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Some results about blow-up and global existence to a semilinear degenerate heat equation.

Jacques GIACOMONI

Abstract

In this paper, we are dealing with the following degenerate parabolic problem :

$$(P_t)\begin{cases} \partial_t u - |x|^2\Delta u = g(u) \text{ in } \mathbb{R}^+ \times B_1 \\ u(t,x) \equiv 0 \text{ in } \mathbb{R}^+ \times \partial B_1 \; ; \; u(0,x) = u_0 \geq 0 \end{cases}$$

where $B_1 = \{x \in R^N \; ; \; \|x\| = 1\}$ and g is nonlinear.

We are interested in analizying such questions as local and global existence, blow-up in finite time and convergence to a stationary solution for solutions of (P_t).

First, we give some examples of nonlinearities g where the blow up in $L^2(\frac{dx}{|x|^2}) \cap L^\infty(B_1)$ occurs. In a second part of this work, we present two cases of global existence of solutions to (P_t) which converge in $L^\infty(B_1)$ to a stationary solution of (P_t) when $t \to \infty$.

1 Introduction

In this work, we study the following problem :

$$(P_t)\begin{cases} \partial_t u - |x|^2\Delta u = g(u) \text{ in } \mathbb{R}^+ \times B_1 \\ u(t,x) \equiv 0 \text{ in } \mathbb{R}^+ \times \partial B_1 \; ; \; u(0,x) = u_0 \geq 0, \end{cases}$$

where g is nonlinear and B_1 is the unit ball in $\mathbb{R}^N$.

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First, using Hille-Yosida theory, we prove for all $u_0 \in L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$[1] and $g \in W^{1,\infty}_{loc}(\mathbb{R}^N)$, the local existence and the uniqueness of the solution $u(t) = S(t)u_0$ of (P_t), where $S(t)$ is the semigroup associated to (P_t). Then, we are interested in the behaviour of the solution $u(t)$ as t increases. Precisely, under different assumptions of g and u_0, we give on one hand, some examples of blow-up in finite time and on the other hand, some examples of global existence of solutions to (P_t) which converge to a stationary solution of (P_t).

Throughout this work, we keep in mind the results of [7] and [8] which deal with the stationary problem (P) :

$$(P)\begin{cases} -|x|^2\Delta u = g(u) \text{ in } B_1 \\ u \in H^1_0(B_1)/\{0\} \; ; \; u \geq 0 \end{cases}$$

Precisely, in [7], the authors prove the nonexistence of nontrivial solutions to (P) in the case where g satisfies the following assumptions :

(GS1) $\lambda - (\frac{N-2}{2})^2 + \lim\limits_{s \to +\infty} \dfrac{g(s)}{s} > 0.$

(GS2) $\forall s > 0 \; , \; G(s) \leq \frac{g(s)s}{2}.$

Otherwise, in [8], the authors give some results about the existence of nontrivial solutions of (P) in the case where g is sublinear. They treat three cases :

1. $g(u) \sim \lambda u + u^p - u^q$ where $1 < p < q$
2. $g(u) \sim \lambda u - u^p$ where $p > 1$ and $\lambda > (\frac{N-2}{2})^2$
3. $g(u) \sim u^\alpha + \lambda u$ where $0 < \alpha < 1$ and $\lambda < (\frac{N-2}{2})^2$

It is worth noting that in all cases, an unbounded connected branch of positive solutions in either $H^1_0(B_1)$ or $L^\infty(B_1)$ exists and in the second and third case, there is uniqueness of the nontrivial solution in $H^1_0(B_1)$. Then, it is very natural to see in which cases the nonexistence of nontrivial solutions of (P) implies the blow-up in finite time for solutions of (P_t) and when the uniqueness of the solution of (P) implies the convergence to a stationary solution for solutions to (P_t) when $t \to +\infty$. In this work, we prove some results in these directions.

So, the outline of the present paper is as follows :

[1] $L^2(\frac{dx}{|x|^2}) := \{u / \int_{B_1} \frac{|u|^2}{|x|^2} dx < \infty\}$

1. Local existence of solutions to (P_t) in $\mathbb{R}^+ \times L^\infty \cap L^2(\frac{dx}{|x|^2})$.
2. Some examples of blow up in finite time for solutions to (P_t)
   (a) The case $g(0) = 0$
   (b) The case $g(0) > 0$
3. Two examples of existence of global solutions and convergence to a stationary solution.

Precisely, in Section 2, we apply Hille-Yosida theory in $L^\infty \cap L^2(\frac{dx}{|x|^2})$. In Section 3, we start adapting a classical spectral method (see for instance [4]) to prove the blow-up in finite time when g satisfies :

(B1) g is convex and positive in $\mathbb{R}^+$.

(B2) $(\frac{N-2}{2})^2 < \lim\limits_{s\to 0^+} \dfrac{g(s)}{s} = \lambda < +\infty$.

(B3) There exists $s_0 > 0$ such that $\int_{s_0}^{+\infty} \frac{ds}{g(s)-\lambda s} < +\infty$

Next, we use a well known "energy method" (see for instance [4]). For this, we assume the following hypothesis :

(B4) $\lambda = \lim\limits_{s\to 0^+} \dfrac{g(s)}{s} < +\infty$ and there exists $\alpha > 0$, $C > 0$ such that $h(s) = g(s) - \lambda s \geq C s^{\alpha+1}$, for all $s \geq 0$.

(B5) There exists $\epsilon > 0$ such that $(2+\epsilon) \int_0^s h(t)\, dt \leq s h(s)$, $\forall\, s > 0$.

Then, we prove that if u_0 satisfies $\int_{B_1} \frac{|\nabla u_0|^2}{2} - \int_{B_1} \frac{G(u_0)}{|x|^2} < 0$, where $G(s) = \int_0^s g(t)\, dt$, the solution $u(t)$ to (P_t) blows up in finite time. Finally, we conclude the section with the case $g(0) > 0$. Precisely, we apply a method from [3] which links directly the blow-up and the nonexistence of stationary solutions. For this, we assume :

(B6) $g(0) > 0$, $g \in C^1([0, +\infty[)$, convex and increasing.

(B7) There exists $x_0 > 0$ such that $\int_{x_0}^{+\infty} \frac{ds}{g(s)} < \infty$.

Then, for any $u_0 \geq 0$, the solution $u(t) = S(t)u_0$ blows up in finite time.

In Section 4, we give some results concerning the existence of global solutions to (P_t). First, proving the radial symmetry of the solution to (P_t) when u_0 is radially symmetric, we exhibit the heat kernel of $-|x|^2\Delta$ in $H_0^1(B_1)$. Then, using a method due to Fujita, we prove the existence of a global solution of (P_t) for small initial data when $g(t) \sim \lambda t + t^p$, $p > 1$ and $\lambda < 0$. Moreover, we prove that $u(t)$ converges to 0 in $L^\infty(B_1)$ with an exponentional decay when $t \to +\infty$.

Finally, assuming the following hypothesis :

(G3) $s \to \frac{g(s)}{s}$ is continuous and strictly decreasing,

(G4) $\frac{g(s)}{s} \overset{s\to+\infty}{\longrightarrow} -\infty$,

(G5) $\lim_{s\to 0+} \frac{g(s)}{s} = \lambda > (\frac{N-2}{2})^2$,

we show that for any $u_0 > 0$ satisfying $u_0 \in L^\infty \cap L^2(\frac{dx}{|x|^2})$, $\|u_0\|_{L^\infty} \leq f^{-1}(0)$ and $u_0 \not\equiv f^{-1}(0)$ where $f(t) := \frac{g(t)}{t}$, the solution $u(t)$ of (P_t) is global and converges to the unique nontrivial stationary solution of (P_t) in $L^\infty(B_1) \cap H_0^1(B_1)$.

2 Local existence

Throughout this section, we assume that $g \in W_{loc}^{1,\infty}(\mathbb{R})$. Our goal is to study the local existence of a solution to (P_t). Precisely, we show that we can apply Hille-Yosida theory in $L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$. Consequently, for every $u_0 \in L^\infty \cap L^2(\frac{dx}{|x|^2})$, the uniqueness of solutions of (P_t) follows. First, we remark :

Proposition 2.1. *Let $A = -|x|^2\Delta$. Then, A is a self adjoint maximal monotone operator in $L^2(\frac{dx}{|x|^2})$. Moreover, $\mathcal{D}(A) = \{u \in L^2(\frac{dx}{|x|^2}) / u \in H_0^1(B_1)$ and $|x|^2\Delta u \in L^2(\frac{dx}{|x|^2})\}$.*

Proof. For this, notice that for every $u \in \mathcal{D}(A)$ and $\lambda > 0$:

$$\begin{aligned} \|u - \lambda\,|x|^2\Delta\, u\|^2_{L^2(\frac{dx}{|x|^2})} &= \|u\|^2_{L^2(\frac{dx}{|x|^2})} + 2\lambda\|\nabla\, u\|^2_{L^2} \\ &+ |\lambda|^2 \int_{B_1} |x|^2|\Delta\, u|^2 \geq \|u\|^2_{L^2(\frac{dx}{|x|^2})} \end{aligned}$$

which implies that A is dissipative in $L^2(\frac{dx}{|x|^2})$. Then, it suffices to show that A is maximal. Taking $f \in L^2(\frac{dx}{|x|^2})$, we consider the following minimization problem :

$$I_\lambda = \inf_{u \in H_0^1(B_1)} \mathcal{E}(u)$$
$$\text{where} \quad \mathcal{E}(u) = \frac{1}{2}\int_{B_1} (\frac{|u|^2}{|x|^2} + \lambda|\nabla u|^2)\, dx - \int_{B_1} \frac{f\,u}{|x|^2}$$

By Cauchy-Schwarz's inequality,

$$I_\lambda \geq \inf_{u \in H_0^1(B_1)} \frac{1}{2}\int_{B_1} (\frac{|u|^2}{|x|^2} + \lambda|\nabla\, u|^2)\, dx - (\int_{B_1} \frac{|f|^2}{|x|^2})^{\frac{1}{2}} (\int_{B_1} \frac{|u|^2}{|x|^2})^{\frac{1}{2}} > -\infty$$

then, considering a minimizing sequence $\{u_n\}_{n \in \mathbb{N}} \subset H_0^1(B_1) \cap L^2(\frac{dx}{|x|^2})$, it follows that $\|u_n\|_{H_0^1 \cap L^2(\frac{dx}{|x|^2})} \leq C$. And by standard compactness arguments, there exists $u \in H_0^1(B_1) \cap L^2(\frac{dx}{|x|^2})$ such that up to subsequences :

$$u_n \underset{n\to\infty}{\rightharpoonup} u \text{ weakly in } H_0^1(B_1),\, u_n \underset{n\to\infty}{\rightharpoonup} u \text{ weakly in } L^2(\frac{dx}{|x|^2})$$

and

$$\int_{B_1} \frac{f\,u_n}{|x|^2} \overset{n\to\infty}{\longrightarrow} \int_{B_1} \frac{f\,u}{|x|^2}$$

Therefore, I_λ is achieved by u and the proof is complete.

■

We deduce immediatly the following corollary :

Corollary 2.2. *A is maximal monotone in $L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$. Moreover, $\mathcal{D}(A) = \{u \in H_0^1(B_1) \cap L^\infty / |x|^2 \Delta u \in L^2(\frac{dx}{|x|^2}) \cap L^\infty\}$.*

Proof. Let $f \in L^2(\frac{dx}{|x|^2}) \cap L^\infty$ and $\lambda > 0$. By Proposition 2.1, there exists $u \in H_0^1 \cap L^2(\frac{dx}{|x|^2})$ such that

$$u - \lambda|x|^2 \Delta\, u = f \text{ in } B_1 \tag{2.1}$$

Thus, it suffices to show that $u \in L^\infty(B_1)$. Multiplying (2.1) by $(u - \|f\|_{L^\infty})^+$, we obtain :

$$\int_{B_1} \frac{(u - \| f \|_{L^\infty})^{+2}}{| x |^2} dx + \lambda \int_{B_1} | \nabla (u - \| f \|_{L^\infty})^+$$
$$= \int_{B_1} (f - \| f \|_{L^\infty}) \frac{(u - \| f \|_{L^\infty})^+}{| x |^2} \leq 0$$

which yields $(u - \|f\|_{L^\infty})^+ \equiv 0$ and $u \leq \|f\|_{L^\infty}$. By the same arguments, we show that $u \geq -\|f\|_{L^\infty}$. This ends the proof of Corollary 2.2.

■

Remark. For $N \geq 3$, $L^\infty(B_1) \subset L^2(\frac{dx}{|x|^2})$. And in this case, to prove Corollary 2.2, it suffices to show the maximality of A in L^∞.

Now, we apply Hille-Yosida theory (see [15]) and we deduce the following proposition :

Proposition 2.3. *Let $u_0 \in L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$. Then, there exists a unique solution $u(t) = S(t)u_0$ to (P_t) in a maximal interval $[0, T[, T > 0$ such that*

(i) $u(\cdot) \in C^0([0, T[, L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})) \cap C^1(]0, T[, L^2(\frac{dx}{|x|^2}))$.

(ii) For all t in $]0, T[$, $u(t) \in H_0^1(B_1) \cap L^\infty \cap L^2(\frac{dx}{|x|^2})$ and $|x|^2 \Delta u(t) \in L^2(\frac{dx}{|x|^2})$.

(iii) If $u_0 \geq 0$, then $u(t) \geq 0$ for all $t > 0$.

(iv) If $u_0 \in L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$ satisfies $|x|^2 \Delta u_0 \in L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$, then $u(t) \in C^1([0, T[, L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2}))$.

Proof. By Proposition 2.1, Corollary 2.2 and since $g \in W^{1,\infty}_{loc}$ we can apply Theorems 3.7 and 3.9 of [4]. This proves assertions (i), (ii) and (iv). Now, let us prove assertion (iii). For every $T_0 < T$, we multiply the equation in (P_t) by $\frac{(-u)^-}{|x|^2}$ and integrate by parts to obtain for every $t \in [0, T_0]$:

$$\frac{1}{2} \frac{d}{dt} \int_{B_1} \frac{|u^-|^2}{|x|^2} dx = - \int_{B_1} |\nabla u^-|^2 - \int_{B_1} \frac{g(u) u^-}{|x|^2} \leq C(T_0) \int_{B_1} \frac{|u^-|^2}{|x|^2}$$

which implies by Gronwall's lemma that $u^- \equiv 0$. This completes the proof of Proposition 2.3.

∎

As a consequence of Hille-Yosida Theory, we have the following alternative for $u(t) = S(t)u_0$:

Corollary 2.4. *If $u_0 \in L^2(\frac{dx}{|x|^2}) \cap L^\infty$, then, either $T = T(\|u_0\|_{\mathcal{D}(A)}) = +\infty$ and the solution $u(\cdot) = S(\cdot)u_0$ is global, or $T < +\infty$ and the solution blows up in finite time which means that*

$$\|u(t)\|_{L^\infty} + \|u(t)\|_{L^2(\frac{dx}{|x|^2})} \overset{t \to T^-}{\longrightarrow} +\infty$$

Proof. See [4].

Remarks. If $g \equiv 0$ and $u_0 \in H^1 \cap L^\infty \cap L^2(\frac{dx}{|x|^2})$ then $u(t) = S(t)u_0$ is global and satisfies :

$$\int_{B_1} \frac{|u(t)|^2}{|x|^2} \le e^{-(\frac{N-2}{2})^2 t} \|u_0\|^2_{L^2(\frac{dx}{|x|^2})}. \tag{2.2}$$

The proof is based upon Hardy's inequality. First, observe that since $g \equiv 0$, (P_t) is linear. Therefore, $u(t) = S(t)u_0$ is global. Moreover, multiplying (P_t) by $u(t)e^{(\frac{N-2}{2})^2 t}$ and integrating by parts, we have :

$$\frac{d}{dt} \int_{B_1} \frac{|u^-|^2}{|x|^2} e^{(\frac{N-2}{2})^2 t}\, dx = 2 \left(\frac{N-2}{2}\right)^2 \int_{B_1} \frac{|u^-|^2}{|x|^2} e^{(\frac{N-2}{2})^2 t}\, dx - 2$$
$$\int_{B_1} |\nabla u|^2 e^{(\frac{N-2}{2})^2 t}\, dx \le 0$$

by Hardy's inequality. Thus, integrating on $[0, t]$, we deduce (2.2).

Now, we deal with the behaviour of the solution to (P_t). In the next section, we give some examples of blow-up in finite time of solutions to (P_t).

3 Blow up in finite time in $L^2(\frac{dx}{|x|^2}) \cap L^\infty$

Throughout this section, we assume that g belongs to $W^{1,\infty}_{loc}$, $u_0 \in L^\infty \cap L^2(\frac{dx}{|x|^2})$ and $G(s) = \int_0^s g(t)\, dt$.

3.1 Main results

We consider three classes of functions g. First, we adapt a classical "spectral method" (see for instance [4]). Precisely, we prove the following theorem :

Theorem 3.1. *Assuming $N \geq 3$ and*

(B1) g is convex and positive in $\mathbb{R}^+$,

(B2) $(\frac{N-2}{2})^2 < \lambda := \lim_{s \to 0^+} \frac{g(s)}{s} < +\infty$,

(B3) There exists $s_0 > 0$ such that $\int_{s_0}^{+\infty} \frac{ds}{h(s)} < \infty$ where $h(s) = g(s) - \lambda s$.

Then, for any $u_0 \geq 0$ in $L^\infty \cap L^2(\frac{dx}{|x|^2})$, $u(t) = S(t)u_0$ satisfies : $\exists T \in \mathbb{R}^+$ such that

$$\lim_{t \to T^-} \int_{B_1} \frac{|u(t)|^2}{|x|^2} = +\infty \quad \text{and} \quad \lim_{t \to T^-} \|u(t)\|_{L^\infty} = +\infty$$

The second blow-up case is based upon an "energy method" (see for instance [4]).

Theorem 3.2. *Assume that u_0 satisfies (*) $\int_{B_1} \frac{|\nabla u_0|^2}{2} - \int_{B_1} \frac{G(u_0)}{|x|^2} < 0$ and that g has the following properties :*

(B4) $\lambda := \lim_{s \to 0^+} \frac{g(s)}{s} \in \mathbb{R}$ and there exists $\alpha > 0$, $C > 0$ such that $h(s) = g(s) - \lambda s \geq C\, s^{\alpha+1}$ for all $s \geq 0$,

(B5) There exists $\epsilon > 0$ such that for all $s \geq 0$, $(2+\epsilon)H(s) \leq sh(s)$ where $H(t) = \int_0^t h(s)\, ds$.

Then, $u(t) = S(t)u_0$ satisfies : $\exists T > 0$ such that $\lim_{t \to T^-} \int_{B_1} \frac{|u(t)|^2}{|x|^2} = +\infty$.

Remarks.

1. If $g(s) = \lambda\, s + s^p$ with $\lambda > \left(\frac{N-2}{2}\right)^2$ and $p > 1$, (B1), (B2) and (B3) are satisfied.

2. If $g(s) = \lambda\, s + s^p$ with $p > 1$, (B4) and (B5) are satisfied.

3. Let $\phi \in L^\infty \cap H_0^1$. Then, by (B4), there exists $M > 0$, large enough, such that $u_0 = M\,\phi$ satisfies (*).

4. If $u_0 \geq 0$ is a radially decreasing nontrivial subsolution of (P) and belongs to $H_0^1(B_1) \cap L^\infty$, then, a simple computation based upon a "Pohozaev's equality type" shows that (*) is satisfied for $N > 2$. Indeed, multiplying $-|x|^2 \Delta u_0 \leq g(u_0)$ by $\frac{x}{|x|^2} \cdot \nabla\, u_0$ and integrating by parts, we obtain :

$$-\left(\frac{N-2}{2}\right) \int_{B_1} |\nabla u_0|^2 - \frac{1}{2} \int_{\partial B_1} |\frac{\partial u_0}{\partial n}|^2\, ds \geq (2-N) \int_{B_1} \frac{G(u_0)}{|x|^2}$$

which implies :

$$\int_{B_1} \frac{|\nabla u_0|^2}{2} - \int_{B_1} \frac{G(u_0)}{|x|^2} \leq -\frac{1}{2(N-2)} \int_{\partial B_1} |\frac{\partial u_0}{\partial n}|^2\, ds < 0$$

Finally, we deal with the case $g(0) > 0$. In this case, we adapt a method from [3]. And we use the results of nonexistence of solutions to the problem (P).

Theorem 3.3. *Assume that $N \geq 3$ and the following assumptions on g :*

(B6) $g > 0$ is convex, increasing and belongs to $C^1([0, +\infty[)$,

(B7) There exists $s_0 > 0$ such that $\int_{s_0}^{+\infty} \frac{ds}{g(s)} < \infty$.

Then, for all $u_0 \geq 0$ in $L^\infty \cap L^2(\frac{dx}{|x|^2})$ and nontrivial, $u(t) = S(t)u_0$ blows up in finite time in L^∞ and in $L^2(\frac{dx}{|x|^2})$.

Remarks.

1. It is worth noting that in Theorems 3.1 and 3.3, no additional assumption is required for u_0. Here, the nonexistence of weak nontrivial solutions of the stationary problem (P) implies the blow-up in finite time for any initial data in $L^\infty \cap L^2(\frac{dx}{|x|^2})$.

2. The assumptions (B3) and (B7) prevent the existence of unbounded global solutions (i.e. which blow up when $t \to \infty$).

Now, we prove Theorem 3.1:

Proof of Theorem 3.1. Let us consider ψ_ϵ the eigenfunction associated with the first eigenvalue λ_ϵ^1 of $-(|x|^2+|\epsilon|^2)\Delta$ in $H_0^1(B_1)$ such that $\int_{B_1}\frac{\psi_\epsilon}{|x|^2}=1$ (for this, notice that $N\geq 3$ implies that $L^2(\frac{dx}{|x|^2})\subset L^1(\frac{dx}{|x|^2})$). It is easy to prove that $\lambda_\epsilon^1\longrightarrow(\frac{N-2}{2})^2$ when $\epsilon\to 0$. Therefore, by (B2), there exists $\epsilon>0$ small enough such that $\lambda_\epsilon^1<\lambda$. Thus, multiplying (P_t) by $\frac{\psi_\epsilon}{|x|^2}$, we obtain :

$$\frac{d}{dt}\int_{B_1}\frac{u(t)\,\psi_\epsilon}{|x|^2}+\lambda_\epsilon^1\int_{B_1}\frac{u(t)\,\psi_\epsilon}{|x|^2+|\epsilon|^2}=\int_{B_1}\frac{g(u(t))\psi_\epsilon}{|x|^2}$$

Since g is convex (which implies that f is convex), by Jensen's inequality, we have :

$$\frac{d}{dt}\int_{B_1}\frac{u(t)\,\psi_\epsilon}{|x|^2}\geq(\lambda-\lambda_\epsilon^1)\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2}+h(\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2})$$

From which it follows :

$$\frac{d}{dt}\left(\int_0^{\phi(t)}\frac{ds}{h(s)}\right)\geq 1\quad\text{where }\phi(t)=\int_{B_1}\frac{u(t)\,\psi_\epsilon}{|x|^2}\tag{3.1}$$

Integrating (3.1), one has $\int_0^{\phi(t)}\frac{ds}{h(s)}\geq t+C$ which together with (B3) implies that $\phi(\cdot)$ blows up in finite time. Finally, noting that for $N\geq 3$, the injection $L^\infty\hookrightarrow L^2(\frac{dx}{|x|^2})$ is continuous, the proof of Theorem 3.1 is complete.

■

Next, we give the proof of Theorem 3.2:

Proof of Theorem 3.2. Suppose that the solution $u(t)=S(t)u_0$ is global. Let us consider $E(t)=\frac{1}{2}\int_{B_1}|\nabla u(t)|^2-\int_{B_1}\frac{G(u(t))}{|x|^2}$. Then, multiplying (P_t) by $\frac{u_t}{|x|^2}$ and integrating by parts, we obtain :

$$\int_{B_1}\frac{|u_t|^2}{|x|^2}=-\frac{1}{2}\frac{d}{dt}\int_{B_1}|\nabla u(t)|^2+\frac{d}{dt}\int_{B_1}\frac{G(u(t))}{|x|^2}=-\frac{d}{dt}(E(t))$$

Thus, $E(t)$ is decreasing and $E(t)\leq E(0)<0$. Now, multiplying the equation in (P_t) by $\frac{u(t)}{|x|^2}$ and integrating by parts :

$$\frac{1}{2}\frac{d}{dt}\int_{B_1}\frac{|u(t)|^2}{|x|^2}=-\int_{B_1}|\nabla u(t)|^2+\int_{B_1}\frac{g(u(t))u(t)}{|x|^2}\tag{3.2}$$

By using (B5), and taking $H(s) = \int_0^s h(t)\,dt$, we prove that :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{B_1}\frac{|u(t)|^2}{|x|^2} &\geq -\int_{B_1}|\nabla u(t)|^2 + (2+\epsilon)\int_{B_1}\frac{H(u(t))}{|x|^2} + \lambda\int_{B_1}\frac{|u(t)|^2}{|x|^2}\\ &\geq -2E(t) + \epsilon\int_{B_1}\frac{H(u(t))}{|x|^2}\\ &\geq -2E(0) + \epsilon\int_{B_1}\frac{H(u(t))}{|x|^2} \qquad (3.3)\end{aligned}$$

Thus, (3.3) and (*) imply that $\lim\limits_{t\to\infty}\int_{B_1}\dfrac{|u(t)|^2}{|x|^2} = +\infty$. Then, by (3.3) :

$$\frac{1}{2}\frac{d}{dt}\int_{B_1}\frac{|u(t)|^2}{|x|^2} \geq \epsilon\int_{B_1}\frac{H(u(t))}{|x|^2} \geq C\epsilon\int_{B_1}\frac{|u(t)|^{2+\alpha}}{|x|^2} \geq C\epsilon\left(\int_{B_1}\frac{|u(t)|^2}{|x|^2}\right)^{\frac{2+\alpha}{2}}$$

Taking $\phi(t) = \int_{B_1}\frac{|u(t)|^2}{|x|^2}$, we have :

$$\frac{d}{dt}\phi(t) \geq 2\epsilon C\phi(t)^{\frac{\alpha+2}{2}} \qquad (3.4)$$

Integrating (3.4) on $[t_0, t]$, we obtain :

$$\frac{1}{\phi(t)^{\frac{\alpha}{2}}} - \frac{1}{\phi(t_0)^{\frac{\alpha}{2}}} \geq C(t - t_0)$$

which contradicts that $u(\cdot)$ is a global solution of (P_t). This completes the proof of Theorem 3.2.

■

Finally, we prove Theorem 3.3. Here, we use an approach from [3] : the nonexistence of stationary weak solutions implies the nonexistence of global, bounded solution of (P_t) for every $u_0 \geq 0$.

First, we adapt the definition of a weak stationary solution of (P_t) from [3] :

Definition 3.1. *A weak stationary solution of (P_t) is a function $u \in L^1(B_1)$ such that $\frac{g(u)}{|x|^2}\delta(x) \in L^1(B_1)$ (where $\delta(x) =$ dist$(x, \partial B_1)$) and*

$$\forall\,\xi \in C^2(\bar{B}_1) \quad -\int_{B_1} u\Delta\,\xi\,dx = \int_{B_1}\frac{g(u)}{|x|^2}\,\xi\,dx$$

Then, we have the following result :

Proposition 3.4. *Assume that g satisfies (B6) and (B7). Then, there is no weak stationary solution of* (P_t). **Proof .**

We apply a method from [3]. Precisely, for all η such that $0 \leq \eta < 1$, we define :

$$(P_\eta)\begin{cases} -|x|^2\Delta u = (1-\eta)g(u) \text{ in } B_1 \\ u \geq 0\,, \quad u \in \mathrm{H}_0^1(B_1) \end{cases}$$

As in [3], we define $h(u) = \int_0^u \frac{ds}{g(s)}$, $\tilde{h}(u) = \frac{1}{1-\eta}h(u)$ and $\Phi(u) = \tilde{h}^{-1}(h(u))$. It is easy to prove the following assertions (see [3]) :

(i) $\Phi(0) = 0$ and $0 \leq \Phi(u) \leq u$.

(ii) Φ is increasing and concave. Moreover, $\Phi'(u) \leq 1$.

(iii) $\Phi \in L^\infty$ and $\Phi(u)$ satisfies :

$$\forall\, \xi \in C^2(B_1) \quad -\int_{B_1} (\Delta\Phi(u))\,\xi \geq (1-\eta)\int_{B_1} \frac{g(\Phi(u))\xi}{|x|^2}$$

which means that $\Phi(u)$ is a "weak supersolution" of (P_η).

Forall $\xi \in C_0^2(\bar{B}_1)$, let us consider the following iterative scheme :

$$\begin{cases} -\int_{B_1} u_{n+1}\Delta\xi = (1-\eta)\int_{B_1} \frac{g(u_n)\xi}{|x|^2} \text{ in } B_1 \\ u_0 = \Phi(u)\,, \quad u \in \mathrm{H}_0^1(B_1) \end{cases}$$

Then, noting that $\Phi(u) \in L^\infty$ implies that for $N \geq 3$, $\frac{g(\Phi(u))}{|x|^2} \in L^1$ and by the fact that 0 is a strict subsolution to (P_η), we prove, by the maximun principle, that in L^∞, $\{u_n\}_{n\geq 1}$ is a decreasing sequence of weak supersolutions of (P_η) and $u_n \leq \Phi(u)$. Thus, $v_\eta = \lim_{n\to\infty} u_n \in L^\infty$ is a weak solution of (P_η). Now, consider for all ϵ in $]0,1[$, the following problem :

$$(P_{\epsilon,\eta})\begin{cases} -(|x|^2 + |\epsilon|^2)\Delta\, v = (1-\eta)g(v) \text{ in } B_1 \\ v \geq 0\,,\ v \in \mathrm{H}_0^1(B_1) \end{cases}$$

As in [1], we prove the existence of a minimal solution of $(P_{\epsilon,\eta})$, $v_{\epsilon,\eta}$, such that $v_{\epsilon,\eta} \leq v_\eta \leq \Phi(u)$.

Putting $w_{\epsilon,\eta}(x) := v_{\epsilon,\eta}(\epsilon x)$, for $x \in B_{\frac{1}{\epsilon}}$, we have :

$$\begin{cases} -(|x|^2+1)\Delta\, w_{\epsilon,\eta} = (1-\eta)g(w_{\epsilon,\eta}) \ \text{ in } \ B_{\frac{1}{\epsilon}} \\ w_{\epsilon,\eta} \geq 0\,, \quad w_{\epsilon,\eta} \in \ \mathrm{H}^1_0(B_{\frac{1}{\epsilon}}) \end{cases}$$

As above, we can show that $\epsilon \longrightarrow w_{\epsilon,\eta}$ is increasing in L^∞. Passing to the limit $\epsilon \to 0$, it is easy to prove that $w := \lim\limits_{\epsilon\to 0} w_\epsilon$ satisfies $\|w\|_{L^\infty} \leq \|v_\eta\|_{L^\infty}$ and is the minimal non trivial solution of the following problem :

$$\begin{cases} -(|x|^2+1)\Delta\, w = (1-\eta)g(w) \ \text{ in } \ \mathbb{R}^N \\ w \geq 0 \end{cases}$$

Therefore, $w(x) = \frac{1}{C_N|x|^{N-2}} * \frac{g(w)}{|x|^2+1}$ where $C_N = (N-2)|\sigma_{N-1}|$ and $|\sigma_{N-1}|$ the surface area of the unit sphere. Thus,

$$\begin{aligned} w(0) \ &= \ \int_{\mathbb{R}^N} \frac{C_N\, g(w)}{|x|^{N-2}(|x|^2+1)}\, dx \geq \inf_{s\,\in\,[0,\|w\|_{L^\infty}]} g(s) \\ &\int_{\mathbb{R}^N} \frac{1}{(|x|^2+1)|x|^{N-2}} = +\infty \end{aligned}$$

This contradicts the boundedness of w and the proof of Proposition 3.4 is now complete.

■

Proof of Theorem 3.3. First, note that by the maximum principle, it suffices to prove Theorem 3.3 in the case $u_0 \equiv 0$ (note that since g is increasing, $u_0 \leq w_0 \Rightarrow \forall t \geq 0\,,\ S(t)u_0 \leq S(t)w_0$). Moreover, $g(0) > 0 \Rightarrow u_t > 0$ for t small. Then, for $\delta > 0$ small,

$$u(t+\delta) = S(t+\delta)0 = S(t)\circ S(\delta)0 \geq S(t)0 = u(t) \text{ and } u_t \geq 0\,,\ \forall t \geq 0$$

Now, taking $\phi \in C^2_0(\bar{B}_1)$, multiplying the equation in (P_t) by $\frac{\phi}{|x|^2}$ and integrating by parts, we obtain :

$$\frac{d}{dt}\int_{B_1} \frac{u(t)\phi}{|x|^2} - \int_{B_1} u\Delta\phi = \int_{B_1} \frac{g(u(t))\phi}{|x|^2} \tag{3.5}$$

Therefore, choosing $\phi = \psi_\epsilon$ (defined in the proof of Theorem 3.1) we have :

$$\frac{d}{dt}\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2}+\lambda_\epsilon^1\int_{B_1}\frac{u\psi_\epsilon}{|x|^2+|\epsilon|^2}=\int_{B_1}\frac{g(u(t))\psi_\epsilon}{|x|^2}$$

Thus,

$$\frac{d}{dt}\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2}\geq\int_{B_1}\left(\frac{g(u(t))}{u(t)}-\lambda_\epsilon^1\right)\frac{u(t)\psi_\epsilon}{|x|^2}$$

which provides the following alternative :

1. either there exists $M > 0$ such that $\int_{B_1}\frac{g(u(t))\psi_\epsilon}{|x|^2}$, $\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2}\leq M$ for all $t \geq 0$, or

2. $\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2}\overset{t\to+\infty}{\longrightarrow}+\infty$.

Let us suppose that the second case holds. Then, by Jensen's inequality, we have for t large enough :

$$\frac{d}{dt}\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2}\geq\frac{1}{2}\int_{B_1}\frac{g(u(t))\psi_\epsilon}{|x|^2}\geq\frac{1}{2}g\left(\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2}\right)$$

Hence,

$$\int_0^{f(t)}\frac{ds}{g(s)}\geq\frac{1}{2}t+C\quad\text{where}\quad f(t)=\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2}$$

which contradicts (B7). And $u(t) = S(t)0$ blows up in finite time.

Finally, suppose that the first case occurs. And let ζ denote the unique solution of the following problem :

$$\begin{cases}-(|x|^2)\Delta\zeta=1 & \text{in } B_1\\ \zeta\equiv 0 & \text{in } \partial B_1\end{cases}$$

For $N \geq 3$, it is easy to prove that $\zeta \in W^{2,p}(B_1)$ for all $p < \frac{N}{2}$ which by Hardy's inequality and by Sobolev's embedding implies that $\zeta \in L^2(\frac{dx}{|x|^2}) \cap H_0^1(B_1)$. Hence, there exists $\{\zeta_n\}_{n\in\mathbb{N}} \subset C_0^\infty(B_1)$ such that :

$$\Delta\zeta_n\overset{L^1}{\longrightarrow}\Delta\zeta\quad\text{and}\quad\zeta_n\overset{L^2(\frac{dx}{|x|^2})}{\longrightarrow}\zeta \tag{3.6}$$

Choosing $\phi := \zeta_n$ in (3.5) and integrating in $[t, t+1]$, we have :

$$\left[\int_{B_1} \frac{u(s)\zeta_n}{|x|^2} \right]_t^{t+1} + \int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\zeta_n) = \int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\zeta_n}{|x|^2} \tag{3.7}$$

Passing to the limit $n \to \infty$, we obtain by (3.6) :

$$\int_{B_1} \frac{u(s)\zeta_n}{|x|^2} \overset{n\to\infty}{\longrightarrow} \int_{B_1} \frac{u(s)\zeta}{|x|^2}$$

Moreover, by Lebesgue theorem and by (3.6) :

$$\int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\zeta_n}{|x|^2} \overset{n\to\infty}{\longrightarrow} \int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\zeta}{|x|^2}$$

and

$$\int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\zeta_n) \overset{n\to\infty}{\longrightarrow} \int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\zeta)$$

Therefore,

$$\left[\int_{B_1} \frac{u(s)\zeta}{|x|^2} \right]_t^{t+1} + \int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\zeta) = \int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\zeta}{|x|^2}.$$

Now, since $u_t \geq 0$,

$$\begin{aligned} \int_{B_1} \frac{u(t)}{|x|^2} &\leq \int_t^{t+1} ds \int_{B_1} \frac{u(s)\zeta}{|x|^2} = \int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\zeta) \\ &= \int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\zeta}{|x|^2} - \left[\int_{B_1} \frac{u(s)\zeta}{|x|^2} \right]_t^{t+1} \\ &\leq \int_{B_1} \frac{g(u(t+1))\zeta}{|x|^2} \leq M \end{aligned}$$

Therefore, by monotone convergence, there exists $w \in L^1(\frac{dx}{|x|^2})$ such that $u(t) \overset{t\to+\infty}{\longrightarrow} w$ in $L^1(\frac{dx}{|x|^2})$. It implies that for all $\phi \in C_0^2(\bar{B}_1)$:

$$\int_{B_1} \frac{u(t)\phi}{|x|^2} \overset{t\to+\infty}{\longrightarrow} \int_{B_1} \frac{w\phi}{|x|^2}, \int_t^{t+1} ds \int_{B_1} u(s)(-\Delta\phi) \overset{t\to\infty}{\longrightarrow} \int_{B_1} w(-\Delta\phi) \text{ and}$$

$$\int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\phi}{|x|^2} \overset{t\to\infty}{\longrightarrow} \int_{B_1} \frac{g(w)\phi}{|x|^2}$$

(For this, note that $\int_t^{t+1} ds \int_{B_1} \frac{g(u(s))\phi}{|x|^2} \leq 2\int_{B_1} \frac{w\phi}{|x|^2} < +\infty$). Therefore, for all $\phi \in C_0^2(\bar{B}_1)$:

$$-\int_{B_1} w\Delta\phi = \int_{B_1} \frac{g(w)\phi}{|x|^2}$$

which contradicts the nonexistence of weak stationary solutions to (P_t). This completes the proof of Theorem 3.3.

■

Remarks. Consider $g \in C^1$ convex , increasing function satisfying $\lim_{s\to 0^+} \frac{g(s)}{s} > \left(\frac{N-2}{2}\right)^2$, (B7) and for all $s \geq 0$, $\frac{sg(s)}{2} \geq G(s) = \int_0^s g(t)\,dt$. Then, we can apply the previous method. Precisely, for all $u_0 > 0$, $u(\cdot) = S(\cdot)u_0$ blows up in finite time in $L^2(\frac{dx}{|x|^2})$.

It suffices to modify the proof of Theorem 3.3 as follows :

1. 0 is replaced by $\epsilon\phi_\epsilon$ which is a subsolution of (P) and $\epsilon\phi_\epsilon < u_0$, for ϵ small enough.
2. The nonexistence of stationary solutions of (P_t) is provided by the results from [7].

4 Global existence of solutions to (P_t) and convergence to a stationary solution

4.1 Main results

In this section, we give two examples of global existence of solutions to (P_t) which converge to a stationary solution when $t \to \infty$. In each case, we obtain an exponential control of the convergence either in L^∞ or in $H_0^1(B_1)$. Here, it is worth to underline that the convergence to a stationary solution is related to the uniqueness of the solution to (P). First, we prove the following :

Theorem 4.1. *Assume that $N \geq 2$ and the following hypothesis :*

(G1) $\lim_{s\to 0^+} \frac{g(s)}{s} = \lambda < 0,$

(G2) There exists $\epsilon > 0$, *such that* $|g(s) - \lambda s| \leq C|s|^{1+\epsilon}$.

Then, for u_0 *such that* $\|u_0\|_{L^\infty}$ *small enough,* $u(\cdot) = S(\cdot)u_0$ *is global and there exists* $C > 0$ *such that* $\|u(t)\|_{L^\infty} \leq Ce^{\lambda t}$ *for all* $t \geq 0$.

In the second part of the section, we prove the following theorem :

Theorem 4.2. *Assume that* $N \geq 3$ *and* g *satisfies the following assumptions :*

(G3) $s \to \frac{g(s)}{s}$ *is continuous and strictly decreasing,*

(G4) $\frac{g(s)}{s} \overset{s\to+\infty}{\longrightarrow} -\infty$,

(G5) $\frac{g(s)}{s} \overset{s\to0^+}{\longrightarrow} \lambda > (\frac{N-2}{2})^2$,

Then, for any u_0 *such that* $0 < u_0 \leq f^{-1}(0)$ *and* $u_0 \not\equiv f^{-1}(0)$, *with* $f(s) := \frac{g(s)}{s}$, $u(t) = S(t)u_0$ *is global and converges to the unique non-trivial solution of* (P), w_λ, *when* $t \to \infty$. *Moreover, if we suppose, in addition, that* $-g$ *is strictly convex, there exists* $K > 0$ *such that* $\|u(t) - w_\lambda\|_{H_0^1(B_1)} \leq C e^{-Kt}$ *for all* $t \geq 0$.

We start by proving a proposition which provides the heat kernel of $-|x|^2\Delta$:

Proposition 4.3. *Consider* $u = T(t)u_0 \in L^\infty(B_1) \cap L^2(\frac{dx}{|x|^2})$ *solution of*

$$\begin{cases} u_t - |x|^2\Delta u = \lambda u \;\; in \;\; B_1 \\ \qquad u(t,x) = 0 \;\; in \;\; \mathbb{R}^+ \times \partial B_1 \;, \;\; u(0,x) = u_0 \end{cases}$$

where u_0 *is radial. Then,* $u(t)$ *is radial and if* $v(t,s) := u(t,x)$ *with* $s = -ln\,|x|$ *and* $\lambda_N = (\frac{N-2}{2})^2$, *then,*

$$v(t,s) = \frac{e^{\frac{N-2}{2}s - (\lambda_N - \lambda)t - \frac{|s|^2}{4t}}}{(4\pi t)^{\frac{1}{2}}} * v(0,s).$$

Proof. First, we remark that the radial symmetry of u follows from the uniqueness of the solution to (P_t). Then, to compute the heat kernel of

$-|x|^2\Delta$, we use a method from [7]. Indeed, put $w(t,s) := e^{-\frac{N-2}{2}s}v(t,s)$. We show that w satisfies :

$$(P_w)\begin{cases} w_t - w_{ss} = (\lambda - \lambda_N)w \text{ in } \mathbb{R}^+ \times (0,+\infty) \\ w(t,0) = 0 \ , \ w(t,s) \overset{s\to+\infty}{\longrightarrow} 0 \ , \ w(0,s) = v(0,s)e^{-\frac{N-2}{2}s} \end{cases}$$

Taking $w(t,-s) = -w(t,s)$ for all $s \geq 0$, we have that (P_w) is satisfied in $\mathbb{R}^+ \times \mathbb{R}$. And we can apply Fourier transform. Indeed, for $N \geq 2$, $w(t,\cdot)$ belongs to $L^2(\mathbb{R})$ (for $N > 2$, it is obvious since $v \in L^\infty$ and for $N = 2$, it suffices to remark that $\int_{B_1} \frac{|u|^2}{|x|^2} < \infty \Rightarrow \int_0^{+\infty} w^2\,ds < \infty$).

A simple computation shows that $\hat{w}(t,x) = \hat{w}_0 e^{-(|x|^2+(\lambda_N-\lambda)t)}$. Using inverse Fourier transform, one has :

$$w(t,s) = w_0 * \frac{e^{-(\lambda_N-\lambda)t-\frac{|s|^2}{4t}}}{(4\pi t)^{\frac{1}{2}}} \quad \text{and} \quad v(t,s) = v_0 * \frac{e^{\frac{N-2}{2}s-(\lambda_N-\lambda)t-\frac{|s|^2}{4t}}}{(4\pi t)^{\frac{1}{2}}}.$$

This completes the proof of Proposition 4.3.

■

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Here, we apply a method from [4]. First, remark that by the maximum principle, it suffices to prove Theorem 4.1 when u_0 is radially symmetric. Then, by Proposition 4.3, $T(t)u_0$ is radially symmetric and

$$\begin{aligned} \|T(t)u_0\|_{L^\infty} &= \left\| v_0 * \frac{e^{\frac{N-2}{2}s-(\lambda_N-\lambda)t-\frac{|s|^2}{4t}}}{(4\pi t)^{\frac{1}{2}}} \right\|_{L^\infty} \\ &\leq \|v_0\|_{L^\infty} \left\| \frac{e^{\frac{N-2}{2}s-(\lambda_N-\lambda)t-\frac{|s|^2}{4t}}}{(4\pi t)^{\frac{1}{2}}} \right\|_{L^1} \end{aligned} \tag{4.1}$$

Now, using Laplace transform $f(y) := \int_{-\infty}^{+\infty} e^{yx-\frac{|x|^2}{4t}}\,dx$, we show that $\|e^{\frac{N-2}{2}s-\frac{s^2}{4t}}\|_{L^1} =$
$e^{t\lambda_N}(4\pi t)^{\frac{1}{2}}$. Therefore, by (4.1), $\|T(t)u_0\|_{L^\infty} \leq e^{\lambda t}\|u_0\|_{L^\infty}$. Now, we apply a method from [4]. First, we define $\Theta(\cdot)$ such that $\Theta(x) = \frac{C}{\epsilon|\lambda|}x^{1+\epsilon} - x$ with C defined in (G2) and $\delta > 0$ satisfy

$$\min \Theta(x) + \delta < 0 \ , \ \Theta(\delta) + \delta \geq 0 \ \text{ and } \ \Theta'(\delta) \leq 0$$

Let us choose $u_0 \in L^\infty \cap L^2(\frac{dx}{|x|^2})$ such that $||u_0||_{L^\infty} \leq \delta$. Then, $u = S(t)u_0$ satisfies :

$$\begin{aligned} ||u(t)||_{L^\infty} \leq ||T(t)u_0||_{L^\infty} &+ C\int_0^t e^{\lambda(t-s)}||u(s)||_{L^\infty}^{1+\epsilon} \leq e^{\lambda t}\delta \\ &+ Ce^{\lambda t}\int_0^t e^{\epsilon\lambda s}(e^{-\lambda s}||u(s)||_{L^\infty})^{1+\epsilon}\, ds \end{aligned}$$

Putting $\phi(t) = \sup_{[0,t]} e^{-\lambda s}||u(s)||_{L^\infty}$ which is an increasing function, we have :

$$\phi(t) \leq \delta + C\int_0^t \phi^{1+\epsilon}(s)e^{\epsilon\lambda s}\, ds \leq \delta + \frac{C}{\epsilon|\lambda|}\phi(t)^{1+\epsilon}.$$

If $\mu = \inf\{x > 0\, /\Theta(x) + \delta \leq 0\} \geq \delta$, it is easy to prove that $\phi(t) \leq \mu$ for all $t \in [0, T[$, where T is defined in Proposition 2.3. Moreover, $||u(t)||_{L^\infty} \leq e^{\lambda t}\mu$, which implies that u is global and $T = \infty$. This completes the proof of Theorem 4.1.

∎

Remarks.

1. If $p \in]1, +\infty[$, the function $g : s \to s^p$ satisfies the hypothesis of Theorem 4.1. Therefore, Theorems 3.1 and 4.1 show that the behaviour of the solution of (P_t) depends on the initial data.

2. It is worth noticing that for $N = 2$, we obtain almost a complete description of the behaviour of solutions of (P_t). Precisely, $\lambda_N = (\frac{N-2}{2})^2 = 0$ is the "blow-up critical parameter" (see [13]) which means that for $\lambda < \lambda_N$, there exists global solutions of (P_t) for small initial data and if $\lambda > \lambda_N$ then for all $u_0 \not\equiv 0$, $u(t) = S(t)u_0$ blows up in finite time. However, we do not know what happens in the case $\lambda = \lambda_N$. Moreover, since the heat kernel of $-|x|^2\Delta$ does not vanish at the boundary, we cannot apply a method due to Fujita (see [9]) which would furnish the answer. For $N \geq 3$, we suspect that λ_N still remains the critical blow-up parameter.

Now, we give the proof of Theorem 4.2.

Proof of Theorem 4.2.

Since there is a unique nontrivial solution to (P), it suffices to prove Theorem 4.2 when u_0 is radially decreasing. In this case, $S(t)u_0$ is also radially decreasing. Indeed, choosing $\epsilon \in]0,1[$, we remark that $u(t,\epsilon x) := u_\epsilon(t)$ is solution to

$$(P_{\epsilon,t}) \begin{cases} u_t - |x|^2\Delta u = g(u) & \text{in } \mathbb{R}^+ \times B_{\frac{1}{\epsilon}} \\ u(t,x) = 0 \ \text{ in } \mathbb{R}^+ \times \partial B_{\frac{1}{\epsilon}} & , u(0,x) = u_0(\epsilon x) \end{cases}$$

Since $u_0(\epsilon x) \geq u_0(x)$, by the maximum principle, for any $\epsilon \in]0,1[$, $u_\epsilon(t) \geq S(t)u_0$ which proves that $S(t)u_0$ is radially decreasing.

Now, as above we prove that :

$$\frac{d}{dt}\int_{B_1} \frac{|u(t)|^2}{|x|^2} = -\int_{B_1} |\nabla u(t)|^2 + \int_{B_1} \frac{g(u(t))u(t)}{|x|^2}$$

Moreover, $E(u(t)) = \frac{1}{2}\int_{B_1} |\nabla u(t)|^2 - \int_{B_1} \frac{G(u(t))}{|x|^2}$ satisfies

$$\frac{d}{dt}E(u(t)) < 0 \quad \text{and } E(u(t)) < E(u_0). \tag{4.2}$$

Futhermore, multiplying the equation in (P_t) by $\frac{(u-f^{-1}(0))^+}{|x|^2}$ we obtain :

$$\frac{d}{dt}\int_{B_1} \frac{\left((u(t)-f^{-1}(0))^+\right)^2}{|x|^2} + \int_{B_1} |\nabla(u(t)-f^{-1}(0))^+|^2$$
$$= \int_{B_1} \frac{((\lambda - f(u(t)))(u-f^{-1}(0))^+)^2}{|x|^2} \leq 0$$

which implies that for all $t \geq 0$, $u(t) \leq f^{-1}(0)$ and therefore $\cup_{t\geq 0}\{u(t)\}$ is uniformly bounded in $L^\infty(B_1)$. By (4.2), for $N \geq 3$, it follows that :

$$\int_{B_1} \frac{|\nabla u(t)|^2}{2} \leq E(u_0) - G(f^{-1}(0))\int_{B_1} \frac{1}{|x|^2}\,dx \leq C$$

Therefore, $\cup_{t\geq 0}\{u(t)\}$ is bounded in $L^\infty(B_1) \cap H_0^1(B_1)$. Then, for any sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to +\infty$, there is $w \in L^\infty(B_1) \cap H_0^1(B_1)$ (depending a priori on $\{t_n\}_{n\in\mathbb{N}}$) satisfying

$$u(t_n) \overset{n\to\infty}{\rightharpoonup} w \quad \text{in } H_0^1(B_1)\ ,\ u(t_n) \overset{n\to\infty}{\to} w \quad \text{in } L^2(\frac{dx}{|x|^2})$$

and

$$G(u(t_n) \overset{n\to\infty}{\to} G(w) \text{ in } L^1(\frac{dx}{|x|^2}).$$

For this, notice that on one hand

$$\int_{B_1} \frac{|u(t_n) - w|^2}{|x|^2} \leq (\int_{B_1} |u(t_n) - w|^{p'})^{\frac{1}{p'}} (\int_{B_1} \frac{1}{|x|^{2p}})^{\frac{1}{p}}$$

where $p < \frac{N}{2}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. On the other hand, since $\cup_{t\geq 0}\{u(t)\}$ and w are uniformly bounded in L^∞,

$$\int_{B_1} \frac{|G(u(t_n) - G(w)|^2}{|x|^2} \leq C \int_{B_1} \frac{|u(t_n) - w|^2}{|x|^2}$$

Let us show that

$$u(t_n) \overset{H_0^1(B_1)}{\longrightarrow} w \quad \text{when} \quad n \to \infty$$

For this, it suffices to prove that $\int_{B_1} |\nabla u(t_n)|^2 \overset{n\to\infty}{\longrightarrow} \int_{B_1} |\nabla w|^2$. Let us prove that $\int_{B_1} |\nabla u(t_n)|^2$ does not concentrate in $x = 0$. First, for any $\delta < 1$, multiplying the equation in (P_t) by $\frac{u(t)}{|x|^2}$ in B_δ, we have :

$$\frac{d}{dt}\int_{|x|\leq\delta} \frac{|u|^2}{|x|^2} + \int_{|x|\leq\delta} |\nabla u(t)|^2 - \int_{|x|=\delta} \frac{\partial\, u(t)}{\partial n} u(t)\, ds = \int_{|x|\leq\delta} \frac{g(u(t))u(t)}{|x|^2}$$

Since $u(t)$ is radially decreasing,

$$\frac{d}{dt}\int_{|x|\leq\delta} \frac{|u|^2}{|x|^2} + \int_{|x|\leq\delta} |\nabla u(t)|^2 \leq \int_{|x|\leq\delta} \frac{g(u(t))u(t)}{|x|^2} \tag{4.3}$$

Integrating (4.3) in $[t, t+1]$, we obtain :

$$\left[\int_{|x|\leq\delta} \frac{u(s)^2}{|x|^2}\right]_t^{t+1} + \int_t^{t+1} ds \int_{|x|\leq\delta} |\nabla u(s)|^2 \leq C \int_{|x|\leq\delta} \frac{1}{|x|^2}$$

where C is independent of t. Then, for all $\epsilon > 0$, there is $\delta(\epsilon) > 0$ small enough such that for all $\delta \leq \delta(\epsilon)$, we have :

$$0 \leq \int_t^{t+1} \int_{|x|\leq\delta} |\nabla u(t)|^2 \leq \epsilon \tag{4.4}$$

To conclude the proof, suppose that $\int_{B_1} |\nabla w|^2 < \lim_{n\to\infty} \int_{B_1} |\nabla u(t_n)|^2$. Then, by (4.2) $E(w) < E_\infty = \lim_{t\to\infty} E(u(t))$. However, by (4.4), it is easy to prove that

$$\int_t^{t+1} ds \int_{B_1} |\nabla u(t_n+\tau)|^2 \overset{n\to\infty}{\longrightarrow} \int_t^{t+1} \int_{B_1} |\nabla (S(\tau)w)|^2 \tag{4.5}$$

Indeed, by the boundedness of $\{u(t)\}_{t\geq 0}$ in $L^\infty \cap H_0^1(B_1)$,

$$\begin{aligned} \| \, |x|^2 \Delta(S(t+\tau)u_0)\|_{L^2(\frac{dx}{|x|^2})} &\leq \| \, |x|^2\Delta(T(t)u_0)\|_{L^2(\frac{dx}{|x|^2})} \\ &+ \int_t^{t+\tau} \| \, |x|^2 \Delta T(t+\tau-s) g(u(s))\|_{L^2(\frac{dx}{|x|^2})}\, ds \end{aligned}$$

from which it follows :

$$\begin{aligned} \| \, |x|^2 \Delta S(t+\tau)u_0\|_{L^2(\frac{dx}{|x|^2})} &\leq \frac{C}{\tau}\|u(t)\|_{L^\infty} \\ &+ C\int_t^{t+\tau} \frac{ds}{(t+\tau-s)^{\frac12} s^{\frac12}} \|u_0\|_{L^\infty} \leq \frac{C}{\tau} \end{aligned}$$

(for this, using a method from [4] Lemma 3.10, we prove that

$$\| \, |x|^2 \Delta T(t) u_0\|_{L^2(\frac{dx}{|x|^2})} \leq \frac{1}{t^{\frac12}} \|u_0\|_{H_0^1(B_1)}$$

and

$$\|T(t)u_0\|_{H_0^1(B_1)} \leq t^{-\frac12}\|u_0\|_{L^2(\frac{dx}{|x|^2})} \leq C t^{-\frac12}\|u_0\|_{L^\infty}$$

for $N \geq 3$)

Finally, (4.4) and the compactness of the embedding $H^2(\delta \leq |x| \leq 1) \hookrightarrow H^1(\delta \leq |x| \leq 1)$ imply (4.5). Now, using that $E(\cdot)$ is decreasing, we have :

$$\int_t^{t+1} E(S(\tau)w)\, d\tau \leq E(w) < E_\infty$$

which contradicts (4.5). Thus,

$$\int_{B_1} |\nabla u(t_n)|^2 \overset{n\to\infty}{\longrightarrow} \int_{B_1} |\nabla w|^2$$

and for any

$$t \geq 0 E(S(t)(w)) = E(w) = E_\infty.$$

This implies that w is a stationary solution of $(P_t.u_0)$ and either $w \equiv 0$, or $w \equiv w_\lambda$ which is the nontrivial solution of (P).

Now, let us prove that $u(t) \overset{t\to\infty}{\longrightarrow} w$ in $L^\infty(B_1)$. By a bootstrap argument (see [12]), it is easy to prove that for any $\delta > 0$,

$$\|u(t) - w\|_{L^\infty(|x|\geq\delta)} \overset{t\to\infty}{\longrightarrow} 0 \tag{4.6}$$

We consider $u_0 := \epsilon\, \psi_\epsilon$ which satisfies

$$|x|^2 \Delta u_0 + g(u_0) \geq 0 \quad \text{if } \epsilon \text{ is small enough} \tag{4.7}$$

We recall that ψ_ϵ is the eigenfunction of $-(|x|^2 + |\epsilon|^2)\Delta$ defined in the proof of Theorem 3.1. Note that by (G5) and (G4), (4.7) is satisfied for ϵ small enough. Then, u_0 is a strict subsolution of (P) and as above, it implies that $\frac{d}{dt}S(t)u_0 \geq 0$ for all $t \geq 0$. Hence, $u(\cdot)$ is increasing. Hence, $w = v_\lambda$. Then, by Dini's theorem and (4.6), we have for all $\delta > 0$:

$$\|u(t) - w_\lambda\|_{L^\infty(|x|\geq\delta)} \overset{t\to\infty}{\longrightarrow} 0 \tag{4.8}$$

Moreover, from [8] we know that $w_\lambda(0) = f^{-1}(0)$. Therefore, since $\{u(t)\}_{t\geq 0} \cup \{w_\lambda\}$ are radially decreasing

$$\begin{aligned}
\limsup_{t\to\infty} \|w_\lambda - u(t)\|_{L^\infty(B_1)} &\leq \limsup_{t\to\infty} \|u(t) - w_\lambda\|_{L^\infty(|x|\leq\delta)} \\
&+ \limsup_{t\to\infty} \|u(t) - w_\lambda\|_{L^\infty(|x|\geq\delta)} \\
&= \limsup_{t\to\infty} \|u(t) - w_\lambda\|_{L^\infty(|x|\leq\delta)} \leq f^{-1}(0) \\
&- \lim_{t\to\infty} \lim_{|x|\to 0} u(t,x)
\end{aligned}$$

Thus, suppose that $c_\lambda := \lim_{t\to\infty} \lim_{|x|\to 0} u(t,x) < f^{-1}(0)$. Then, since $u(t,\cdot)$ is radially decreasing, for any x_δ such that $|x| = \delta > 0$, $w_\lambda(x_\delta) = \lim_{t\to\infty} u(t, x_\delta) \leq c_\lambda$. This contradicts that w_λ is continuous.

Now, considering any u_0 such that $0 < u_0 \leq f^{-1}(0)$ and $u_0 \neq f^{-1}(0)$, there exists $\epsilon > 0$ small enough such that $\epsilon\psi_\epsilon < u_0$. It implies that

$$S(t)(\epsilon\psi_\epsilon) < S(t)(u_0) \leq f^{-1}(0) \quad \text{and} \quad \|S(t)u_0 - w_\lambda\|_{L^\infty(B_1)} \overset{t\to\infty}{\longrightarrow} 0 \tag{4.9}$$

To conclude the proof of Theorem 4.2, let us prove that if we suppose, in addition, that $-g$ is strictly convex, then, there exists $K > 0$ such that $\|u(t) - w_\lambda\|_{H_0^1(B_1)} \leq C e^{-Kt}$ for all $t \geq 0$. First, note that

$$\frac{d}{dt}\int_{B_1} \frac{|w_\lambda - u(t)|^2}{|x|^2} + \int_{B_1} |\nabla(w_\lambda - u(t))|^2 = \int_{B_1} \frac{(g(w_\lambda) - g(u(t)))(w_\lambda - u(t))}{|x|^2}$$

By (4.9), for t large enough, we have :

$$\frac{d}{dt}\int_{B_1} \frac{|w_\lambda - u(t)|^2}{|x|^2} + \frac{\lambda_1}{2}(-\Delta - \frac{g'(w_\lambda)}{|x|^2})\int_{B_1} \frac{(w_\lambda - u(t))^2}{|x|^2} \leq 0 \quad (4.10)$$

where $\frac{\lambda_1}{2}(-\Delta - \frac{g'(w_\lambda)}{|x|^2})$ is the first eigenvalue of $(-\Delta - \frac{g'(w_\lambda)}{|x|^2})$ in $H_0^1(B_1)$. Then, the strict convexity of $-g$ implies (see [8]) :

$$\lambda_1(-\Delta - \frac{g'(w_\lambda)}{|x|^2}) > \lambda_1(-\Delta - \frac{1}{|x|^2}\frac{g(w_\lambda)}{w_\lambda}) = 0$$

Thus, from (4.10), it is easy to prove that :

$$\int_{B_1} \frac{|w_\lambda - u(t)|^2}{|x|^2} \leq Ce^{-\frac{\lambda_1 t}{2}}$$

Using (4.10) and putting $K = \frac{\lambda_1}{2}$, we have for all t :

$$\int_{B_1} |\nabla(w_\lambda - u(t))|^2 \leq Ce^{-Kt} \quad (4.11)$$

This completes the proof of Theorem 4.2.

■

Remarks.

1. If $g(s) := \lambda s - |s|^{p-1}s$ where $\lambda > (\frac{N-2}{2})^2$ and $p > 1$, then, g satisfies the assumptions of Theorem 4.2.

2. Suppose that $-g$ is strictly convex and satisfies the assumptions (G3) to (G5). Then, taking $\delta > 0$, (4.11) and a bootstrap argument show that for all $t \geq 0$:

$$\|u(t) - w_\lambda\|_{L^\infty(|x|\geq\delta)} \leq C(\delta)e^{-Kt}.$$

However, we don't know if that remains valid for $\delta = 0$.

3. The assumption (G5) and the second part of Assumption (G4) suffice to prevent that $u(t) = S(t)u_0$ converges to 0 in $L^\infty(B_1)$ when $t \to \infty$ and when $u_0 \neq 0$.

Indeed, suppose that $\|S(t)u_0\|_{L^\infty(B_1)} \overset{t\to\infty}{\longrightarrow} 0$. Then, adapting a method from [14], we consider ϵ small enough such that $\lambda_\epsilon^1 < \lambda$. Then, multiplying the equation in (P_t) by ψ_ϵ :

$$\frac{d}{dt}\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2} + \lambda_\epsilon^1\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2+|\epsilon|^2} = \int_{B_1}\frac{g(u(t))\psi_\epsilon}{|x|^2}$$

from which it follows for t large enough :

$$\begin{aligned}\frac{d}{dt}\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2} &\geq -\lambda_\epsilon^1\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2} + \int_{B_1}\frac{g(u(t))\psi_\epsilon}{|x|^2}\\ &\geq \frac{1}{2}(g'(0)-\lambda_\epsilon^1)\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2} \qquad (4.12)\end{aligned}$$

Moreover, assumption (G5) implies that for ϵ small enough, $g'(0) = \lambda > \lambda_\epsilon^1$. Thus, by (4.12), we have $\int_{B_1}\frac{u(t)\psi_\epsilon}{|x|^2} \geq Ce^{\frac{(\lambda-\lambda_\epsilon^1)t}{2}} \overset{t\to\infty}{\longrightarrow} +\infty$ which contradicts the uniform boundedness of $\{u(t)\}_{t\geq 0}$.

4. In [8], the authors show the existence and the uniqueness of the solution, u_ϵ, to the following pertubed problem :

$$(P_\epsilon)\begin{cases} -|x|^2\Delta u = g(u) + \epsilon f(u) & \text{in } B_1\\ u \in H_0^1(B_1) \ , \ u \geq 0\end{cases}$$

where g satisfies (G3) to (G5), f is a positive function in $\mathbb{R}^+$ and belongs to $C^1(\mathbb{R}^+)$ such that $\lim_{s\to+\infty} f(s) + g(s) = -\infty$ and $\epsilon > 0$ small enough. Moreover, $\lambda_1(-\Delta - \frac{(g'(u_\epsilon)+\epsilon f'(u_\epsilon))}{|x|^2}) > 0$. Then, Theorem 4.2 holds for (P_ϵ).

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CEREMADE (URA.CNRS N 749),
Université de Paris-Dauphine,
75775 Paris Cedex 16, France
e-mail: giacomo@ceremade.dauphine.fr

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