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The nonreduced order spectrum of a commutative ring.

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Abstract

A *nonreduced order* of a ring A is a subset α satisfying i) $\alpha + \alpha \subset \alpha$, ii) $\alpha \cdot \alpha \subset \alpha$, iii) $-\alpha \cup \alpha = A$ (without any restriction on $-\alpha \cap \alpha$). The purpose of this paper is to organize these objects into a topological space or spectrum, to show relations to existing concepts, to analyze some examples, and to motivate our inquiry by questions of multiplicity.

1 Nonreduced Orders

The concept of *order* in a field and its generalizations to commutative rings play a central role in real algebraic and semialgebraic geometry. This paper studies another such generalization, that of a *nonreduced order* of a ring. (By ring we understand without further mention commutative ring containing 1/2.) The nonreduced order seems to be the weakest possible notion of *total order* in a ring that is compatible with the ring structure. By this we mean the following: each total order on a ring A is determined by the subset α consisting of elements which are nonnegative in the order. (We freely use the single term "order" or more explicitly "nonreduced order" to refer either to this subset or the relation $\geq$ induced on A according to $b \geq a$ if and only if $b - a \in \alpha$.) This nonnegative cone should have some obvious properties. It should be closed under addition and multiplication and contain all squares. In other language it should be a "quadratic semiring" or "preordering".

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Also for the order to be total it is necessary that $-\alpha \cup \alpha = A$. What seems less obvious is that these conditions *alone* suffice to give an interesting class of objects. The purpose of this paper is to support this claim by introducing a corresponding spectrum, showing relations to existing concepts, analyzing some examples, and motivating our inquiry by questions of multiplicity. (Although the unadorned term "order" has other uses in ring theory, our use here is distinct enough to be unambiguous even if we omit the qualifier *"nonreduced".*) If we were to require also that the elements order-equivalent to 0 (that is, which are both ≥ 0 and ≤ 0) form a prime ideal, we would arrive at the concept of prime cone or prime ordering and the related fundamental notion of the real spectrum of a ring $Spec_r(A)$ [2]. However, our aim is to stop short of this with the following weaker definition.

Definition 1.1. *An order of a ring A is a subset α satisfying*

i) $\alpha + \alpha \subset \alpha$

ii) $\alpha \cdot \alpha \subset \alpha$

iii) $-\alpha \cup \alpha = A$.

We note that properties ii) and iii) together imply that α contains the squares. The book [3] surveys a number of spectra associated with ordered structures such as the Keimel spectrum for F-rings.

It is often convenient to regard A itself, which satisfies these conditions, as the *improper order*. All other orders we consider *proper*. It is easy to see that an order is proper if and only if it does not contain -1. The *support* of an order $supp(\alpha)$ is the ideal $-\alpha \cap \alpha$. We call elements of this subset "order-equivalent to 0", a property which we indicate by ~ 0. The term "nonreduced" refers to the quotient ring $A/supp(\alpha)$, which in general contains nilpotents. However, if α is an element of the real spectrum then this ring is reduced. Thus, a more accurate term would be the (absurd) "not-necessarily-reduced order" according to which the real spectrum is the subset of prime orders. The *order spectrum* of A, denoted by $Ord(A)$, is the set of proper orders of A.

We equip $Ord(A)$ with the *ordinary topology* using the subbasis of *principal basic closed sets* of the form

$$\{\alpha \mid a \in \alpha, \alpha \text{ a proper order }\}$$

for $a \in A$. We understand *constructible sets* to be members of the algebra of sets generated by the principal basic sets and freely define sets by inequalities according to $\{a \geq 0\} = \{\alpha \mid a \in \alpha\}$ and $\{a > 0\} = \{\alpha \mid a \in \alpha, a \notin -\alpha\}$. We note that there is a certain danger in becoming too comfortable with the inequality symbolism since here $a > 0$ and $b > 0$ do not imply $ab > 0$ but only $ab \geq 0$. However algebraic operations respect in a familiar sense the *weak signs* ≥ 0 and ≤ 0.

We also have the finer *constructible topology* for which the constructible sets form a subbasis. It is easy to see that constructible sets are finite unions of sets of the form $\{f_1 = 0, \cdots, f_m = 0, g_1 > 0, \cdots, g_k > 0\}$. This is the simplest form in the order spectrum as opposed to the real spectrum, where several equalities can be replaced by a single degenerate equality. But in general,

$$\{f^2 + g^2 = 0\} = \{f^2 = fg = g^2 = 0\} \neq \{f = g = 0\}.$$

Open constructible sets are defined as finite unions of sets of the form $\{f_1 > 0, \cdots, f_m > 0\}$ and closed constructibles are finite unions of sets of the form $\{f_1 \geq 0, \cdots, f_m \geq 0\}$.

Why study nonreduced orders? Our motivation is to obtain intrinsically semialgebraic numerical attributes of points, sets or functions by using information residing in nilpotent elements. Crudely put, in the study of varieties we know how to distinguish the equations $x = 0$ and $x^3 = 0$. We wish, at the very least, to give analogous distinctions in the study of inequalities for distinguishing $x \geq 0$ and $x^3 \geq 0$. In algebraic geometry we start with algebraic data such as the coordinate ring $A(V)$ of an affine variety V. From this we strive to derive geometric information. The key device of the modern abstract approach is to assemble directly from A algebraically defined images of V which articulate V for different purposes with varying degrees of explicitness. Here, there are many possible choices: prime spectra, maximal spectra, real spectra, valuation spectra etc., each depending upon which features are to be brought into view [5]. In the same spirit, in defining nonreduced orders and assembling them into the order spectrum we make yet another decision about how much and what sort of algebraic information is to be recast in geometric form. In this we have in mind a kind of loose comparison with the situation of schemes, which has both positive and negative aspects.

We know from complex geometry that it is important to study schemes both as geometric objects and as objects equipped with certain algebraic data. The latter point of view distinguishes sets which are geometrically the same but which differ significantly in the data which define them and is vital for assigning notions of multiplicity to subsets. One device for this is the closed subscheme [6]. In this approach the underlying sets can be the same but are equipped with different sheaves that reflect differences in the defining algebraic data. If we were to view V as not just the set of its points but as the set of its closed subschemes, we would shift our attention from the set of prime ideals of A to the set of all ideals. This set is probably too large in most situations to view geometrically. The idea of the closed subscheme amounts to taking these one at a time, foregoing further geometrization by using only prime ideals as points and retaining other information in algebraic form in the associated sheaf.

In contrast, and somewhat in the face of this warning example, we propose enlarging the set of real points. We are emboldened to take this step for several reasons. First, in special cases the order spectrum appears to be an interesting geometric object. Second, in the real case, which requires inequalities as well as equations, the natural counterpart of the set of all ideals seems to be the complete set of preorderings. Hence, our notion of order spectrum seems to have no obvious counterpart in the case of schemes. In fact, for very simple rings it is possible to geometrize all preorderings collectively. However, for rings of geometric interest this set is typically huge, even grossly infinite-dimensional. Finally, continuing to reckon size, the order spectrum gives us instead, surprisingly small objects of intermediate size. We will find that $Ord(A)$ has a natural fibration over $Spec_r(A)$ and, again comparing with affine schemes and ideals, that we are geometrizing a set related to a highly restricted subset of the ideals of A.

First we develop some algebraic properties of $Ord(A)$. Then, to fix our ideas, we use these properties in sections 3 and 4 to determine and analyze some simple order spectra. Section 5 gives mainly topological results, especially relating the order spectrum to the real spectrum. Section 6 treats the special case of finite dimensional algebras over the reals.

2 Algebraic Properties

Definition 2.1. *If α and β are orders and $\alpha \subset \beta$ then we say β is a specialization of α and α is a generization of β. If $a, b \in \alpha$ and $x \in A$ and $x(a+b) \in q \Rightarrow xa, xb \in q$, then q is an α-absolutely-convex ideal. If $\alpha \subset \beta$ and $a(\beta) > 0$ then $a(\alpha) > 0$.*

Theorem 2.2. *Let α be a nonreduced order of A, $q = supp(\alpha)$ and $p = r(q)$. Then:*

i) If $ab \in q$ then $a^2 \in q$ or $b^2 \in q$. In particular p is prime.

ii) q is an α-absolutely-convex ideal.

iii) β is a proper order extending α if and only if there exists a proper α-convex ideal, I, such that $\beta = \alpha + I$ and $supp(\beta) = I$.

iv) The set of nonreduced orders extending α is linearly ordered by inclusion. The support of α is the smallest α-convex ideal. Therefore there is a one-to-one correspondence between α-convex ideals and specializations of α.

v) The function

$$\phi : \alpha \rightarrow \alpha + p$$

maps $Ord(A)$ to $Spec_r(A)$ and induces a fibration of $Ord(A)$ over $Spec_r(A)$ by trees of nonreduced orders partially ordered by inclusion.

vi) $\phi(\alpha)$ is the smallest element of $Spec_r(A)$ containing α.

vii) If A is Noetherian then $Ord(A)$ satisfies an ascending chain condition.

Proof.

i) If $ab \in supp(\alpha)$ then by replacing a or b by $-a$ or $-b$ if necessary, we can suppose that a and $b \in \alpha$. Then, since α is total, by permuting the roles of a and b if necessary we can suppose that $a - b \in \alpha$. Then $ab - b^2 \in \alpha \Rightarrow -b^2 \in \alpha - ab \subset \alpha + \alpha \subset \alpha$. Hence $b^2 \in -\alpha \cap \alpha = q$. Hence if $ab \in q$ then $a^2 \in q$ or $b^2 \in q$. Now suppose $ab \in p$. Then $a^k b^k \in q \Rightarrow a^{2k} \in q$ or $b^{2k} \in q \Rightarrow a \in p$ or $b \in p$.

ii) If $x(a+b) \in q$ then since α is a total order we can assume that $x \in \alpha$. Then $xa, xb, -xa - xb \in \alpha \Rightarrow -xa, -xb \in \alpha \Rightarrow xa, xb \in q$.

iii) Suppose $\alpha \subset \beta$. Then set $I = supp(\beta)$. I is certainly proper. If $b \in \beta \setminus \alpha$ then $b \in -\alpha \subset -\beta \Rightarrow b \in supp(\beta)$. This shows that $\beta = \alpha + I$. To see that I is α-convex, suppose that $a > b > 0$ relative to α and $a \in I$. Then these inequalities hold in β and $supp(\beta)$ is β-convex. So,

$b \in I$. Conversely, it is elementary that $\alpha + I$ is an order. Clearly, $I \subseteq supp(\alpha + I)$. For the reverse inclusion, suppose $z \in supp(\alpha + I)$. Then $z = a + x$ and $-z = b + y$, where $a, b \in \alpha$ and $x, y \in I$. Hence $a + b \in I$ and by convexity $a, b \in I$. Hence, $supp(\alpha + I) \subseteq I$. In particular, $\alpha + I$ is proper.

iv) If β and $\tilde{\beta}$ extend α then $\beta \cap \tilde{\beta}$ is also an order. Suppose, seeking a contradiction, that $\beta \setminus \tilde{\beta}$ and $\tilde{\beta} \setminus \beta$ are each nonempty, containing elements b and $\tilde{b}$ respectively. Then $-b \in \tilde{\beta}$ and $-\tilde{b} \in \beta$. Hence $\tilde{b} - b \in \tilde{\beta}$ and $b - \tilde{b} \in \beta$. Permuting the roles of β and $\tilde{\beta}$, if necessary, we can suppose that $b - \tilde{b} \in \beta \cap \tilde{\beta} \subset \tilde{\beta}$, which implies $b \in \tilde{b} + \tilde{\beta} \subset \tilde{\beta}$, a contradiction. Hence, at least one of $\beta \setminus \tilde{\beta}$ and $\tilde{\beta} \setminus \beta$ must be empty, that is, either $\beta \subset \tilde{\beta}$ or $\tilde{\beta} \subset \beta$.

v) Let $\beta = \alpha + p$. We show that $p = supp(\beta)$. It is obvious that $p \subset supp(\beta)$. To establish the reverse inclusion suppose $x \in supp(\beta)$. Then $x = a + r = -a' - r'$ where $a, a' \in \alpha$ and $r, r' \in p$. Hence, $a + a' \in p$. This implies $(a + a')^k \in q$, which by α-convexity of q shows $a^k \in q$ and hence $x \in p$. Thus $supp(\beta)$ is prime and β belongs to $Spec_r(A)$. If $\gamma \in Spec_r(A)$ and $\alpha \in \phi^{-1}(\gamma)$ then by property iv) the set of orders containing α is linearly ordered by inclusion. This gives $Ord(A) \cup \{A\}$ the structure of a tree with root the improper order.

vi) If $\alpha \subset \beta \in Spec_r(A)$ then

$$\alpha + p \subset \beta + radical(supp(\beta)) = \beta.$$

vii) If A is Noetherian then by property iii) extensions of α require extensions of support which satisfy the chain condition.

As is usual in the case of spectra, homomorphisms of rings contravariantly induce mappings of order spectra. However, part ii) of the following shows that for surjective mappings the direct image of a nonreduced order is again an order with the caveat that it could be improper.

Theorem 2.3. *If $\varphi : A \to B$ is a homomorphism then:*

i) $\varphi^{-1}(Ord(B)) \subset Ord(A)$;

ii) If φ is surjective then $\varphi(Ord(A)) \subset Ord(B) \cup \{B\}$.

Proof.

i) If $\beta \in Ord(B)$ then $\alpha = \varphi^{-1}(\beta)$ is a subsemiring of A. Also $A = \varphi^{-1}(B) = \varphi^{-1}(-\beta \cup \beta) = \varphi^{-1}(-\beta) \cup \varphi^{-1}(\beta) = -\alpha \cup \alpha$. If α were improper then $-1 \in \alpha$ would imply $\varphi(-1) = -1 \in \beta$ contradicting that β is proper. Hence α is a proper order.

ii) If $\alpha \in Ord(A)$ then $\beta = \varphi(\alpha)$ is a subsemiring of B. Also, $B = \varphi(A) = \varphi(-\alpha \cup \alpha) = \varphi(-\alpha) \cup \varphi(\alpha) = -\beta \cup \beta$. Hence β is an order, possibly improper.

The functor Ord has very simple behavior with respect to direct sums.

Proposition 2.4.

$$Ord(A \oplus B) = \{Ord(A) \oplus B\} \cup \{A \oplus Ord(B)\}.$$

Proof.

If $\gamma \in Ord(A \oplus B)$ then $(-1, 1)$ or $(1, -1) \in \gamma$. If the former, then $(-1, 1)(1, 0)^2 = (-1, 0) \in \gamma$. By theorem 2.3 ii) the projection $\pi_1\gamma$ on A is an order, in this case improper, that is, $\pi_1\gamma = A$. Similarly $\pi_2\gamma = \beta \in Ord(B) \cup \{B\}$. Also, the squares $(1, 0)$ and $(0, 1) \in \gamma$ which implies $A \oplus \beta \subset \gamma$. Since $\gamma \subset A \oplus \beta \subset \gamma$ we conclude that $\gamma = A \oplus \beta$. Finally, if β were improper then $\gamma = A \oplus B$ would be improper also, contrary to hypothesis.

Next, our main concern is to show that various operations on proper orders lead again to proper orders. By an *ordered ring* we mean a pair (A, α) consisting of a ring A and an element α of $Ord(A)$.

Definition 2.5. *Let (A, α) and (B, β) be ordered rings. Let $\varphi : A \to B$ be a ring homomorphism. Define β^c, the contraction of β, to be $\varphi^{-1}(\beta)$ and α^e, the extension of α via φ, to be the semiring in B generated by $\varphi(\alpha)$ and the squares in B. If φ is surjective then $\alpha^e = \varphi(\alpha)$.*

Proposition 2.6. *Let (A, α) and (B, β) be ordered rings. Let $\varphi : A \to B$ be a surjective ring homomorphism.*

i) If (A, m, α) is a local ordered ring and m is α-convex then α^e is a proper order.

ii) $\beta = \beta^{ce}$.

iii) If $ker(\varphi) \subseteq \alpha$ then $\alpha = \alpha^{ec}$. In addition, if α is a prime order then α^e is also prime.

Proof.

i) Suppose that $-1 \in \alpha^e$. That is, $-1 = \varphi(a)$ for some $a \in \alpha$. Hence, $0 = \varphi(1+a)$. If $a \in m$ then $1+a$ is a unit and $\varphi(1) = 0$, a contradiction. If $a \notin m$ then $1+a \notin m$ by convexity. So $1+a$ is a unit and again $\varphi(1) = 0$.

ii) The proof is elementary.

iii) Clearly $\alpha \subseteq \alpha^{ec}$. Now, let $x \in \alpha^{ec}$. Then $y = \varphi(x) \in \alpha^e$. That is, $y = \varphi(x')$ for some $x' \in \alpha$. Since $\varphi(x - x') = 0$, $x - x' \in \alpha$. Hence $x \in \alpha$. Suppose that $supp(\alpha)$ is prime and that $xy \in supp(\alpha^e)$. Then there exist a and b in A such that $\varphi(a) = x$ and $\varphi(b) = y$. Hence $ab \in supp(\alpha^{ec}) = supp(\alpha)$, which implies that either $a \in supp(\alpha)$ or $b \in supp(\alpha)$. Hence, $supp(\alpha^e)$ is prime.

Proposition 2.7. *Let (A, α) be an ordered ring and T a multiplicative set in A. Define α_T in A_T as*

$$\alpha_T = \{\frac{a}{t^2} | \ a \in \alpha \ \ for \ some \ t \in T\}.$$

Then

i) (A_T, α_T) is an ordered ring and $\alpha_T^c = \{b \mid bt^2 \in \alpha \ \ for \ some \ t \in T\}$.

ii) Define $i_T : A \to A_T$ by $a \to \frac{a}{1}$. Then i_T preserves order.

iii) If $T \cap r(supp(\alpha)) = \emptyset$ then α_T is a proper order provided α is proper.

iv) In particular, if $\wp = r(supp(\alpha))$ and in $A_\wp$ we define $\alpha_\wp := \alpha_{A \setminus \wp}$ then $\alpha_\wp$ is a proper order of $A_\wp$ provided α is proper.

Proof.

i) α_T is clearly closed under addition and multiplication. To see that it is total, let $\frac{a}{t} \in A_T$ with $a \in A$ and $t \in T$. Then either $a \in \alpha$ or $-a \in \alpha$ and $t \in \alpha$ or $-t \in \alpha$. Hence either $\frac{at}{t^2} \in \alpha_T$ or $\frac{-at}{t^2} \in \alpha_T$. The identification of α_T^c is straightforward.

ii) If $a \in \alpha$ then $\frac{a}{1} \in \alpha_T$.

iii) Suppose $-1 \in \alpha_T$. That is, $-1 = \frac{a}{t^2}$ where $a \in \alpha$ and $t \in T$. Then $-t^2 \in \alpha$, which implies $t \in r(supp(\alpha))$.

It should be clear by now that consideration of $Ord(A)$ leads to rings containing nilpotents. More precisely we are led to an interesting natural generalization of the ordered field $k(\alpha)$ intrinsically associated to each point α of the real spectrum of A. For a prime order, $k(\alpha)$ is the quotient of the localization of A at the prime ideal $p = supp(\alpha)$ by the image of p, ordered by the image of α. In the case of a nonreduced order we let $q = supp(\alpha)$ and $p = r(q)$. Then, since p is prime, we can form $A(\alpha) = A_p/qA_p$. If A is Noetherian then it is easy to see that this is an ordered Artinian local ring. Up to isomorphism this is an ordered extension of an ordered field by a finite number of nilpotents. This is the motivation for our study of finite dimensional real algebras in section 6.

3 Examples

Here are three examples. Each is a three-dimensional algebra over the reals.

Example 3.1. $A = R[x]/(x^3) \simeq \{a + b\xi + c\xi^2 \mid \xi^3 = 0, a, b, c \in R\}$.

Ord(A) is finite. The real spectrum contains a single point which can be described variously as $\{a \geq 0\}$ or $R^{\geq 0} + (\xi)$ or (regarding elements of A as 'quadratic functions') as $\{f \mid f(0) \geq 0\}$. The nilpotent element ξ is necessarily infinitesimal in the sense that $R^+ > \pm\xi$ with respect to any proper order. To see this, suppose, to the contrary, that $\xi - r > 0$ where $r > 0$. Then, since in any order $\xi^2 + r\xi + r^2 \geq 0$, we infer $\xi^3 - r^3 = -r^3 \geq 0$ which implies that the order is improper. The full order spectrum has five members:

$$\{a \geq 0\}$$

$$\{a > 0\} \cup \{a = 0, b \geq 0\} \qquad \{a > 0\} \cup \{a = 0, b \leq 0\}$$

$$\{a > 0\} \cup \{a = 0, b > 0\} \cup \{a = b = 0, c \geq 0\}\{a > 0\}$$

$$\cup\{a = 0, b < 0\} \cup \{a = b = 0, c \geq 0\}.$$

These nonreduced orders are determined by the sign of ξ and by the lowest of its powers which is order-equivalent to 0. At the top level ξ is order-equivalent to 0 and the support is the prime ideal (x) . At the second level ξ has a sign and ξ^2 is order-equivalent to 0. At the bottom level ξ has a sign and $\xi^2 > 0$. Each of these orders is minimal with support 0, which precludes any proper subset from being a total order.

Example 3.2.

$$A = R[x,y]/(x,y)^2 \simeq \{a + b\xi + c\eta \mid \xi^2 = \xi\eta = \eta^2 = 0, a, b, c \in R\}.$$

Here again we have a single point α_0 in the real spectrum, which can be described as $R^{\geq 0} + (\xi, \eta)$ or, regarding elements of A as "affine functions", as $\{f \mid f(0,0) \geq 0\}$. The squares are $\Sigma = \{a > 0\} \cup \{0\}$. Moreover the multiplicative structure is so weak that any extension of Σ by a convex cone κ in the plane $a = 0$ is a subsemiring of A. If κ is the union of an open half-space in this plane with one of the closed half-rays on its boundary then this extension is an order α. Since $-\alpha \cap \alpha = \{0\}$, this order is minimal. Extending the boundary ray to the boundary line gives a larger order and any further extension gives the prime ordering α_0. This exhausts the possibilities. Parameterizing the half-spaces by S^1 (use the inner unit normal) with the binary choice of boundary ray, the minimal orders correspond precisely to oriented bases for R^2, or the orthogonal group $O(2)$. We thus obtain the following diagram for the order spectrum which, ordering elements by inclusion, becomes a one-dimensional tree.

$$\begin{array}{c} \alpha_0 \\ | \\ S^1 \\ | \\ O(2) \end{array}$$

The ring A obviously admits the action of $O(2)$ and each branch of the tree is equivalent modulo this action to

$$\{a \geq 0\} \supset \{a > 0\} \cup \{a = 0, b \geq 0\} \supset \{a > 0\} \cup \{a = 0, b > 0\} \cup \{a = b = 0, c \geq 0\}.$$

Let $a \gg b$ (a is infinitely larger than b) mean that for all $\lambda \in R$, $a + \lambda b > 0$. Then this order has an alternate description in terms of the relations $\gg$, infinitely larger than, and ~ 0, order-equivalent to 0, in which the sign of each element is precisely determined by the indicated relations.

$$\{1 \gg x \sim y \sim 0\} \supset \{1 \gg x \gg y \sim 0\} \supset \{1 \gg x \gg y \gg 0\}.$$

Example 3.3. $A = R \oplus R \oplus R$ where R is real closed.

By proposition 2.4 on direct sums there are only three orders:

$$R^{\geq 0} \oplus R \oplus R, R \oplus R^{\geq 0} \oplus R \text{ and } R \oplus R \oplus R^{\geq 0}.$$

In this case the order spectrum coincides with the real spectrum.

4 The Augmented Line, $Ord(R[x])$

We recall [1] that the real spectrum of $R[x]$ where R is the field of real numbers $Spec_r(R[x])$ consists of

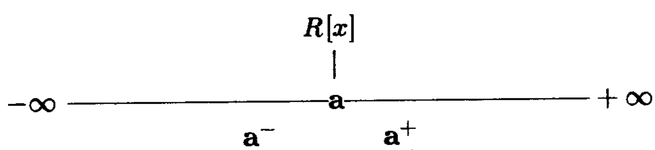
(1) zero-dimensional points, which are maximal orders having the form $\mathbf{a} = \{f \mid f(a) \geq 0\}$ for each $a \in R$,

(2) two infinite points

$$\pm\infty = \{f \mid f(\pm x) \geq 0 \text{ for all } x \text{ sufficiently large and positive}\}$$

(3) one dimensional half-branches $\mathbf{a}^+$ $(\mathbf{a}^-) = \{f \mid f(x) \geq 0$ on an open interval containing a as left (right) endpoint$\}$.

The orders $\mathbf{a}^\pm$ and $\pm\infty$ are minimal nonreduced orders since they have support $\{0\}$ which implies that no smaller preordering can be total. The infinite points are also maximal proper in the sense that any enlargement generates the improper order $R[x]$. These, together with the improper order $R[x]$, can be assembled into the following diagram.



$\mathbf{Spec_r(R[x]) \cup \{R[x]\}}$

The real spectrum is, of course, a subset of the order spectrum but there are many more nonreduced orders. Here $Spec_r(R[x])$ will form a kind of exoskeleton for $Ord(R[x])$. We proceed to determine all nonreduced orders. Let α be any proper nonreduced order. Then α is contained in at least one prime order β. If β is minimal then $\alpha = \beta$ so it suffices to consider $\beta = \mathbf{a}$. Then by theorem 2.2, $\beta \setminus \alpha \subset -\beta \cap \beta$. Since $\alpha \subset \beta$ we have $\alpha = \beta \setminus (\beta \setminus \alpha) \supset \beta \setminus (-\beta)$. Hence $\alpha \supset \{f \mid f(a) > 0\}$. Thus α contains elements negative at any point of R other than a and cannot be contained in any order $\mathbf{b} \neq \mathbf{a}$. By theorem 2.2, $supp(\alpha)$ is an ideal with prime radical in $R[x]$ which is convex with respect to α and hence with respect to the preorder consisting of the sums of squares. $R[x]$ is a PID in which the only primes are linear and irreducible quadratic. The quadratics do not give convex ideals, so the only convex ideals with prime radical are (0) and $(x-b)^m$ and the only ideals of this form possible in **a** are (0) and $(x-a)^m$. We have already classified the cases with support (0). So we can suppose that $\{f \mid f(a) > 0\} + ((x-a)^m) \subset \alpha$ where m is minimal. Since α is total, either $x-a$ or $a-x \in \alpha$. If $x-a \in \alpha$ then any element f of $\mathbf{a}^+$ is either $f = 0$ or there is a positive constant c such that $f = (x-a)^k(c+(x-a)g) \subset \alpha \cdot \{f \mid f(a) > 0\} \subset \alpha \cdot \alpha \subset \alpha$. In either case $f \in \alpha$, that is, $\mathbf{a}^+ \subset \alpha$. So $\mathbf{a}^+ + ((x-a)^m) \subset \alpha$. But $\mathbf{a}^+ + ((x-a)^m)$ is a nonreduced order. By theorem 2.2, if $f \in \alpha \backslash (\mathbf{a}^+ + ((x-a)^m))$ then $f \in supp(\alpha) = (x-a)^m$, which is a contradiction. Hence $\alpha = \mathbf{a}^+ + (x-a)^m$. The case in which $a - x \in \alpha$ similarly leads to $\alpha = \mathbf{a}^- + ((x-a)^m)$. This completes the identification of nonreduced orders. These, considered as points of a geometric object, all fit together in a diagram which elaborates the preceding depiction of the real spectrum by interpolating for each $a \in R$ the nested discrete chains of nonreduced orders $\mathbf{a}^{\pm} + (x-a)^m$, $m = 2, 3...$ between $\mathbf{a}^{\pm}$ at the bottom and **a** at the top.

$$R[x]$$

$$-\infty \text{ ———————— } \mathbf{a} \text{ ———————— } +\infty$$

$$\mathbf{a}^- + (x-a) = \mathbf{a} = \mathbf{a}^+ + (x-a)$$

$$\mathbf{a}^- + (x-a)^2 \qquad \mathbf{a}^+ + (x-a)^2$$

$$\cdots$$

$$\mathbf{a}^- + (x-a)^m \qquad \mathbf{a}^+ + (x-a)^m$$

$$\cdots$$

$$\cdots$$

$$\mathbf{a}^- \qquad\qquad \mathbf{a}^+$$

Ord(R[x]) ∪ {R[x]}

It is instructive to examine some simple semialgebraic sets in $Ord(R[x])$. The algebraic sets $\{x = 0\}$ and $\{x^3 = 0\}$ are quite different, the former consisting of **0** while the latter is:

$$\begin{array}{lcl} \mathbf{0}^- + (x) = \mathbf{0} = \mathbf{0}^+ + (x) \\ \mathbf{0}^- + (x)^2 \qquad\qquad \mathbf{0}^+ + (x)^2 \\ \mathbf{0}^- + (x)^3 \qquad\qquad \mathbf{0}^+ + (x)^3 \end{array}$$

$$\{\mathbf{x^3 = 0}\} \subset \mathbf{Ord(R[x])}.$$

Similarly the sets mentioned in the introduction as a crude paradigm, $\{x \geq 0\}$ and $\{x^3 \geq 0\}$, are different. Less obvious is that the sets $\{x > 0\}$ and $\{x^3 > 0\}$ also are different because of differences with respect to the nonreduced orders with one-dimensional support. In fact

$$\{x > 0\} = \{x^3 > 0\} \cup \{0^+ + (\mathrm{x})^2, 0^+ + (\mathrm{x})^3\}.$$

It is evident that $Ord(R[x])$ contains sufficient structure to count multiplicities. We illustrate this by using it to give a kind of semialgebraic interpretation of counting the zeros of a single real polynomial $f(x)$.

Theorem 4.1. *Given $f \in R[x]$ where R is the field of real numbers, let $\mu(f)$ be the number of real zeros of f counted without multiplicity, $\nu(f)$ the number of real zeros counted with multiplicity and $T(R[x]/f)$ the tree of nonreduced orders of $R[x]/(f)$ including the improper order. Then the width of $T(R[x]/f)$ is $2\mu(f)$, and the sum of the lengths of the branches is $2\nu(f)$.*

Proof. This follows from the action of Ord on direct sums and the following.

i) $T(R[x]/(x-a)^m)$ consists of a two branches of length m.
ii) If f has real zeros x_j with multiplicity m_j then

$$R[x]/(f) \simeq \oplus R[x]/(x-x_j)^{m_j} \oplus R[x]/(g),$$

where $T(R[x]/g)$ is the trivial tree consisting of the root $\{R[x]/g\}$.

iii) If f and g have no zeros in common then it follows from proposition 2.4 that

$$T(R[x]/fg) \simeq T(R[x]/f \oplus R[x]/g)$$

is obtained by tying together the trees $T(R[x]/f)$ and $T(R[x]/g)$ at their roots.

5 Topological Properties

We continue the development of topological properties begun in Section 1. Many properties of the order spectrum parallel those of the real spectrum closely. In fact the proof of the following is verbatim from Becker's paper [1] by using the previously mentioned change to $> 0, < 0$ and $= 0$.

Proposition 5.1. *Let $\alpha, \beta \in Ord(A)$ with the ordinary topology. Then*
i) $\beta \in \overline{\{\alpha\}}$ iff $\alpha \subseteq \beta$.
ii) α, β can be separated by open sets iff $\alpha \not\subseteq \beta$ and $\beta \not\subseteq \alpha$.
iii) A proper order is contained in a unique maximal proper specialization.
iv) α is closed iff α is a maximal proper order.

The relationship between $Ord(A)$ and $Spec_r(A)$ is first established in theorem 2.2 via a mapping, ϕ. This is, $\phi : Ord(A) \to Spec_r(A)$ which maps α to its minimal prime specialization, $\alpha + r(supp(\alpha))$. Note that

$$a \in \phi(\alpha)\backslash -\phi(\alpha) \iff a^k \in \alpha\backslash -\alpha \quad \forall k \geq 1.$$

Hence,

$$\phi^{-1}(a > 0) = \{a > 0, a^2 > 0 \cdots\}.$$

This is not, in general, open in the ordinary topology. However, it is closed in the constructible topology since its complement is

$$\{a \leq 0\} \cup \{a > 0, a^2 \leq 0\} \cup \{a > 0, a^2 > 0, a^3 \leq 0\} \cup \cdots.$$

Similarly, $\phi^{-1}(a \geq 0)$ and $\phi^{-1}(a = 0)$ are not necessarily closed in the ordinary topology but they are, in fact, open in the constructible topology. Hence, ϕ is not necessarily continuous in either topology. However, we do have the following.

Proposition 5.2. *ϕ is a closed function with respect to the ordinary topology.*

Proof. Let $F \subseteq Ord(A)$ be a closed set. Let $\alpha \in \overline{\phi(F)}\backslash\phi(F)$. As an element of $Ord(A)$, $\alpha \notin F$, since $\alpha = \phi(\alpha)$. Hence in $Ord(A)$ there exists a basic open set, $U = \{a_1 > 0, \cdots, a_n > 0\}$ such that $\alpha \in U$ and $U \cap F = \emptyset$. In $Spec_r(A)$, let $\tilde{U} := \{a_1 > 0, \cdots, a_n > 0\}$. Then $\alpha \in \tilde{U}$, which is open in $Spec_r(A)$. Since $\alpha \in \overline{\phi(F)}$, $\tilde{U} \cap \phi(F) \neq \emptyset$. That is, there exists $\beta \in F$ such that $\phi(\beta) =: \hat{\beta} \in \tilde{U} \cap \phi(F)$. Hence, $a_1, \cdots, a_n \in \hat{\beta} \backslash - \hat{\beta}$. In particular, $a_1, \cdots, a_n \in \beta \backslash - \beta$. That is, $\beta \in U \cap F$. But $U \cap F = \emptyset$, a contradiction.

Note: ϕ is not an open map. In $R[x]$, $\phi(x^{2k+1} > 0) = \{x \geq 0\}\ \forall k \geq 0$. It would be interesting to know when images of constructible sets by ϕ are constructible.

Proposition 5.3. *Let $\psi : A \to B$ be a homomorphism. Then ψ induces a continuous mapping, $\psi^* : Spec_r(B) \to Spec_r(A)$ which lifts to a continuous mapping, $\psi^* : Ord(B) \to Ord(A)$ such that the diagram commutes*

$$\begin{array}{ccc} Spec_r(B) & \xrightarrow{\psi^*} & Spec_r(A) \\ i \downarrow & & \downarrow i \\ Ord(B) & \xrightarrow[\psi^*]{} & Ord(A) \end{array}$$

where i, the inclusion map, is continuous.

Proof. The map $\psi^* : Ord(B) \longrightarrow Ord(A)$ is also defined as $\psi^*(\beta) = \beta^c$. To see that ψ^* is continuous it is straightforward that $a \in \beta^c \backslash - \beta^c \iff \psi(a) \in \beta \backslash - \beta$. Hence $\psi^{*-1}(\{a_1 > 0, \cdots, a_k > 0\}) = \{\psi(a_1) > 0, \cdots, \psi(a_k) > 0\}$ and continuity follows. That the diagram commutes and i is continuous are elementary.

If the ring homomorphism is surjective, there is an associated covariant map, which is defined as follows. Let $\psi : A \to B$ be surjective. First, consider $U = \{\alpha |\ \alpha^e \text{ is improper}\}$. Then U is an open set. For if $\alpha \in U$ then there exists $a \in \alpha$ such that $\psi(a) = -1$. By convexity, $2a + 1 > 0$, relative to α. This, together with $\psi(2a + 1) = -1$, shows that $\alpha \in \{2a + 1 > 0\} \subset U$. That is, U is open. Now, let $C = \{\alpha |\ \alpha^e \text{ is proper}\}$. Then C is a closed set, which we consider with the relative topology. Define $\psi_* : C \to Ord(B)$ by $\psi_*(\alpha) = \alpha^e$. We use this map to recover a familiar principle exhibited by the Zariski spectrum.

Proposition 5.4. *Let $\psi : A \to B$ be a surjective homomorphism. Then ψ^* gives a canonical homeomorphism of $Ord(B)$ to a closed subset, K of $Ord(A)$, which maps $Spec_r(B)$ to the closed set $K \cap Spec_r(A)$.*

Proof. Let $K = \{\alpha | Ker(\psi) \subseteq \alpha\}$. It is closed, since $K = \bigcap_{a \in Ker(\psi)} \{a = 0\}$. From proposition 2.6, $K \subset C$. For, $Ker(\psi) \subseteq \alpha$ implies $\alpha = \alpha^{ec}$.

In particular, α^e is proper. Now, consider ψ_* restricted to K. From proposition 2.6, we also see that $\psi^*(Ord(B)) = K$, $\psi^* \circ \psi_* = id_K$, $\psi_* \circ \psi^* = id_{Ord(B)}$ and

$$b \in \alpha^e \backslash -\alpha^e \iff \forall a \in \psi^{-1}(b), a \in \alpha \backslash -\alpha.$$

This last property is used to show that ψ_* is continuous. This establishes the homeomorphism, since ψ^* is continuous. The continuity of ψ_* follows from checking the inverse of basic open sets, that is,

$$\psi_*^{-1}(\{b_1 > 0, \cdots, b_k > 0\}) = \{\alpha \in K | \psi_*(\alpha) \in \{b_1 > 0, \cdots, b_k > 0\}\}$$

$$= \{\alpha \in K | \alpha^e \in \{b_1 > 0, \cdots, b_k > 0\}\} = \bigcup \{\alpha \in K | \alpha \in \{a_1 > 0, \cdots, a_k > 0\}\}$$

where the union is over $a_1, \cdots, a_k \in \psi^{-1}(b_1), \cdots, \psi^{-1}(b_k)$ respectively. This is open in K and ψ_* is continuous. Finally, proposition 2.6 also guarantees that ψ^* maps $Spec_r(B)$ onto $K \cap Spec_r(A)$.

Proposition 5.5.. *$Ord(A)$ with the constructible topology is a Stone space in which the clopen sets are exactly the constructible sets. The constructible sets are quasi-compact in the ordinary topology. An open set is constructible if and only if it is quasi-compact.*

The proof in [2] proposition 7.1.12 carries over to this context if one eliminates the requirement that the order is prime.

Proposition 5.6. *Let α be an order in $R[x_1, \cdots, x_n]$ such that $dim(\alpha) = n - 1$(here dimension means $dim(R[x_1, \cdots, x_n]/supp(\alpha))$). Then there exists a prime order β such that $dim(\beta) = n$ and $\beta \subseteq \alpha$.*

Proof. Let γ be the minimal prime specialization of α. Then $dim(\alpha) = dim(\gamma)$ and $supp(\gamma)$ is generated by an irreducible element p such that $p \in \alpha$. Since $\alpha \subseteq \gamma$ it must be that $supp(\alpha) = (p^k)$ for some $k \geq 1$. We invoke proposition 10.2.6 from [2], which states that γ is the specialization of two distinct n-dimensional prime orders. Let β be the prime order in which $p > 0$. We show that $\beta \subseteq \alpha$. Clearly, $\beta \setminus (p) = \alpha \setminus (p)$. So, let $f \in \beta \cap (p)$. Then $f = gp^m$ where p does not divide g. Then $g \in \beta$. For otherwise $g \in -\beta$ would imply that $\pm f \in \beta$ which, since β is n-dimensional, would imply $f = 0$. Hence $g \in \beta \backslash (p) = \alpha \backslash (p)$ and $f \in \alpha$.

The following example shows, contrary to the special conclusion of this result, that a nonreduced order need not contain a prime order even if the ring is a domain. It also incidentally illustrates that the forward image under a surjective map of a prime order need not be a prime order.

Example 5.7. In $R[x,y]$ let α be the prime order $\alpha = \{p \mid p([-a,0),0) \geq 0 \text{ for some } a > 0\}$ and let β be the image of α under the quotient mapping $R[x,y] \to R[x,y]/(y^2 - x^3)$. Then theorem 2.3 ensures that β is an order, possibly improper, of the quotient ring. We show that $\bar{x}$, the image of x, is not in β. If $\bar{x} \in \beta$ then $x = a + g(y^2 - x^3)$ with $a \in \alpha$. For $y = 0$ this gives the relation $x + x^3 g(x,0) = a(x,0)$ which, by the definition of α, is impossible for x small and negative. So β is proper. Now suppose that β contains a prime order γ. Then $\bar{x} \notin \gamma$. Hence $-\bar{x} \in \gamma$ and $-\bar{x}^3 \in \gamma$. But in the quotient ring $\bar{x}^3 = \bar{y}^2$ and hence is in every order. Hence $\bar{x}^3 \in supp(\gamma)$, and since γ is a prime order we arrive at the contradiction $\bar{x} \in \gamma$. Thus β is a nonreduced order of $R[x,y]/(y^2 - x^3)$ containing no prime order.

6 Finite Dimensional Algebras Over the Real Numbers

We now consider the case where A is a finite dimensional real algebra which corresponds in geometry to a zero dimensional real variety specified by an ideal of definition. We begin with some strong inferences using arguments from the theory of convexity in R^N, which in their simplest form apply only to the real numbers [4] and to no other ordered field. Choosing a vector basis for A over R we can identify A with some R^N equipped with an associative, commutative multiplication. In this situation any order α must be a total convex cone, that is a convex cone satisfying $-\alpha \cup \alpha = R^N$. Although, in general, there are many more such cones than nonreduced orders, still the following classification shows that they are not too numerous.

Theorem 6.1. *Let α be a proper total convex cone in R^N. Let $(.,.)$ be a nondegenerate inner product. Then there is a sequence of $k \leq N$*

mutually orthogonal nonzero vectors $e_1, e_2, ..., e_k$ *such that* α *is*

$$\begin{aligned} &\{(e_1,x) > 0\} \\ \cup&\{(e_1,x) = 0, (e_2,x) > 0\} \\ &\dots \\ &\dots \\ \cup&\{(e_1,x) = (e_2,x) \ldots = (e_{k-1},x) = 0, (e_k,x) \geq 0\} \end{aligned}\quad .$$

Proof. Any proper convex cone in R^N is contained in a closed half-space $\{(e_1,x) \geq 0\}$. Then $\alpha \subset \{(e_1,x) \geq 0\}$ implies

$$\alpha = \{(e_1,x) \geq 0\} \setminus (\{(e_1,x) \geq 0\} \setminus \alpha).$$

Since α is total, this implies

$$\alpha \supset \{(e_1,x) \geq 0\} \setminus -\alpha \supset \{(e_1,x) \geq 0\} \setminus \{(e_1,x) \leq 0\} = \{(e_1,x) > 0\}.$$

Thus, α lies between an open half-space and its closure. Also, this half-space is unique since if α contained the union of two distinct open half-spaces it could not be proper. This open half-space determines e_1 modulo R^+. If α is the closed half-space then $\alpha = \{(e_1,x) \geq 0\}$ and we are done. Otherwise repeat this argument inside the vector space orthogonal to e_1 with the convex cone which is its intersection with α to determine the next vector. Continuing until the process terminates yields the required sequence.

We shall refer to the integer k as the *depth* of the order. For the study of examples we will represent such orders by choosing a basis for A, identifying an element with its vector of components in R^N, and using the usual inner product $e^T f$ to measure orthogonality. We then can represent orders α in the form

$$\alpha = \left\{ \begin{matrix} e_1 \\ e_2 \\ . \\ . \\ e_k \end{matrix} \right\} or \left\{ \begin{matrix} e_{11} & \ldots & e_{1N} \\ e_{21} & \ldots & e_{2N} \\ . & & \\ . & & \\ e_{k1} & \ldots & e_{kN} \end{matrix} \right\}.$$

This representation embodies weak necessary conditions for a cone to be an order which reflect only the linear structure. However there are some further necessary conditions that use the multiplicative structure.

Theorem 6.2. *Let*

$$\alpha = \left\{ \begin{array}{c} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_k \end{array} \right\}$$

be an order in a finite dimensional real algebra. Let $e_{k+1}, \ldots, e_N$ fill out an orthogonal linear basis. Then, for $1 \leq j \leq k$, e_j is orthogonal to the ideal $(e_{j+1}, \ldots, e_N)$.

Proof. By induction on j. Whatever the real parameters $\lambda_2, \ldots, \lambda_N$, the element $x = e_1 + \lambda_2 e_2 + \ldots + \lambda_N e_N$ satisfies the simplest test for strict positivity, namely, positive inner product with e_1. Hence for any element a the weak sign ($\geq$ *or* $\leq$) of ax is independent of these parameters. Thus

$$(e_1, ax) = (ae_1, e_1) + \lambda_2(ae_2, e_1) + \ldots + \lambda_N(ae_N, e_1)$$

cannot assume both strict signs. This is possible only if

$$(ae_2, e_1) = (ae_3, e_1) = \ldots = (ae_N, e_1) = 0.$$

This implies the case $j = 1$. Next, assuming the conclusion for $1, 2, \ldots j - 1$, we repeat the above argument with $x = e_j + \lambda_{j+1} e_{j+1} + \ldots + \lambda_N e_N$. By orthogonality to $e_1, \ldots, e_{j-1}$ and positive inner product with e_j, x is strictly positive. By the induction hypothesis, for any a, ax is also orthogonal to $e_1, \ldots, e_{j-1}$ and again, whatever the parameters, cannot have inner products of both signs with e_j. This is possible only if the conclusion of the theorem holds.

From these properties we draw the conclusion that any order on a finite dimensional real algebra has a simple finitary order-theoretic description.

Corollary 6.3. *Let*

$$\alpha = \left\{ \begin{array}{c} e_1 \\ e_2 \\ \cdot \\ e_k \end{array} \right\}$$

be an order in a finite dimensional real algebra. Then the sign of any element is determined by the infinitesimal and null relations

$$e_1 \gg e_2 \gg \ldots \gg e_k \gg e_{k+1} \sim e_{k+2} \sim \ldots e_N \sim 0.$$

Proof. Apply the criteria for nonnegativity to the orthogonal expansion of the element in the e_j.

Another necessary consequence of the multiplicative structure is that the linear span of $e_{k+1}, \ldots, e_N$ is an ideal since it is the support of the order.

Corollary 6.4. *If for $k > 1$*

$$\alpha = \left\{ \begin{array}{c} e_1 \\ e_2 \\ \cdot \\ e_k \end{array} \right\}$$

is a proper order in a finite dimensional real algebra then so is

$$\left\{ \begin{array}{c} e_1 \\ e_2 \\ \cdot \\ e_{k-1} \end{array} \right\} = \alpha + (e_k).$$

Proof. Let $\beta = \left\{ \begin{array}{c} e_1 \\ e_2 \\ \cdot \\ e_{k-1} \end{array} \right\}$ and suppose $b \in \beta$. If this membership is determined by inner products above depth $k-1$ or by a positive inner product with e_{k-1} then $b \in \alpha$. If all inner products with $e_1, \ldots, e_{k-1}$ vanish then $b = \lambda_k e_k + \ldots + \lambda_N e_N$, which exhibits b as an element of

$\alpha + (e_k)$. Thus $\beta \subset \alpha + (e_k)$. To establish the reverse inclusion suppose $b \in \alpha + (e_k)$. Then $b = a + ce_k$ with $a \in \alpha$. By the orthogonality of (e_k) to $e_1, \ldots, e_{k-1}$, b satisfies exactly the conditions for membership in β that a satisfies. Thus, $\alpha + (e_k) \subset \beta$. To see that β is proper, suppose $-1 \in \beta$. Then $-1 = a + be_k$. Since $k > 1$, $-(1, e_1) = (a, e_1)$. Opposite sides of this equation have opposite weak signs and so both must vanish. This implies $-1 \in (e_2, \ldots, e_N)$, which implies in turn that $e_1 \in (e_2, \ldots, e_N)$. This means that e_1 is self-orthogonal, a contradiction.

Reconsidering the earlier example $A = R[x,y]/(x,y)^2$ in the light of these results we use the basis $\{1, x, y\}$, which we represent by the usual basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ of R^3 and the usual inner product. Then any choice e_2 and e_3 of vectors filling out an orthogonal basis gives a descending chain of orders

$$\left\{ \begin{matrix} 1 & 0 & 0 \end{matrix} \right\} \supset \left\{ \begin{matrix} 1 & 0 & 0 \\ 0 & e_{22} & e_{23} \end{matrix} \right\} \supset \left\{ \begin{matrix} 1 & 0 & 0 \\ 0 & e_{22} & e_{23} \\ 0 & e_{32} & e_{33} \end{matrix} \right\}$$

with corresponding chain of supporting ideals

$$(x,y) \supset (e_{32}x + e_{33}y) \supset (0).$$

The order-theoretic description of these orders using the relations $\gg$ of infinitely larger and $\sim$ of order-equivalence gives, for an order of depth 2,

$$1 \gg e_{22}x + e_{23}y > 0 \text{ and } e_{32}x + e_{33}y \sim 0;$$

and for an order of depth 3,

$$1 \gg e_{22}x + e_{23}y \gg e_{32}x + e_{33}y > 0;$$

with only 0 order-equivalent to 0.

Example 6.5.

$$A = R[x,y]/(x^3, y^2) \cong \{\text{linear span}\{1, x, y, x^2, xy, x^2y\} \mid x^3 = y^2 = 0\}.$$

Let the indicated linear basis correspond to the usual basis of R^6. Then, as in the preceding example, there is a single prime order of depth one with support the prime ideal (x,y). First, since

$$\pm x^2 y = (x^2 \pm y/2)^2,$$

x^2y must be order equivalent to 0 in any order. Accordingly, there are no orders of depth 6. Also, there is a perfect correspondence with the orders of

$$\tilde{A} = R[x,y]/(x^3, x^2y, y^2) \cong \{\text{linear span}\{1, x, y, x^2, xy\} \mid x^3 = x^2y = y^2 = 0\}$$

and it is these that we shall classify according to depth.

Granting these preliminaries, the single order of depth 1 is the prime cone

$$\alpha_1 = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \end{matrix} \right\}.$$

By corollary 6.4 any order of depth 2 must specialize to an order of depth 1. Hence the representation

$$\alpha_2 = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & B & C & D & E \end{matrix} \right\}$$

will account for all orders of depth 2. B and C cannot both be zero since then $x \sim 0$ and $y \sim 0 \Rightarrow x^2 \sim xy \sim 0 \Rightarrow D = E = 0$ which would reduce α_2 to α_1. Thus we can suppose that $B^2 + C^2 = 1$. We next show that D and E both vanish. If $D = 0$ then $x^2 \sim 0 \Rightarrow \pm xy \sim (x \pm y/2)^2 \Rightarrow E = 0$. If $D \neq 0$ then $Dx - Bx^2 \sim 0 \Rightarrow Dx^2 \sim 0 \Rightarrow x^2 \sim 0 \Rightarrow D = 0$, a contradiction. Hence, at depth 2

$$\alpha_2 = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & B & C & 0 & 0 \end{matrix} \right\} \text{ where } B^2 + C^2 = 1.$$

This form is only a necessary condition for a cone to be an order of depth 2, but it is easy to check that it is also sufficient.

Descending to depth 3, the orthogonality of rows in our representation allows us to suppose that

$$\alpha_3 = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & B & C & 0 & 0 \\ 0 & -pC & pB & D & E \end{matrix} \right\}.$$

The nonnegativity of $(x + \lambda y)^2 = x^2 + 2\lambda xy$ is decided at depth 3 by $D + 2\lambda E \geq 0$, which is true for all λ only if $E = 0$ and $D \geq 0$. Also $pCx - pBy > 0, xy \sim 0$ and $1 + \lambda x > 0$ together imply

$$(1 + \lambda x)(pCx - pBy) \sim pCx - pBy + \lambda pCx^2 \geq 0.$$

This requires that $p^2(B^2+C^2)+\lambda pCD \geq 0$ for all λ, which is possible only if $pCD=0$. We distinguish three cases depending on which factor vanishes.

If $p=0$ then D must be strictly positive, and we can suppose that

$$\alpha_3 = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & B & C & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} \right\}.$$

If this cone were an order then $Cx - By \sim xy \sim 0 \Rightarrow Cx^2 \sim 0$. Since x^2 is strictly positive, this is possible only if $C=0$, and we can suppose that the cone has the form

$$\alpha_{31} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} \right\}.$$

Again it is easy to check that these are in fact orders.

If $p \neq 0$ but $D=0$ then we can normalize p to ± 1. This gives the form

$$\alpha_{32} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & B & C & 0 & 0 \\ 0 & \pm C & \mp B & 0 & 0 \end{matrix} \right\}$$

which is parameterized by $O(2)$.

If p and D are not 0 but $C=0$ then the cone can be normalized to

$$\alpha_{33} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & p & 1 & 0 \end{matrix} \right\}.$$

Thus, since the orders α_{31} fit together smoothly with the family α_{33}, the orders of depth 3 consist of the two families α_{32} and α_{33} parameterized by $O(2)$ and R respectively.

At depth 4 we encounter a new phenomenon. Not all orders of depth 3 can be generalized to depth 4. Any generization of α_{32} will have the form

$$\alpha_{42} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & p_{11} & p_{12} & 0 & 0 \\ 0 & p_{21} & p_{22} & 0 & 0 \\ 0 & 0 & 0 & D & E \end{matrix} \right\}.$$

The preceeding argument with $(x + \lambda y)^2$ again shows that $D \geq 0$ and $E = 0$. The subcase $D = 0$ collapses to the case of depth 3. Hence we can suppose that $D = 1$ and

$$\alpha_{42} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & p_{11} & p_{12} & 0 & 0 \\ 0 & p_{21} & p_{22} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} \right\}.$$

However, if $p_{11} \neq 0$ then this form determines $p_{11}x > 0$ at depth 2, and in any case determines $p_{21}x + p_{22}y \geq 0$ at depth 3. Together these determine at depth 4 that $p_{11}p_{21} \geq 0$. This necessary condition also suffices for this generization from depth 3 to depth 4.

In the other case of depth 3, the element $y - px^2$ has weak sign determined below depth 3. This is the λ-independent weak sign of $(1 + \lambda x)(y - px^2)$ which, if E is not zero, is the sign of λE. Hence $E = 0$, and we have the necessary form

$$\alpha_{43} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & p & 1 & 0 \\ 0 & 0 & \pm 1 & \mp p & 0 \end{matrix} \right\}.$$

At depth 5 orthogonality determines the forms

$$\alpha_{52} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & p_{11} & p_{12} & 0 & 0 \\ 0 & p_{21} & p_{22} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \pm 1 \end{matrix} \right\}$$

in which the last row determines the sign of xy. Its sign is that of $p_{11}p_{12}$ if that is not 0, of $p_{21}p_{12}$ if $p_{11} = 0$, and of $p_{11}p_{22}$ if $p_{21} = 0$. Finally, generizations of α_{43} have the form

$$\alpha_{53} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & p & 1 & 0 \\ 0 & 0 & \epsilon_2 & -\epsilon_2 p & 0 \\ 0 & 0 & 0 & 0 & \epsilon_3 \end{matrix} \right\}$$

where the ϵ_j are $+1$ or -1 and ϵ_3 is constrained by the signs determined at shallower depth according to $\epsilon_3 = \epsilon_1 sign(p)$ if $p \neq 0$ and $\epsilon_3 = \epsilon_1\epsilon_2$ if $p = 0$.

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