Quantitative estimates for interpolated operators by multidimensional methods.

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Abstract

We describe the behaviour of ideal variations under interpolation methods associated to polygons.

0 Introduction

The behaviour of weakly compact operators under interpolation methods for $N$-tuples defined by means of polygons has been considered by Cobos, Fernández-Martínez and Martínez [5] and by Carro and Nikolova [4]. Among other things, they showed that the interpolated operator acting between two $K$-spaces or two $J$-spaces is weakly compact provided that all but two restrictions of $T$ (located in adjacent vertices of the polygon) are weakly compact. Moreover, a similar result holds for other operator ideals sharing certain properties with weakly compact operators (see [5], Remark 2.9).

In this paper we investigate how far the interpolated operator can be from being weakly compact. In a more general way, we estimate the distance of the interpolated operator to a given operator ideal. In the case of the classical real method for Banach couples, this question has been recently studied by Cobos, Manzano and Martínez [9] and Cobos.

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and Martínez [10], [11], where they have established estimates for the measures $\gamma_T$, $\beta_T$ related to a given operator ideal $I$. We consider here similar questions in the multidimensional context of interpolation spaces associated to polygons. Our techniques use some ideas introduced in [9] combined with the geometrical elements which are natural to the interpolation methods that we deal with.

We start by reviewing in Section 1 some basic facts on ideal variations and on $J$- and $K$-methods associated to polygons. Then, in Section 2, we establish estimates for $\gamma_T$ and $\beta_T$ when one of the $N$-tuples of Banach spaces degenerates into a single space. Finally, in Section 3, we deal with the case of general $N$-tuples assuming that the operator ideal $I$ satisfies the $\Sigma_q$-condition (see [14]).

1 Preliminaries

Let $A$ and $B$ be Banach spaces. By $\mathcal{L}(A, B)$ we denote the collection of all bounded linear operators from $A$ into $B$, endowed with the usual operator norm. The closed unit ball of $A$ is designated by $A_{\text{c}}$, and $A^*$ stands for the dual of $A$. We put $\ell_1(U_A)$ for the Banach space of all absolutely summable families of scalars $(\lambda_a)_{a \in U_A}$ with $U_A$ as index set. The map $Q_A : \ell_1(U_A) \rightarrow A$ defined by $Q_A(\lambda_a) = \sum_{a \in U_A} \lambda_a a$ is a metric surjection. The space $\ell_\infty(U_{B^*})$ is formed by all bounded families of scalars indexed by the elements of $U_{B^*}$. Write $J_B : B \rightarrow \ell_\infty(U_{B^*})$ for the isometric embedding given by $J_B b = (\langle f, b \rangle)_{f \in U_{B^*}}$.

A class $I$ of bounded linear operators is said to be an operator ideal if each component $I \cap \mathcal{L}(A, B) = I(A, B)$ is a linear subspace of $\mathcal{L}(A, B)$ that contains the finite rank operators and satisfies that $STR \in I(E, F)$ whenever $R \in \mathcal{L}(E, A)$, $T \in I(A, B)$ and $S \in \mathcal{L}(B, F)$. The ideal $I$ is called closed if each component $I(A, B)$ is closed in $\mathcal{L}(A, B)$. The ideal $I$ is said to be surjective if for every $T \in \mathcal{L}(A, B)$ it follows from $TQ_A \in I(\ell_1(U_A), B)$ that $T \in I(A, B)$. The ideal $I$ is called injective if for every $T \in \mathcal{L}(A, B)$ it follows from $J_B T \in I(A, \ell_\infty(U_{B^*}))$ that $T \in I(A, B)$. Compact operators $K$ or weakly compact operators $W$ are examples of closed injective and surjective operator ideals. Strictly singular operators $S$ is an ideal which is closed and injective but it is not surjective, while strictly cosingular operators $C$ is closed and surjective but it is not injective (see [17]).
Given an operator ideal $\mathcal{I}$, we put $\tilde{\mathcal{I}}^*$ for its closed surjective hull, that is, the smallest closed surjective operator ideal containing $\mathcal{I}$. For $T \in \mathcal{L}(A, B)$, it turns out that $T$ belongs to $\tilde{\mathcal{I}}^*(A, B)$ if and only if for every $\varepsilon > 0$ there is a Banach space $E$ and an operator $R \in \mathcal{I}(E, B)$ such that

$$T(U_A) \subseteq R(U_E) + \varepsilon U_B \quad \text{(see [15]).}$$

The characterization for the elements of the closed injective hull $\tilde{\mathcal{I}}$ of $\mathcal{I}$ is as follows: Let $T \in \mathcal{L}(A, B)$. The operator $T$ belongs to $\tilde{\mathcal{I}}(A, B)$ if and only if for every $\varepsilon > 0$ there is a Banach space $F$ and an operator $S \in \mathcal{I}(A, F)$ such that

$$\|Tx\|_B \leq \|Sx\|_F + \varepsilon \|x\|_A, \ x \in A.$$

It is natural then to associate with $\mathcal{I}$ the functionals defined for each $T \in \mathcal{L}(A, B)$ by

$$\gamma_{\mathcal{I}}(T) = \gamma_{\mathcal{I}}(T_{A, B}) = \inf \{\sigma > 0 : T(U_A) \subseteq \sigma U_B + R(U_E),
R \in \mathcal{I}(E, B), E \text{ any Banach space}\},$$

$$\beta_{\mathcal{I}}(T) = \beta_{\mathcal{I}}(T_{A, B}) = \inf \{\sigma > 0 : \text{there is a Banach space } F \text{ and}
S \in \mathcal{I}(A, F) \text{ such that } \|Tx\|_B \leq \sigma \|x\|_A + \|Sx\|_F, x \in A\}.$$

The (outer) measure $\gamma_{\mathcal{I}}$ was introduced by Astala in [1], and it shows the deviation of $T$ from $\tilde{\mathcal{I}}^*$ in the sense that

$$\gamma_{\mathcal{I}}(T) = 0 \text{ if and only if } T \in \tilde{\mathcal{I}}^*(A, B).$$

The (inner) measure $\beta_{\mathcal{I}}$ was introduced by Tylli in [19] and it gives the deviation of $T$ from $\tilde{\mathcal{I}}$. These functionals are subadditive

$$\gamma_{\mathcal{I}}(S + T) \leq \gamma_{\mathcal{I}}(S) + \gamma_{\mathcal{I}}(T), \quad \beta_{\mathcal{I}}(S + T) \leq \beta_{\mathcal{I}}(S) + \beta_{\mathcal{I}}(T)$$

submultiplicative

$$\gamma_{\mathcal{I}}(ST) \leq \gamma_{\mathcal{I}}(S)\gamma_{\mathcal{I}}(T), \quad \beta_{\mathcal{I}}(ST) \leq \beta_{\mathcal{I}}(S)\beta_{\mathcal{I}}(T)$$

satisfy that

$$\max \{\gamma_{\mathcal{I}}(T), \beta_{\mathcal{I}}(T)\} \leq \|T\|$$
and moreover the following minimal properties hold

\[ \gamma_I(J_BT) = \min \{ \gamma_I(jT) : j : B \rightarrow F \text{ isometric embedding} \} \quad (1) \]

\[ \beta_I(TQ_A) = \min \{ \beta_I(T\pi) : \pi : E \rightarrow A \text{ metric surjection} \} \quad (2) \]

(see [1], pag. 21 and [9], § 2).

Let us see now some concrete cases. Choose \( I = \mathcal{K} \), the ideal of compact operators, so \( \mathcal{K}^i = \mathcal{K}^s = \mathcal{K} \). It can be checked that \( \gamma_{\mathcal{K}}(T) \) coincides with the (ball) measure of non-compactness of \( T \)

\[ \gamma_{\mathcal{K}}(T) = \inf \{ \sigma > 0 : \text{there exists a finite number of elements} \]

\[ b_1, \ldots, b_k \in B \text{ such that } T(U_A) \subseteq \bigcup_{j=1}^k \{b_j + \sigma U_B\} \}

while \( \beta_{\mathcal{K}}(T) = \lim_{n \to \infty} c_n(T) \), where \( (c_n(T)) \) is the sequence of the Gelfand numbers of \( T \). The measures \( \gamma_{\mathcal{K}} \) and \( \beta_{\mathcal{K}} \) are equivalent. More precisely

\[ \frac{1}{2} \gamma_{\mathcal{K}}(T) \leq \beta_{\mathcal{K}}(T) \leq 2 \gamma_{\mathcal{K}}(T) \quad \text{(see [16]).} \]

Take next \( I = \mathcal{W} \), the ideal of weakly compact operators. Again \( \mathcal{W}^i = \mathcal{W}^s = \mathcal{W} \). The measure \( \gamma_{\mathcal{W}}(T) \) is equal to the measure of weak non-compactness introduced by De Blasi [13]

\[ \gamma_{\mathcal{W}}(T) = \inf \{ \sigma > 0 : \text{there is a weakly compact set } W \text{ in } B \]

\[ \text{such that } T(U_A) \subseteq W + \sigma U_B \} \]

As in the previous example, \( \beta_{\mathcal{W}}(T) = \gamma_{\mathcal{W}}(T^*) \), but this time \( \gamma_{\mathcal{W}} \) and \( \beta_{\mathcal{W}} \) are not equivalent (see [2]).

For \( I = \mathcal{S} \), the ideal of strictly singular operators, one has \( \mathcal{S}^i = \mathcal{S} \) and \( \mathcal{S}^s = \mathcal{R} \), where \( \mathcal{R} \) stands for the ideal of Rosenthal operators (see [17]). The functional \( \beta_{\mathcal{S}} \) is the relevant one to show the deviation of an operator from being strictly singular, while \( \gamma_{\mathcal{S}} = \gamma_{\mathcal{R}} \) gives the deviation of an operator from being Rosenthal.

Cosingular operators \( \mathcal{C} \) satisfy that \( \mathcal{C}^s = \mathcal{C} \) and \( \mathcal{C}^i = \mathcal{R} \). The relevant functional to work with \( \mathcal{C} \) is then \( \gamma_{\mathcal{C}} \).

Next we review the definition and some basic results on interpolation methods defined by means of polygons.

Let \( \Pi = P_1 \ldots P_N \) be a convex polygon in the plane \( \mathbb{R}^2 \), with vertices \( P_j = (x_j, y_j) \), \( j = 1, \ldots, N \). By a Banach \( N \)-tuple we mean a family \( \tilde{A} = \)
\{A_1, \ldots, A_N\} \text{ of } N \text{ Banach spaces } A_j \text{ which are continuously embedded in a common Hausdorff topological space. It will be useful to imagine each space } A_j \text{ as sitting in the vertex } P_j.

By means of the polygon \( \Pi \), we define the following family of norms on \( \Sigma(\bar{A}) = A_1 + \cdots + A_N \)

\[
K(t, s; a) = \inf \left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j, a_j \in A_j \right\}, \quad t, s > 0.
\]

The corresponding family of norms on \( \Delta(\bar{A}) = A_1 \cap \cdots \cap A_N \) is

\[
J(t, s; a) = \max_{1 \leq j \leq N} \left\{ t^{x_j} s^{y_j} \|a_j\|_{A_j} \right\}, \quad t, s > 0.
\]

Given any interior point \((\alpha, \beta)\) of \( \Pi \) \( [(\alpha, \beta) \in \text{Int } \Pi \text{ and any } 1 \leq q \leq \infty, \text{ the } K\text{-space } \tilde{A}_{(\alpha, \beta), q; K} \text{ consists of all } a \text{ in } \Sigma(\bar{A}) \text{ which have a finite norm}

\[
\|a\|(\alpha, \beta), q; K = \left( \sum_{(m,n) \in \mathbb{Z}^2} 2^{-\alpha m - \beta n} K(2^m, 2^n; a) \right)^{\frac{1}{q}} \quad (\text{if } q < \infty)
\]

\[
\|a\|(\alpha, \beta), \infty; K = \sup_{(m,n) \in \mathbb{Z}^2} \left\{ 2^{-\alpha m - \beta n} K(2^m, 2^n; a) \right\}.
\]

The \( J\)-space \( \tilde{A}_{(\alpha, \beta), q; J} \) is formed by all those elements \( a \) in \( \Sigma(\bar{A}) \) which can be represented as

\[
a = \sum_{(m,n) \in \mathbb{Z}^2} u_{m,n} \quad \text{(convergence in } \Sigma(\bar{A}))
\]

with \( u_{m,n} \in \Delta(\bar{A}) \) and

\[
\left( \sum_{(m,n) \in \mathbb{Z}^2} \left( 2^{-\alpha m - \beta n} J(2^m, 2^n; u_{m,n}) \right)^{q} \right)^{\frac{1}{q}} < \infty
\]

(the sum should be replaced by the supremum if \( q = \infty \)). The norm in \( \tilde{A}_{(\alpha, \beta), q; J} \) is

\[
\|a\|(\alpha, \beta), q; J = \inf \left\{ \left( \sum_{(m,n) \in \mathbb{Z}^2} \left( 2^{-\alpha m - \beta n} J(2^m, 2^n; u_{m,n}) \right)^{q} \right)^{\frac{1}{q}} \right\}
\]
where the infimum is taken over all representations \((u_{m,n})\) of \(a\) as above.

These interpolation spaces were introduced by Cobos and Peetre in [12]. One can find there continuous characterizations of \(\tilde{A}(\alpha, \beta), q; K\) and \(\tilde{A}(\alpha, \beta), q; J\), using integrals instead of sums, but they will not be required here. An important difference with the classical real method for couples, where \(K\)- and \(J\)-spaces coincide to within equivalence of norms (see [3] and [18]), is that in general \(\tilde{A}(\alpha, \beta), q; K \neq \tilde{A}(\alpha, \beta), q; J\). We only have now that \(\tilde{A}(\alpha, \beta), q; J\) is continuously embedded in \(\tilde{A}(\alpha, \beta), q; K\) (see [12], Thm. 1.3).

Let \(\tilde{B} = \{B_1, \ldots, B_N\}\) be another Banach \(N\)-tuple which we also imagine as sitting on the vertices of another copy of the polygon \(\Pi\). By \(T \in \mathcal{L}(\tilde{A}, \tilde{B})\) we mean a linear operator from \(\Sigma(\tilde{A})\) into \(\Sigma(\tilde{B})\) whose restriction to each \(A_j\) defines a bounded operator from \(A_j\) into \(B_j\), \(j = 1, \ldots, N\). Let \(M_j = \|T\|_{A_j, B_j}\).

If \(T \in \mathcal{L}(\tilde{A}, \tilde{B})\), then the restriction of \(T\) to \(\tilde{A}(\alpha, \beta), q; K\) gives a bounded linear operator \(T : \tilde{A}(\alpha, \beta), q; K \to \tilde{B}(\alpha, \beta), q; K\). The norm of this interpolated operator has been computed in [8], Thm. 1.9. It turns out that

\[
\|T\|_{\tilde{A}(\alpha, \beta), q; K, \tilde{B}(\alpha, \beta), q; K} \leq C_1 \max \{M_i^c, M_k^c, M_r^c : \{i, k, r\} \in \mathcal{P}\}. \tag{3}
\]

Here \(C_1\) is a constant depending only on \(\Pi\) and \((\alpha, \beta)\), \(\mathcal{P}\) stands for the set of all those triples \(\{i, k, r\}\) such that \((\alpha, \beta)\) belongs to the triangle with vertices \(P_i, P_k, P_r\), and \((c_i, c_k, c_r)\) are the barycentric coordinates of \((\alpha, \beta)\) with respect to \(P_i, P_k, P_r\). A similar estimate holds for \(J\)-spaces.

When the interpolated operator is considered from a \(J\)-space into a \(K\)-space then a better estimate is valid. Namely

\[
\|T\|_{\tilde{A}(\alpha, \beta), q; J, \tilde{B}(\alpha, \beta), q; K} \leq C_2 \prod_{j=1}^N M_j^{\theta_j}. \tag{4}
\]

Here \(0 < \theta_1, \ldots, \theta_N < 1\) with \(\sum_{j=1}^N \theta_j = 1\) and \(\sum_{j=1}^N \theta_j P_j = (\alpha, \beta)\) (that is, \(\theta = (\theta_1, \ldots, \theta_N)\) are some barycentric coordinates of \((\alpha, \beta)\) with respect to the vertices \(P_1, \ldots, P_N\)), and \(C_2\) is a constant depending only on \(\theta\) (see [8], Thm. 3.2).

Estimate (1.4) implies that

\[
\|a\|_{(\alpha, \beta), q; K} \leq C_3 \prod_{j=1}^N \|a\|_{A_j}^{\theta_j}, \quad a \in \Delta(\tilde{A}). \tag{5}
\]
On the other hand, inequality (1.3) in the case of $J$-spaces yields that
\[
\|a\|_{(\alpha, \beta), q; J} \leq C_4 \max \left\{ \|a\|_{\tilde{A}_i} \|a\|_{\tilde{A}_k} \|a\|_{\tilde{A}_r} : \{i, k, r\} \in \mathcal{P} \right\}, \quad a \in \Delta(\tilde{A}).
\] (6)

2 Estimates for degenerated cases

The following result describes the behaviour of the ideal variations when one of the $N$-tuples reduces to a single Banach space.

**Theorem 2.1.** Let $I$ be an operator ideal, let $\Pi = \overline{P_1 \ldots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$, let $(\alpha, \beta) \in \text{Int} \Pi$ and $1 \leq q \leq \infty$. Define $\mathcal{P}$ and $\theta = (\theta_1, \ldots, \theta_N)$ as before. Assume that $\tilde{A} = \{A_1, \ldots, A_N\}$ is a Banach $N$-tuple and that $B$ is a Banach space.

If $T \in \mathcal{L}(\Sigma(\tilde{A}), B)$ then

a) $\gamma(T_{\tilde{A}(\alpha, \beta); q, K; B})$
\[
\leq D_1 \max \left\{ \gamma(T_{A_i, B})^c_i \gamma(T_{A_k, B})^c_k \gamma(T_{A_r, B})^c_r : \{i, k, r\} \in \mathcal{P} \right\}.
\]

b) $\gamma(T_{\tilde{A}(\alpha, \beta); q, J; B})$ \leq D_2 \prod_{j=1}^{N} \gamma(T_{A_j, B})^{\theta_j}.$

If $T \in \mathcal{L}(B, \Delta(\tilde{A}))$ then

c) $\beta(T_{B, \tilde{A}(\alpha, \beta); s, J})$
\[
\leq D_3 \max \left\{ \beta(T_{B, A_i})^c_i \beta(T_{B, A_k})^c_k \beta(T_{B, A_r})^c_r : \{i, k, r\} \in \mathcal{P} \right\}.
\]

d) $\beta(T_{B, \tilde{A}(\alpha, \beta); q, K})$ \leq D_4 \prod_{j=1}^{N} \beta(T_{B, A_j})^{\theta_j}.$

Here $D_1$ and $D_3$ are constants depending only on $\Pi$ and $(\alpha, \beta)$, while $D_2$ and $D_4$ are other constants that only depend on $\theta$.

**Proof.** Since $\tilde{A}(\alpha, \beta); q, K \hookrightarrow \tilde{A}(\alpha, \beta); \infty, K$ with norm less than or equal to 1, in order to establish a) it is enough to consider the case $q = \infty$. Observe that there is a constant $C$, depending only on $\Pi$ and $(\alpha, \beta)$, such that
\[
\sup_{t, s > 0} \left\{ t^{-\alpha} s^{-\beta} K(t, s; a) \right\} \leq C \|a\|_{(\alpha, \beta); \infty, K}, \quad a \in \tilde{A}(\alpha, \beta); \infty, K.
\]
Hence, given any \( \varepsilon, t, s > 0 \) and \( a \in U_{A(\alpha, \beta), \infty; K} \), we can find a decomposition \( a = \sum_{j=1}^{N} a_j \) with \( a_j \in A_j \) and \( \|a_j\|_{A_j} \leq (1 + \varepsilon)Ct^{\alpha - x_j} s^{\beta - y_j} \), \( 1 \leq j \leq N \). So

\[
U_{A(\alpha, \beta), \infty; K} \subseteq \sum_{j=1}^{N} (1 + \varepsilon)Ct^{\alpha - x_j} s^{\beta - y_j} U_{A_j}.
\]

Let \( \sigma_j > \gamma_{x}(T_{A_{j}, B}) \). According to the definition of \( \gamma_{x} \), there exists a Banach space \( E_j \) and an operator \( R_j \in \mathcal{I}(E_j, B) \) so that

\[
T \left( U_{A_j} \right) \subseteq \sigma_j U_B + R_j \left( U_{E_j} \right), \quad 1 \leq j \leq N.
\]

Therefore

\[
T \left( U_{A(\alpha, \beta), \infty; K} \right)
\]

\[
\subseteq \sum_{j=1}^{N} (1 + \varepsilon)C\sigma_j t^{\alpha - x_j} s^{\beta - y_j} U_B + \sum_{j=1}^{N} (1 + \varepsilon)Ct^{\alpha - x_j} s^{\beta - y_j} R_j(U_{E_j})
\]

\[
\subseteq (1 + \varepsilon)C \left( \sum_{j=1}^{N} t^{\alpha - x_j} s^{\beta - y_j} \sigma_j \right) U_B + R_{\varepsilon, t, s}(U_E).
\]

Here \( E = \{(z_1, \ldots, z_N) : z_j \in E_j \} \) normed by \( \|(z_1, \ldots, z_N)\|_E = \max\{\|z_j\|_{E_j} : 1 \leq j \leq N\} \) (i.e., \( E = (\oplus_{j=1}^{N} E_j)_{t_{\infty}} \)), and \( R_{\varepsilon, t, s} : E \rightarrow B \) is the operator defined by \( R_{\varepsilon, t, s}(z_1, \ldots, z_N) = (1 + \varepsilon)C \sum_{j=1}^{N} t^{\alpha - x_j} s^{\beta - y_j} R_j z_j \). Ideal property of \( \mathcal{I} \) implies that \( R_{\varepsilon, t, s} \in \mathcal{I}(E, B) \). Hence

\[
\gamma_{x}(T_{A(\alpha, \beta), \infty; K}, B) \leq C \inf_{t, s > 0} \left\{ \sum_{j=1}^{N} t^{\alpha - x_j} s^{\beta - y_j} \gamma_{x}(T_{A_j}, B) \right\}
\]

\[
\leq NC \inf_{t, s > 0} \left\{ \max_{1 \leq j \leq N} \left\{ t^{\alpha - x_j} s^{\beta - y_j} \gamma_{x}(T_{A_j}, B) \right\} \right\}
\]

\[
= NC \max \{ \gamma_{x}(T_{A_i}, B)^{c_i} \gamma_{x}(T_{A_k}, B)^{c_k} \gamma_{x}(T_{A_r}, B)^{c_r} : \{i, k, r\} \in \mathcal{P} \}
\]

where we have used [8], Thm. 1.9, in the last equality. This establishes a).
To prove b) let again $\sigma_j > \gamma_z(T_{A_j,B})$, and consider the following norm on $\Sigma(\tilde{A})$

$$\|a\| = \inf \left\{ \sum_{j=1}^{N} \sigma_j \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j, a_j \in A_j \right\}.$$

Take any $a \in U_{\tilde{A}(\alpha, \beta), \eta, \gamma}$ and $\varepsilon > 0$. Using the Hahn-Banach theorem, we can find $f \in (\Sigma(\tilde{A}), \| \cdot \|)^*$ such that $f((1 + \varepsilon)^{-1}a) = \| (1 + \varepsilon)^{-1}a \|$ and $\|f\|_{\Sigma(\tilde{A})} \leq \sigma_j$, $1 \leq j \leq N$. By (4), the norm $\|f\|_{(\tilde{A}(\alpha, \beta), \eta, \gamma)^*}$ of the restriction of $f$ to $\tilde{A}(\alpha, \beta), \eta, \gamma$ is less than or equal to $C \prod_{j=1}^{N} \sigma_j^{\theta_j}$. Whence

$$\|a\| = (1 + \varepsilon)\|f((1 + \varepsilon)^{-1}a)\| \\
\leq (1 + \varepsilon)C \prod_{j=1}^{N} \sigma_j^{\theta_j} \|((1 + \varepsilon)^{-1}a)\|_{(A_{\alpha, \beta}, \eta, \gamma)} < (1 + \varepsilon)C \prod_{j=1}^{N} \sigma_j^{\theta_j}.$$

This allows us to find a representation $a = \sum_{j=1}^{N} a_j$ of $a$ with $\|a_j\|_{A_j} \leq \sum_{j=1}^{N} \sigma_j^{\theta_j} \leq (1 + \varepsilon)C \sigma_1^{\theta_1} \ldots \sigma_j^{\theta_j} \ldots \sigma_N^{\theta_N}$, $1 \leq j \leq N$. Choosing again Banach spaces $E_j$ and operators $R_j \in \mathcal{T}(E_j, B)$ with

$$T(U_{A_j}) \subseteq \sigma_j U_B + R_j(U_{E_j}), \quad 1 \leq j \leq N,$$

it follows that

$$T(U_{\tilde{A}(\alpha, \beta), \eta, \gamma}) \subseteq (1 + \varepsilon)C \sum_{j=1}^{N} \sigma_1^{\theta_1} \ldots \sigma_j^{\theta_j} \ldots \sigma_N^{\theta_N} T(U_{A_j})$$

$$\subseteq (1 + \varepsilon)CN \sigma_1^{\theta_1} \ldots \sigma_N^{\theta_N} U_B + (1 + \varepsilon)C \sum_{j=1}^{N} \sigma_1^{\theta_1} \ldots \sigma_j^{\theta_j} \ldots \sigma_N^{\theta_N} R_j(U_{E_j})$$

$$\subseteq (1 + \varepsilon)CN \sigma_1^{\theta_1} \ldots \sigma_N^{\theta_N} U_B + R(U_E)$$

where $E = \left( \oplus_{j=1}^{N} E_j \right)_{\ell_{\infty}}$ and $R \in \mathcal{T}(E, B)$ is the operator defined by

$$R(z_1, \ldots, z_N) = (1 + \varepsilon)C \sum_{j=1}^{N} \sigma_1^{\theta_1} \ldots \sigma_j^{\theta_j} \ldots \sigma_N^{\theta_N} R_j z_j.$$
Consequently
\[ \gamma_\pi(T_{A(\alpha,\beta),q,J,B}) \leq CN \prod_{j=1}^{N} \gamma_\pi(T_{A_j,B})^{\theta_j}. \]

To proceed to c) and d), assume that \( T \in \mathcal{L}(B, \Delta(\bar{A})) \) and let \( \sigma_j > \beta_\pi(T_{B,A_j}), 1 \leq j \leq N \). By the definition of \( \beta_\pi \), we can find Banach spaces \( F_j \) and operators \( S_j \in \mathcal{I}(B, F_j) \) so that
\[ \|Tb\|_{A_j} \leq \sigma_j \|b\|_B + \|S_j b\|_{F_j}, \quad b \in B. \]

Put \( F = \left( \oplus_{j=1}^{N} F_j \right)_{\ell_1}, \sigma = \min\{\sigma_1, \ldots, \sigma_N\} \) and let \( S \in \mathcal{I}(B, F) \) be the operator defined by
\[ Sb = \max \left\{ \sigma_i^{\xi_i} \sigma_k^{\xi_k} \sigma_r^{\xi_r} : \{i, k, r\} \in \mathcal{P} \right\} \sigma^{-1}(S_1b, \ldots, S_Nb). \]

Using (6) we get that
\[ \|Tb\|_{(\alpha,\beta),q,J} \leq C \max \left\{ \|Tb\|_{A_i}^{\xi_i}, \|Tb\|_{A_k}^{\xi_k}, \|Tb\|_{A_r}^{\xi_r} : \{i, k, r\} \in \mathcal{P} \right\} \]
\[ \leq C \max \{ \sigma_i^{\xi_i} \sigma_k^{\xi_k} \sigma_r^{\xi_r} : \{i, k, r\} \in \mathcal{P} \} \|b\|_B + C \|Sb\|_F, \]
and c) follows.

Finally, working with the operator \( V \in \mathcal{I}(B, F) \) given by
\[ Vb = \sigma^{-1}(\prod_{j=1}^{N} \sigma_j^{\theta_j})(S_1b, \ldots, S_Nb) \]
and using (5), we derive that
\[ \|Tb\|_{(\alpha,\beta),q,K} \leq C \prod_{j=1}^{N} \|Tb\|_{A_j}^{\theta_j} \leq C \prod_{j=1}^{N} \left( \sigma_j \|b\|_B + \|S_j b\|_{F_j} \right)^{\theta_j} \]
\[ \leq C \prod_{j=1}^{N} \sigma_j^{\theta_j} \left( \|b\|_B + \frac{1}{\sigma} \|R_j b\|_{F_j} \right)^{\theta_j} \leq C \left( \prod_{j=1}^{N} \sigma_j^{\theta_j} \right) \|b\|_B + C \|Vb\|_F. \]

This implies d) and completes the proof.
Writing down Theorem 2.1 for the case \( \mathcal{I} = \mathcal{W} \), the ideal of weakly compact operators, we get a quantitative version of Thms 2.3 and 2.4 in [5]. For \( \mathcal{I} = \mathcal{K} \), the ideal of compact operators, we obtain estimates for the measure of non-compactness of the interpolated operator that are analogous to those proved in [7], Prop. 3.1 and 3.3 for entropy numbers. Recall that the measure of non-compactness is the limit of the sequence of entropy numbers. Theorem 2.1 can be also applied to derive results on strict singularity and cosingularity.

3 Estimates for the general case

We deal now with the case of non-degenerated \( N \)-tuples. It is not difficult to show by means of examples that Theorem 2.1 fails in this general case. However, assuming an extra condition on the operator ideal \( \mathcal{I} \), we shall be able to describe the behaviour of the ideal variations.

Given any sequence of Banach spaces \( (Z_{m,n})_{(m,n) \in \mathbb{Z}^2} \), any sequence of non-negative numbers \( (\lambda_{m,n})_{(m,n) \in \mathbb{Z}^2} \) and \( 1 < q < \infty \), we denote by \( \ell_q(\lambda_{m,n}Z_{m,n}) \) the vector-valued \( \ell_q \) space defined by

\[
\ell_q(\lambda_{m,n}Z_{m,n}) = \left\{ z = (z_{m,n}) : z_{m,n} \in Z_{m,n} \quad \text{and} \quad \| z \|_{\ell_q(\lambda_{m,n}Z_{m,n})} = \left( \sum_{(m,n) \in \mathbb{Z}^2} (\lambda_{m,n} \| z_{m,n} \|^q) \right)^{\frac{1}{q}} < \infty \right\}.
\]

Any operator \( T \in \mathcal{L}(\ell_q(\lambda_{m,n}Z_{m,n}), \ell_q(\mu_{m,n}Y_{m,n})) \) between two vector-valued \( \ell_q \) spaces can be imagined as an infinite matrix with entries \( Q_{r,s}TP_{u,v} \). Here \( P_{u,v} : \lambda_{u,v}Z_{u,v} \longrightarrow \ell_q(\lambda_{m,n}Z_{m,n}) \) is the embedding \( P_{u,v}z = (\delta_{m,n}^{u,v}z) \), where

\[
\delta_{m,n}^{u,v} = \begin{cases} 1 & \text{if } m = u, n = v, \\ 0 & \text{otherwise} \end{cases}
\]

and \( Q_{r,s} : \ell_q(\mu_{m,n}Y_{m,n}) \longrightarrow \mu_{r,s}Y_{r,s} \) is the projection \( Q_{r,s}(y_{m,n}) = y_{r,s} \).

For \( 1 < q < \infty \), we say that the operator ideal \( \mathcal{I} \) satisfies the \( \Sigma_q \)-condition if for any sequences of Banach spaces

\( (\lambda_{m,n}Z_{m,n}), (\mu_{m,n}Y_{m,n}) \) and any \( T \in \mathcal{L}(\ell_q(\lambda_{m,n}Z_{m,n}), \ell_q(\mu_{m,n}Y_{m,n})) \), it follows from \( Q_{r,s}TP_{u,v} \in \mathcal{I}(\lambda_{u,v}Z_{u,v}, \mu_{r,s}Y_{r,s}) \) for any \( r, s, u, v \) that \( T \in \mathcal{I}(\ell_q(\lambda_{m,n}Z_{m,n}), \ell_q(\mu_{m,n}Y_{m,n})) \).
Weakly compact operators, Rosenthal operators, Banach-Saks operators or dual Radon-Nikodym operators are examples of ideals satisfying the $\Sigma_q$-condition (see [14]). All of them are also injective surjective and closed.

The following result shows the behaviour of the measure $\gamma_Z$ with $K$-spaces.

**Theorem 3.1.** Let $\Pi = P_1 \ldots P_N$ be a convex polygon with vertices $P_j = (x_j, y_j)$, let $(\alpha, \beta) \in \text{Int} \Pi$, $1 < q < \infty$, and let $I$ be an operator ideal which satisfies the $\Sigma_q$-condition. Assume that $\tilde{A} = \{A_1, \ldots, A_N\}$ and $\tilde{B} = \{B_1, \ldots, B_N\}$ are Banach $N$-tuples and let $T \in \mathcal{L}(\tilde{A}, \tilde{B})$. Then for the interpolated operator we have

\[
\gamma_Z \left( \left[ J_{B(\alpha, \beta), q, K}^T \right] A(\alpha, \beta), q, K, \ell_\infty(U_{B(\alpha, \beta), q, K}) \right)
\]

\[
\leq D \max \left\{ \gamma_Z(T_{A_i, B_i})^c, \gamma_Z(T_{A_k, B_k})^c, \gamma_Z(T_{A_r, B_r})^c : \{i, k, r\} \in \mathcal{P} \right\}
\]

where $D$ is a constant depending only on $\Pi$ and $(\alpha, \beta)$.

**Proof.** Let $F_{m,n} = (B_1 + \ldots + B_N, K(2^m, 2^n; \cdot))$, $(m, n) \in \mathbb{Z}^2$, and form the vector-valued space $\ell_q(2^{-\alpha m - \beta n} F_{m,n})$. The map $j : \tilde{B}(\alpha, \beta), q, K \rightarrow \ell_q(2^{-\alpha m - \beta n} F_{m,n})$ defined by $j(b) = (\ldots, b, b, b, \ldots)$ is an isometric embedding. By (1.1), it is then enough to show the inequality for $jT$.

Let $\sigma_j > \gamma_Z(T_{A_j, B_j})$ and find Banach spaces $E_j$ and operators $R_j \in \mathcal{I}(E_j, B_j)$ so that

\[
T \left( U_{A_j} \right) \subseteq \sigma_j U_{B_j} + R_j \left( U_{E_j} \right), \quad j = 1, \ldots, N.
\]

Put

\[
W_{m,n} = (E_1 \oplus \ldots \oplus E_N)_{\ell_\infty}, \quad (m, n) \in \mathbb{Z}^2
\]

and, for $\delta > 0$ and $(r, s) \in \mathbb{Z}^2$, consider the operator

$R : \ell_q(W_{m,n}) \rightarrow \ell_q(2^{-\alpha m - \beta n} F_{m,n})$ defined by

\[
R \left( z_1^{m,n}, \ldots, z_N^{m,n} \right) = \left( \sum_{j=1}^N (1 + \delta) 2^{(\alpha - \alpha_j)(m+r)} 2^{(\beta - \beta_j)(n+s)} R_j z_j^{m,n} \right).
\]

This operator is bounded because
Quantitative estimates for interpolated operators by...

\[ \| R(z_{1}^{m,n}, \ldots, z_{N}^{m,n}) \|_{E_{q}(2^{-\alpha m - \beta n} F_{m,n})} \leq \left( \sum_{(m,n) \in \mathbb{Z}^{2}} (2^{-\alpha m - \beta n} \sum_{j=1}^{N} (1 + \delta) 2^{m x_{j} + n y_{j}} 2^{(\alpha - x_{j})(m+r)} \right)^{\frac{1}{q}} 
\leq (1 + \delta)^{N} \max_{1 \leq j \leq N} \left\{ 2^{(\alpha - x_{j})q} 2^{(\beta - y_{j})q} \| R_{j} \|_{E_{q}, E_{q}} \| z_{j}^{m,n} \|_{E_{q}} \right\} \| (z_{1}^{m,n}, \ldots, z_{N}^{m,n}) \|_{E_{q}(W_{m,n})} \]

Moreover, since each entry
\[ Q_{t,w} R P_{u,v}(z_{1}, \ldots, z_{N}) = \]

\[
\begin{cases}
0 & \text{if } (t, w) \neq (u, v) \\
\sum_{j=1}^{N} (1 + \delta) 2^{(\alpha - x_{j})(t+r)} 2^{(\beta - y_{j})(w+s)} R_{j} z_{j} & \text{if } (t, w) = (u, v)
\end{cases}
\]

belongs to \( I(W_{u,v}, 2^{-\alpha t - \beta w} F_{t,w}) \), the \( \Sigma_{q} \)-property implies that

\[ R \in I \left( \ell_{q}(W_{m,n}), \ell_{q}(2^{-\alpha m - \beta n} F_{m,n}) \right). \]

We claim that
\[ j T \left( U_{A(\alpha, \beta), q; m} \right) \]

\[ \leq \left[ N(1 + \delta)^{\max_{1 \leq j \leq N} \left\{ 2^{q(\alpha - x_{j})} + q(\beta - y_{j}) \right\}} \right] U_{\ell_{q}(2^{-\alpha m - \beta n} F_{m,n})} + R \left( U_{\ell_{q}(W_{m,n})} \right). \]

Indeed, given any \( a \in U_{A(\alpha, \beta), q; m} \) we can choose \( d_{m,n} = d_{m,n}(a) > 0 \) with

\[ 2^{-\alpha m - \beta n} K(2^{m}, 2^{n}; a) < d_{m,n} \quad \text{and} \quad \sum_{(m,n) \in \mathbb{Z}^{2}} d_{m,n}^{q} \leq (1 + \delta)^{q}. \]

Since

\[ K(2^{m+r}, 2^{n+s}; a) < 2^{(m+r)} 2^{\beta(n+s)} d_{m+r,n+s} \]

we can find a decomposition \( a = \sum_{j=1}^{N} a_{j}^{m,n} \) with \( a_{j}^{m,n} \in A_{j} \) and

\[ 2^{(m+r) x_{j}} 2^{(n+s) y_{j}} \| a_{j}^{m,n} \|_{A_{j}} \leq 2^{(m+r)} 2^{\beta(n+s)} d_{m+r,n+s}. \]
Put
\[ \rho_j^{m,n} = 2^{(m+r)z_j} 2^{(n+s)\nu_j}, \quad 1 \leq j \leq N; \quad \rho_0^{m,n} = 2^{\alpha(m+r)2^{(n+s)}} d_{m+r,n+s}. \]

By (7), we can choose \( z_j^{m,n} \in U_{E_j} \) such that
\[ ||T \left( \frac{\rho_j^{m,n}}{\rho_0^{m,n}} a_j^{m,n} \right) - R_j z_j^{m,n} ||_{B_j} \leq \sigma_j. \]

In other words,
\[ ||T a_j^{m,n} - \frac{\rho_0^{m,n}}{\rho_j^{m,n}} R_j z_j^{m,n} ||_{B_j} \leq \frac{\rho_0^{m,n}}{\rho_j^{m,n}} \sigma_j = 2^{(m+r)(\alpha - x_j)} 2^{(n+s)(\beta - y_j)} \sigma_j d_{m+r,n+s}. \]

Let
\[ z = \left( (1 + \delta)^{-1} d_{m+r,n+s} z_1^{m,n}, \ldots, (1 + \delta)^{-1} d_{m+r,n+s} z_N^{m,n} \right). \]

Then \( z \in U_{\ell_0(W_{m,n})} \) and
\[ ||(jT)a - Rz||_{\ell_0(\mathbb{Z}^2, F_{m,n})} \]
\[ \leq \sum_{(m,n) \in \mathbb{Z}^2} \left[ 2^{-\alpha m - \beta n} \left( \sum_{j=1}^{N} 2^{mz_j + ny_j} ||T a_j^{m,n} - \frac{\rho_j^{m,n}}{\rho_0^{m,n}} R_j z_j^{m,n} ||_{B_j} \right) \right]^q \]
\[ \leq \sum_{(m,n) \in \mathbb{Z}^2} \left[ 2^{-\alpha m - \beta n} \left( \sum_{j=1}^{N} 2^{mz_j + ny_j} 2^{(m+r)(\alpha - x_j) + (n+s)(\beta - y_j)} \sigma_j d_{m+r,n+s} \right) \right]^q \]
\[ \leq \left[ N \max_{1 \leq j \leq N} \left\{ 2^{r(\alpha - x_j) + s(\beta - y_j) \sigma_j} \right\} \right]^q \sum_{(m,n) \in \mathbb{Z}^2} d_{m+r,n+s}^q \]
\[ \leq \left[ N(1 + \delta) \max_{1 \leq j \leq N} \left\{ 2^{r(\alpha - x_j) + s(\beta - y_j) \sigma_j} \right\} \right]^q. \]

Whence
\[ \gamma_z(jT) \leq N(1 + \delta) \max_{1 \leq j \leq N} \left\{ 2^{r(\alpha - x_j) + s(\beta - y_j) \sigma_j} \right\}. \]

Here \( \delta > 0 \) and \( (r, s) \in \mathbb{Z}^2 \) are arbitrary. Therefore we derive that
\[ \gamma_z(jT) \leq N \inf_{(r,s) \in \mathbb{Z}^2} \left[ \max_{1 \leq j \leq N} \left\{ 2^{r(\alpha - x_j) + s(\beta - y_j) \sigma_j} \right\} \right]. \]
\[ D \inf_{t, s > 0} \left[ \max_{1 \leq j \leq N} \left\{ t^{\alpha - x_j} s^{\beta - y_j} \sigma_j \right\} \right] = D \max \left\{ \sigma_i^{c_i} \sigma_k^{c_k} \sigma_r^{c_r} : \{i, k, r\} \in \mathcal{P} \right\} \]

where we have used [8], Thm. 1.9, in the last equality. This implies that
\[ \gamma_z(jT) \leq D \max \left\{ \gamma_z(T_{A_i, B_i})^{c_i} \gamma_z(T_{A_k, B_k})^{c_k} \gamma_z(T_{A_r, B_r})^{c_r} : \{i, k, r\} \in \mathcal{P} \right\} \]

and completes the proof.

The operator \( J_{\mathcal{B}(\alpha, \beta, \epsilon, \kappa)} \) is essential in Theorem 3.1 as we show next by means of an example. We adapt an idea of [9], Remark 3.4.

Let \( \mathcal{I} = \mathcal{W} \) the ideal of weakly compact operators. According to [2], Thm. 4, there is a Banach space \( E \) and a sequence of operators \( (R_n)_{n=1}^{\infty} \subseteq \mathcal{L}(E, c_0) \) such that
\[ \gamma_w(R_n^{**}) \leq \gamma_w(R_n) \leq 1/n, \quad (8) \]
\[ \gamma_w(R_n^*) = 1. \quad (9) \]

Put
\[ T_n = Q_{E^*}^* R_n^* \quad , \quad F = Q_{E^*}^*(E^*) \]
choose \( \Pi \) as the simplex \( \{(0,0), (1,0), (0,1)\} \) and consider the 3-tuples
\[ \tilde{A} = \{\ell_1, \ell_1, \ell_1\} \quad , \quad \tilde{B} = \{F, F, \ell_\infty(U_E)\}. \]

Let \( \alpha > 0, \beta > 0 \) with \( \alpha + \beta < 1 \) (i.e. \( (\alpha, \beta) \in \text{Int} \Pi \)) and \( 1 < q < \infty \).

It is clear that \( \tilde{A}_{(\alpha, \beta), q; \kappa} = \ell_1 \) with equivalence of norms. Moreover \( \tilde{B}_{(\alpha, \beta), q; \kappa} = F \) (equivalent norms) because \( F \) is a closed subspace of \( \ell_\infty(U_E) \). Hence, if Theorem 3.1 would be true without \( J_{\mathcal{B}(\alpha, \beta, \epsilon, \kappa)} \), there would exist a constant \( D > 0 \) such that for any \( n \in \mathbb{N} \)
\[ \gamma_w([T_n]\ell_1, F) \quad (10) \]
\[ \leq D \gamma_w([T_n]\ell_1, F)^{1-\alpha-\beta} \gamma_w([T_n]\ell_1, F)^{\alpha} \gamma_w([T_n]\ell_1, \ell_\infty(U_E))^{\beta}. \]

But \( Q_{E^*}^*: E^* \rightarrow F \) is an isometry onto, so (9) yields
\[ \gamma_w([T_n]\ell_1, F) = \gamma_w([R_n^*]\ell_1, E^*) = 1. \]

On the other hand, by (8) and [1], Cor. 5.3, we get
\[ \gamma_w([T_n]\ell_1, \ell_\infty(U_E)) = \gamma_w(T_n^*) = \gamma_w(R_n^{**}) \leq 1/n. \]
Whence (10) reads

$$1 \leq Dn^{-\beta} \quad \text{for any } n \in N$$

which is impossible.

Our last result describe the behaviour of $\beta_z$ with $J$-spaces.

**Theorem 3.2.** Let $\Pi = P_1 \ldots P_N$ be a convex polygon with vertices $P_j = (x_j, y_j)$, let $(\alpha, \beta) \in \text{Int } \Pi$, $1 < q < \infty$, and let $T$ be an operator ideal which satisfies the $\Sigma_q$-condition. Assume that $A = \{A_1, \ldots, A_N\}$ and $B = \{B_1, \ldots, B_N\}$ are Banach $N$-tuples and let $T \in \mathcal{L}(A, B)$. Then for the interpolated operator we have

$$\beta_z \left( \left[ T^q A_{(\alpha, \beta), q, j} \right]_{l_1(u_{A_{(\alpha, \beta), q, j}}), B_{(\alpha, \beta), q, j}} \right) \leq D \max \left\{ \beta_z(T_{A_1, B_1})^{c_i} \beta_z(T_{A_k, B_k})^{c_k} \beta_z(T_{A_r, B_r})^{c_r} : \{i, k, r\} \in \mathcal{P} \right\}$$

where $D$ is a constant depending only on $\Pi$ and $(\alpha, \beta)$.

**Proof.** Put $G_{m,n} = (A_1 \cap \ldots \cap A_N, J(2^m, 2^n; \cdot))$, $(m, n) \in \mathbb{Z}^2$, and let

$$\pi : \ell_q \left( 2^{-\alpha m - \beta n} G_{m,n} \right) \longrightarrow A_{(\alpha, \beta), q, j}$$

be the metric surjection $\pi(u_{m,n}) = \sum_{m,n} u_{m,n}$. Taking into account (2), it suffices to establish the inequality for $T\pi$.

Let $\sigma_j > \beta_z(T_{A_j, B_j})$. There exist Banach spaces $Z_j$ and operators $S_j \in \mathcal{I}(A_j, Z_j)$ such that

$$\|Tx\|_{B_j} \leq \sigma_j \|x\|_{A_j} + \|S_jx\|_{Z_j}, \ x \in A_j, \ 1 \leq j \leq N. \quad (11)$$

For each $(m, n) \in \mathbb{Z}^2$, let $V_{m,n} = (E_1 \oplus \ldots \oplus E_N)_{l_1}$. Take any $(r, s) \in \mathbb{Z}^2$ and let $S : \ell_q \left( 2^{-\alpha m - \beta n} G_{m,n} \right) \longrightarrow \ell_q(V_{m,n})$ be the operator defined by $S(u_{m,n}) = \left( 2^{(x_1 - \alpha)(m-r)} 2^{(y_1 - \beta)(n-s)} S_1 u_{m,n}, \ldots, 2^{(x_N - \alpha)(m-r)} 2^{(y_N - \beta)(n-s)} S_N u_{m,n} \right)$.

Since

$$\|S(u_{m,n})\|_{\ell_q(V_{m,n})} = \left( \sum_{(m, n) \in \mathbb{Z}^2} \left( \sum_{j=1}^N 2^{(x_j - \alpha)(m-r)} 2^{(y_j - \beta)(n-s)} \|S_j u_{m,n}\|_{Z_j} \right)^q \right)^{\frac{1}{q}}$$
the operator $S$ is bounded. Now, by the $\Sigma_q$-property, it is easy to check that $S \in \mathcal{I} \left( \ell_q \left( 2^{-nm} \Sigma G_{m,n} \right), \ell_q \left( V_{m,n} \right) \right)$. A direct computation using (11) shows that

$$\| T \pi (u_{m,n}) \|_{\mathcal{B}(\ell_q, \ell_q)} \leq \max_{1 \leq j \leq N} \left\{ \| x_j \|_{2} \| y_j \|_{2} \right\} \| u_{m,n} \|_{\ell_q(2^{-nm} \Sigma G_{m,n})} + \| S (u_{m,n}) \|_{\ell_q(V_{m,n})}.$$ 

This implies that

$$\beta_2(T \pi) \leq \max_{1 \leq j \leq N} \left\{ \| x_j \|_{2} \| y_j \|_{2} \right\}.$$ 

Since $(r, s) \in \mathbb{Z}^2$ is arbitrary, taking infimum and using [8], Thm. 1.9, the result follows.

Theorems 3.1 and 3.2 comprise Thm. 2.6 and Remark 2.9 of [5]. In particular, they give quantitative estimates for the weak compactness results mentioned in the Introduction.

Note that Theorems 3.1 and 3.2 do not apply to compact operators because this ideal fails the $\Sigma_q$-condition. This problem has been studied in [6] and [7].

References


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