgroup and therefore the above argument shows that any orientable surface group is $(2g - 1)$-free. A modified argument shows that a non-orientable surface group of genus g is $(g - 1)$-free. G. Baumslag [G.B. 2] generalized this to certain cyclically pinched one-relator groups.

Theorem A1. *[G.B. 2] Let G be a cyclically pinched one-relator group with the property that U and V are not proper powers in the respective free groups on the generators which they involve. Then G is 2-free.*

Using Nielsen and extended Nielsen reduction (see the article [F-R-S 1] for terminology), G.Rosenberger [Ro 1] was then able to give a complete classification of the subgroups of rank ≤ 4 of cyclically pinched one-relator groups.

Theorem A2. *[Ro 1] Let G be a cyclically pinched one-relator group with the property that U and V are not proper powers in the respective free groups on the generators which they involve. Then*

(1) G is 3-free.

(2) Let $H \subset G$ be a subgroup of rank 4. Then one of the following two cases occurs:

(i) H is free of rank 4.

(ii) If $\{x_1, ..., x_4\}$ is a generating system for H then there is a Nielsen transformation from $\{x_1, ..., x_4\}$ to $\{y_1, ..., y_4\}$ with $y_1, y_2 \in zF_1z^{-1}, y_3, y_4 \in zF_2z^{-1}$ for a suitable $z \in G$. Further there is a one-relator presentation for H on $\{x_1, ..., x_4\}$.

We note that the 3-free part of the above theorem was reproven in a different manner by G.Baumslag and P. Shalen [B-S 1]. Further the free-ness part of the above results was extended in the following manner by Fine, Gaglione, Rosenberger and Spellman [F-G-R-S] again using Nielsen reduction techniques.

Theorem A3. *Let $B_1, ..., B_n$ be pairwise disjoint sets of generators, each of size ≥ 2 and for $i = 1, .., n$ let $W_i = W_i(B_i)$ be a non-trivial word in the free group on B_i, neither a proper power nor a primitive element. Let*

$$G = < B_1, ..., B_n; W_1 W_2 ... W_n = 1 > .$$

Then G is n-free.

A similar result can be obtained if the words W_i are proper powers.

Theorem A4. *Let $B_1, ..., B_n$ be pairwise disjoint non-empty sets of generators, and for $i = 1, .., n$ let $W_i = W_i(B_i)$ be a non-trivial word in the free group on B_i. Let*

$$G = < B_1, ..., B_n; W_1^{t_1} W_2^{t_1} ... W_n^{t_n} = 1 >$$

with $t_i \geq 1$. Then G is (n-1)-free.

This result is the best possible since a non-orientable surface group of genus g is $(g-1)$ free but not free.

These results were used in conjuction with a study by Gaglione and Spellman [G-S 1,2,3] and Fine, Gaglione, Rosenberger and Spellman [F-G-R-S] on the universal theory of non-abelian free groups. We'll say more about this in the next section in conjuction with conjugacy pinched one-relator groups.

Recall that a group G is **residually finite** if given $g \in G$ there exists a finite quotient G^* of G with the image of g non-trivial in G^*. G. Baumslag [G.B. 3] has shown that all cyclically pinched one-relator groups are residually finite.

Theorem A5. *Let G be a cyclically pinched one-relator group, then G is residually finite.*

The residual finiteness of one-relator groups in general is undecided. However it has been conjectured by G. Baumslag that one-relator groups with torsion are residually finite. Several special cases of this conjecture have been handled by Allenby and Tang [A1-T1]. In addition to residual finitenss cyclically pinched one-relator groups satisfy several stronger separability properties. A group G is **conjugacy separable** if given elements g, h in G either g is conjugate to h or there exists a finite quotient where they are not conjugate. J. Dyer [Dy] has proved the conjugacy separability of cyclically pinched one-relator groups. Note that conjugacy separability in turn implies residual finiteness.

Theorem A6. *[Dy] A cyclically pinced one-relator group is conjugacy separable, that is two elements of a cyclically pinched one-relator group G are conjugate if and only if they are conjugate in every finite factor group of G.*

It was conjectured that this result could be extended to general Fuchsian groups. Building on work of Stebe[St] and Allenby and Tang [A-T], Fine and Rosenberger [F-R 4] proved the conjugacy separability of general Fuchsian groups.

A group G is **subgroup separable** or **LERF** if H is any finitely generated subgroup of G and $g \in G, g \notin H$, then there exists a finite quotient $G*$ of G such that image of g lies outside the image of H. P.Scott [Sc] proved that surface groups are subgroup separable and then Brunner, Burns and Solitar [Br-B-S] showed that in general cyclically pinched one-relator groups are subgroup separable. Free groups themselves are subgroup separable and Tretkoff [Tr], Gitik[G], Tang[T], Kim[K],Niblo[N],Aab [A] and others have worked on the general question of when free products with amalgamation of subgroups separable groups is again subgroup separable.

Theorem A7. *[Br-B-S] Let G be a cyclically pinched one-relator group. Then G is subgroup separable.*

One-relator groups and groups of F-type in general satisfy many linearity properties - that is properties of linear groups. A result of P.Shalen [Sh] can be used to establish that a cyclically pinched one-relator group where neither U nor V (in presentation 1) is a proper power actually has a faithful presentation in $PSL_2(\mathbb{C})$. Earlier Wehrfritz [We] had established that such a group has a faithful representation over a commutative field.

Theorem A8. *Let G be a cyclically pinched one-relator group with the property that U and V are not proper powers in the respective free group on the generators which they involve. Then G has a faithful representation in $PSL_2(\mathbb{C})$.*

The decision theory of cyclically pinched one-relator groups is also very well determined. Magnus proved [M] that as a consequence of the Freiheitssatz the word problem is solvable in general for one-relator groups. Lipschutz [Li] using small cancellation theory proved that cyclically pinched one-relator groups have solvable conjugacy problem. Subsequently Juhasz [J] using extended small cancellation theory has stated that all one-relator groups have solvable conjugacy problem.

Theorem A9. *[Li] A cyclically pinched one-relator group has a solvable conjugacy problem.*

Rosenberger [Ro 2], again using Nielsen reduction methods, has given a positive solution to the isomorphism problem for cyclically pinched one-relator groups, that is, he has given an algorithm to determine if an arbitrary one-relator group is isomorphic or not to a given cyclically pinched one-relator group.

Theorem A10. *[Ro 2] The isomorphism problem for any cyclically pinched one-relator group is solvable; given a cyclically pinched one-relator group G there is an algorithm to decide in finitely many steps whether an arbitrary one-relator group is isomorphic or not to G.*

More specifically let G be a non-free cyclically pinched one-relator group such that at most one of U and V is a power of a primitive element in F_1 respectively F_2. Suppose $x_1, ..., x_{p+q}$ is a generating system for G. Then one of the following two cases occurs:

(1) There is a Nielsen transformation from $\{x_1, ..., x_{p+q}\}$ to a system $\{a_1, ..., a_p, y_1, .., y_q\}$ with $y_1, ..., y_q \in F_2$ and $F_2 =< V, y_1, .., y_q >$.

2) There is a Nielsen transformation from $\{x_1, ..., x_{p+q}\}$ to a system $\{y_1, ..., y_p, b_1, .., b_q\}$ with $y_1, ..., y_p \in F_1$ and $F_1 =< U, y_1, .., y_p >$.

For $x_1, ..., x_{p+q}$ there is a presentation of G with one-relator. Further G has only finitely many Nielsen equivalence classes of minimal generating systems.

The small cancellation theory used by Lipschutz and Juhasz is closely tied to hyperbolicty in the sense of Gromov (see [G]). As a consequence of a result of Bestvinna and Feighn [Be-F] we obtain that a cyclically pinched one-relator groups where not both U and V are proper powers is hyperbolic.

Theorem A11. *Let G be a cyclically pinched one-relator group with the property that not both U and V are proper powers in the respective free group on the generators which they involve. Then G is word hyperbolic.*

Theorem A11 was further generalized by Juhasz and Rosenberger (see [J-R]).

From a result of G.Baumslag and Shalen [B-S 2] a one-relator group G with at least three generators admits a free product with amalgamation decomposition $G = A \star_C B$ with A, B and C all finitely generated. Such a decomposition is called a **Baumslag-Shalen decomposition.** Clearly cyclically pinched one-relator groups are straightforward examples of such decompositions for one-relator groups. However in general very little is known about the exact nature of the factors. A study of the factors was done by Fine and Peluso [F-P] who gave the following partial converse to the cyclically pinched case.

Theorem A12. *[F-P] Let G be a torsion-free one-relator group with Baumslag-Shalen decomposition $A \star_C B$ with both A and B free groups. Then G must be cyclically pinched if either C has finite index in both factors or C is in the derived group in both factors.*

Further, if C has finite index in both factors then the groups A, B, C are all infinite cyclic and G has a presentation of the form $< a, b; a^n = b^m >$ with $m, n > 1$.

A result of Bieri [Bi] is that if $G = A \star_C B$ is a torsion-free one-relator group with A, B finitely presented and C of finite index in both A and B then A and B must be free groups. Combining this with Theorem A12 gives us:

Corollary A12. *Let G be a torsion-free one-relator group with Baumslag-Shalen decomposition $A \star_C B$. If C is a free group and of finite index in both A and B then the groups A, B, C are all infinite cyclic and G has a presentation of the form $< a, b; a^n = b^m >$ with $m, n > 1$.*

3 Conjugacy Pinched One-Relator Groups

If we return to the surface group T_g

$$T_g = < a_1, b_1, ..., a_g, b_g; [a_1, b_1]...[a_g, b_g] = 1 > \qquad (3.)$$

and we let $b_g = t$ then T_g has the form

$$T_g = < a_1, b_1, ..., a_g, t; tUt^{-1} = V > .$$

where $U = a_g$ and $V = [a_1, b_1]...[a_{g-1}, b_{g-1}]a_g$ and hence T_g is also a conjugacy pinched one-relator group.

The question arises as to which of the general properties of cyclically pinched one-relator groups can be extended to the class of conjugacy pinched one-relator groups. Given the structural similarities of free product with amalgamation and HNN extensions many similarities in properties are to be expected, of course somewhat modified.

Besides the natural ties with surface groups conjugacy pinched one-relator groups arise independently in other contexts as well. Recall that a group acts freely on a tree T if it acts as a group of isometries on T with no fixed points or inversions. If the tree is an ordinary simplicial tree then from Bass-Serre theory G must be a free group. If T is an $\mathbb{R}$-tree (see [B]) then Rips Theorem says that G must be a free product of abelian groups and surface groups. Bass studied the general concept of a free action on a Λ-tree where Λ is an ordered abelian group. As examples of groups which allow free actions on Λ-trees are cyclically pinched one-relator groups and conjugacy pinched one-relator groups except for the Klein bottle group $< a, b; aba^{-1} = b^{-1} >$ (as a subgroup).

Further conjugacy pinched one-relator groups appear in the classification of fully residually free groups and the study of CSA groups (see F-G-M-R-S]). We'll say more of this shortly.

The 2-free and 3-free results for cyclically pinched one-relator groups carry over with modifications to conjugacy pinched one-relator groups. The results for cyclically pinched one-relator groups used Nielsen reduction in free products with amalgamation as developed by Zieschang [Z], Rosenberger[Ro 3,4,5] and others (see [F-R-S 1]). The corresponding theory of Nielsen reduction for HNN groups was developed by Peczynski and Reiwer [P-R] and is used in the analysis of conjugacy pinched one-relator groups. Important for applications of in Peczynski and Reiwer's results is the case where the associated subgroups are malnormal in the base. Recall that $H \subset G$ is malnormal if $xHx^{-1} \cap H = \{1\}$ if $x \notin H$. For a cyclic subgroup $< U >$ of a free group F this requires that U is not a proper power in F. Using this Fine,Roehl and Rosenberger proved the following two-free result.

Theorem B1. *[F-R-R 1] Let $G = < a_1, ..., a_n, t; tUt^{-1} = V >$ be a conjugacy pinched one-relator group. Suppose that neither U nor V are proper powers in the free group on $a_1, ..., a_n$. If $< x, y >$ is a two-generator subgroup of G then one of the following holds:*

(1) $< x, y >$ is free of rank two

(2) $< x, y >$ is abelian

(3) $< x, y >$ has a presentation $< a, b; aba^{-1} = b^{-1} >$.

As a direct consequence of the proof we also obtain the following.

Corollary B1. *Let G be as in Theorem B1 and suppose that U is not conjugate to V^{-1} in the free group on $a_1, ..., a_n$. Then any two-generator subgroup of G is either free or abelian.*

The extension of Theorem B1 to a 3-free result proved to be quite difficult and required some further modifications. A two-generator subgroup N of a group G is **maximal** if rank$N = 2$ and if $N \subset M$ for another two-generator subgroup M of G then $N = M$. A maximal two-generator subgroup $N = < U, V >$ is **strongly maximal** if for each $X \in G$ there is a $Y \in G$ such that $< U, XVX^{-1} > \subset < U, YVY^{-1} >$ and $< U, YVY^{-1} >$ is maximal. Building upon and extending the theory of Peczynski and Reiwer the following is obtained.

Theorem B2. *[F-R-R 2] Let $G = < a_1, ..., a_n, t; tUt^{-1} = V >$ be a conjugacy pinched one-relator group. Suppose that $< U, V >$ is a strongly maximal subgroup of the free group on $a_1, ..., a_n$. Then G is 3-free.*

If $< U, V >$ is not strongly maximal we can further obtain that a subgroup of rank 3 is either free or has a one-relator presentation.

Theorem B2. *[F-R-R 2] Let $G = < a_1, ..., a_n, t; tUt^{-1} = V >$ be a conjugacy pinched one-relator group. Suppose that neither U nor V is a proper power in the free group on $a_1, ..., a_n$ and in this free group U is not conjugate to either V or V^{-1}. Let $H = < x_1, x_2, x_3 > \subset G$. The H is free or has a one-relator presentation on $< x_1, x_2, x_3 >$.*

An analysis of the techniques used in the proof of Theorem B2 leads to several partial solutions of the isomorphism problem for conjugacy pinched one-relator gtoups. First:

Theorem B3. *[F-R-R 2] Let $G =< a_1, ..., a_n, t; tUt^{-1} = V >$ be a conjugacy pinched one-relator group and suppose that neither U nor V is a proper power in the free group on $a_1, ..., a_n$. Suppose further that there is no Nielsen transformation from $\{a_1, ..., a_n\}$ to a system $\{b_1, ..., b_n\}$ with $U \in \{b_1, ..., b_{n-1}\}$ and that there is no Nielsen transformation from $\{a_1, ..., a_n\}$ to a system $\{c_1, ..., c_n\}$ with $V \in \{c_1, ..., c_{n-1}\}$. Then:*

(1) G has rank $n+1$ and for any minimal generating system for G there is a one-relator presentation.

(2) The isomorphism problem for G is solvable, that is it can be decided algorithmically in finitely many steps whether an arbitrary given one-relator group is isomorphic to G.

(3) G is Hopfian

We note that the results of the above theorem hold when $n = 2$ and U, V are elements of F^pF' where F is the free group on a_1, a_2 and F^pF' is the subgroup generated by the pth powers ($p \geq 2$) and the commutators.

The techniques developed for the proofs of Theorem B2 and B3 were also used in the study of the isomorphism problem for a class of para-free groups introduced by G. Baumslag. In particular in [GB 3] Gilbert Baumslag introduced the class of groups $G_{i,j}$ for natural numbers i, j, defined by the presentations

$$G_{i,j} =< a, b, t; a^{-1} = [b^i, a][b^j, t] >$$

This class is of special interest since the groups are **para-free**, that is they share many properties with the free group F of rank 2. In particular, if $\gamma_n(G_{i,j})$ are the terms of the lower central series of $G_{i,j}$, then for all n, $G_{i,j}/\gamma_n(G_{i,j}) \cong F/\gamma_n(F)$ and further the intersection over all n of the $\gamma_n(G_{i,j})$ is $\{1\}$.

Magnus and Chandler [Ch-M] in their History of Combinatorial Group Theory mention the class $G_{i,j}$ to demonstrate the difficulty of the isomorphism problem for torsion-free one-relator groups. They remark that as of 1980 there was no proof showing that any of the groups $G_{i,j}$ are non-isomorphic. S. Liriano [Lir] used representations of $G_{i,j}$ into PSL$(2, p^k)$, $k \in I\!N$, to show that $G_{1,1}$ and $G_{30,30}$ are non-isomorphic.

Notice that if we let $U = a[b^i, a]b^j$ and $V = b^j$ then $G_{i,j}$ is a conjugacy pinched one-relator groups

$$G_{i,j} =< a, b, t; t^{-1}Ut = V > .$$

If in addition $j = 1$ then $< b >$ and $< a[b^i, a]b >$ are maximal cyclic in $< a, b; >$ and hence malnormal and hence the techniques of Theorems B2 and B3 can be applied. Fine,Rosenberger and Stille [F-R-S 2] then proved:

Theorem B4. *[F-R-S 2] Let i be a natural number. Then:*

(1) the isomorphism problem for $G_{i,1}$ is solvable, that is it can be decided algorithmically in finitely many steps whether or not an arbitrary one-relator group is isomorphic to $G_{i,1}$.

(2) $G_{i,1}$ is not isomorphic to $G_{1,1}$ for $i \geq 2$.

(3) if i, k are primes then

$$G_{i,1} \cong G_{k,1} \text{ if and only if } i = k.$$

(4) for all natural numbers i, $G_{i,1}$ is Hopfian, every automorphism of $G_{i,1}$ is induced by an automorphism of the free group $F^\star = F^\star(A, B, T)$ of rank 3, with respect to the epimorphism $A \to a, B \to b, T \to t$, and the automorphism group $Aut\ G_{i,1}$ is finitely generated.

These techniques can also be applied to analyze a second class of groups further extending surface groups. These have the presentations

$$K_{\sigma,\alpha,n} =< a_1, ..., a_n, t; t^{-1}a_1^\alpha...a_n^\alpha t = a_{\sigma(1)}^\alpha...a_{\sigma(n)}^\alpha > \quad (2)$$

where $\alpha \in I\!N, n \in I\!N$ and $\sigma \in S_n$, that is σ is a permutation on $\{1, 2..., n\}$. Notice that if $\alpha = 1, n$ is odd and $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$ then $K_{\sigma,1,n}$ is the surface group of genus $\frac{n+1}{2}$. Notice further that if $\alpha = 1$ the relator in $K_{\sigma,1,n}$ is a quadratic word.

Theorem B5. *[F-R-S 3] Let $K_{\sigma,\alpha,n}$ be as above. If $\{x_1, ..., x_{n+1}\}$ is a generating system of $K_{\sigma,\alpha,n}$ then $\{x_1, ..., x_{n+1}\}$ is Nielsen equivalent to $\{a_1, ..., a_n, t\}$.*

From the Nielsen cancellation method on quadratic words the following corollary is obtained.

Corollary B5. *If $\alpha = 1$ then $K_{\sigma,1,n}$ is a free product of a free group and a surface group. If n is even $K_{\sigma,1,n}$ is never a surface group.*

Kim[K], Niblo [N], Wise [W] and Rosenberger and Sasse[R-S] all considered the residual finitenss and subgroup separability of conjugacy pinched one-relator groups. Let G be a group and u,v elements of infinite order in G. G has **regular quotients** at u if there exists a positive integer m such that for each positive integer s there is a finite index normal subgroup N of G with $N \cap < u > = < u^{ms} >$. G has regular quotients at $\{u, v\}$ if there exists a positive integer m such that for each positive integer s there is a finite index normal subgroup N of G with $N \cap < u > = < u^{ms} >$ and $N \cap < v > = < v^{ms} >$. Niblo [Ni] and Kim [K] proved that if F is a free group and F has regular quotients at $\{U, V\}$ then the corresponding conjugacy pinched one-relator group $< F, t; tUt^{-1} = V >$ is subgroup separable. D.Wise [W] subsequently showed that a free group has regular quotients at a pair of elements unless the elements have conjugate powers. Combining these we have the following.

Theorem B6. *Let $G = < a_1, ..., a_n, t; tUt^{-1} = V >$ be a conjugacy pinched one-relator group with U,V non-trivial elements of infinite order in the free group on $a_1, ..., a_n$ which do not have conjugate powers in this free group. Then G is subgroup separable.*

This result has been extended in various ways. Rosenberger and Sasse [Ro-S] proved the following.

Theorem B7. *Let A be a group and U,V be elements of infinite order in A and let $G = < A, t; tUt^{-1} = V >$ be the HNN extension associating U and V. Suppose further that A has regular quotients at $\{U, V\}$ and that A is both $< U >$-separable and $< V >$-separable. Then G is residually finite.*

D.Wise [W] extended this in the following way. Recall that the Baumslag-Solitar group $BS(n, m)$ is the group $< a, t; ta^n t^{-1} = a^m >$. If $n \neq \pm m$ then $BS(n, m)$ is not subgroup separable (see [Me]).

Theorem B8. *[W] Let*

$$G = < a_1, ..., a_n, t_1, ..., t_m; t_1 U_1 t_1^{-1} = V_1, ..., tU_m t^{-1} = V_m >$$

be a multiple cyclic HNN extension of the free group on $a_1, ..., a_n$ where $U_1, V_1, ..., U_m, V_m$ are non-trivial cyclically reduced words in this free *group. Then G is subgroup separable unless it contains a subgroup isomorphic to $BS(m,n)$ with $n \neq \pm m$.*

Wise and indepedently Aab [A] also considered the subgroup separability of graphs of free groups with cyclic edge groups, continuing the work of Brunner,Burns and Solitar [B-B-S], Tretkoff[T] ,Gitik [Gi], Kim [K] and Niblo [Ni] mentioned in the last section.

Conjugacy pinched one-relator groups also arise in the study of fully residually free groups. This is tied to some questions in logic studied by Gaglione and Spellman (see [G-S 1 2 3] and the references there) and independenty by Myasnikov and Remeslennikov [M-R]. We must define some necessary concepts.

A group G is **residually free** if for each non-trivial $g \in G$ there is a free group F_g and an epimorphism $h_g : G \to F_g$ such that $h_g(g) \neq 1$ and is **fully residually free** provided to every finite set $S \subset G \setminus \{1\}$ of non-trivial elements of G there is a free group F_S and an epimorphism $h_S : G \to F_S$ such that $h_S(g) \neq 1$ for all $g \in S$. Clearly fully residually free implies residually free.

Further a group G is **universally free** if it has the same universal theory as the class of non-abelian free groups (see [F-G-R-S] for a precise formulation) and recall that it is **n-free** for a positive integer n, provided every subgroup of G generated by n or fewer distinct elements is free; **commutative transitive** if commutativity is transitive on the non-identity elements in G; and **tree-free** if G acts freely on some $\Lambda-$tree in the sense of Bass[B] where Λ is some ordered abelian group. Gaglione and Spellman [G-S 6] and independently Remeslennikov [Re], extending a theorem of B.Baumslag [Ba 1] proved that if G is a non-abelian residually free group then the following are equivalent:(1) G is fully residually free; (2) G is commutative transitive; (3) G is universally free. Further they showed that universally free groups are tree free [G-S 5],[Re]. Fine,Gaglione,Rosenberger and Spellman [F-G-R-S] and again independently Remeslennikov [Re] showed that the converse is not true, that is there exist tree-free groups which are not universally free. In [F-G-M-R-S] it was proved every 2-free residually free group is 3-free. Since 2-free groups are commutative transitive (see section 2) it follows from the Baumslag result that 2-free residually free groups are fully residually free. There are fully residually free groups which are not 2-free and there are 3-free, fully residually free groups which are not 4-free

[F-G-R-S]. A very important construction in the development of these results is a special type of HNN extension called a rank one extension of centralizers. If the base group is free this is a special type of conjugacy pinched one-relator group. In particular let $G \neq 1$ be a commutative transitive group,let $u \in G \setminus \{1\}$ and let $M = Z_G(u)$. Then

$$G(u,t) =< G, t; \text{ rel } (G), t^{-1}zt = z, \text{ for all } z \in M >$$

is the **free rank one extension of the centralizer** M of u in G. Free groups are commutative transitive and in a free group centralizers are cyclic. Hence if the base is free the corresponding rank one extension of centralizers has the presentation $< F, t; tUt^{-1} = U >$ which is a conjugacy pinched one-relator group.

Using these ideas Fine,Gaglione,Myasnikov,Rosenberger and Spellman [F-G-M-R-S] were able to give a complete classification of all fully residually free groups of rank three or less. In particular if G is a fully residually free group of rank ≤ 3 then G is either free,abelian or a free rank one centralizer extension of a free group of rank two. Specifically the main results are the following.

Theorem B9. *Every 3-generator, fully residually free group lies in $\mathcal{F}$, where $\mathcal{F}$ is the smallest class of groups containing the infinite cyclic groups and closed under the following four "operators":*

(1) Isomorphism

(2) Finitely Generated Subgroups

(3) Free Products of Finitely Many Factors

(4) Free Rank One Extensions of Centralizers.

Theorem B10. *Every 2-free, residually free group is 3-free.*

We note that the proof of this theorem as well as the next using the Nielsen reduction techniques in HNN groups developed in [F-R-R 2].

Theorem B11. *Let G be a fully residually free group. Then*

(1) if rank$(G) = 1$ then G is infinite cyclic.

(2) if rank$(G) = 2$ then either G is free of rank 2 or free abelian of rank 2

(3) if rank$(G) = 3$ then either G is free of rank 3, free abelian of rank 3 or a free rank one extension of centralizers of a free group of rank 2. That is G has a one-relator presentation

$$G =< x_1, x_2, x_3; x_3^{-1} v x_3 = v >$$

where $v = v(x_1, x_2)$ is a non-trivial element of the free group on x_1, x_2 which is not a proper power.

Finally we mention hyperbolicity. Two elements U, V of a group G are conjugacy separated if no conjugate of $< U >$ intersects non-trivially with a conjugate of $< V >$. Kharlamopovich and Myasnikov [K-M] extended the result of Bestvinna and Feighn to HNN groups. From their results we get the following.

Theorem B12. *Let $G =< a_1, ..., a_n, t; tUt^{-1} = V >$ be a conjugacy pinched one-relator group. If neither U nor V is a proper power and U, V are conjugacy separated in the free group on $a_1, ..., a_n$ then G is hyperbolic.*

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