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Nonlinear elliptic equations involving critical Sobolev exponent on compact riemannian manifolds in presence of symmetries.

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Abstract

In this paper, we study a nonlinear elliptic equation with critical exponent, invariant under the action of a subgroup G of the isometry group of a compact riemannian manifold. We obtain some existence results of positive solutions of this equation, and under some assumptions on G, we show that we can solve this equation for supercritical exponents.

1 Introduction

1.1. Let (M,g) be a compact, smooth riemannian n-manifold, $n \geq 3$. Let also $q \in (1; \frac{n+2}{n-2})$ real, and a, f and h be three smooth functions on M. In a previous paper, Djadli [15], we were concerned with the existence of smooth, positive solutions u to the equation

$$(E) \qquad \Delta_g u + au = fu^{\frac{n+2}{n-2}} + hu^q$$

The goal here is to study the same problem, but in presence of symmetries. More precisely, we set $Isom_g(M)$ the isometry group of M for the metric g, and G a subgroup of $Isom_g(M)$. We assume in the rest of the article that a, f and h are three smooth G-invariant functions. The goal here is to study the existence of smooth, positive, G-invariant solutions to (E).

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1.2. Let us now present the framework. We denote by $C_G^\infty(M)$ the set of smooth, G-invariant functions on M, that is

$$C_G^\infty(M) = \{u \in C^\infty(M), \forall \sigma \in G \quad u \circ \sigma = u\}$$

where $C^\infty(M)$ is the set of smooth functions defined on M. We will have to consider the Sobolev space $H_{1,G}^2(M)$, the completion of $C_G^\infty(M)$ with respect to the norm

$$\|u\|_{H_{1,G}^2(M)} = \left(\int_M |\nabla u|^2 dv(g)\right)^{\frac{1}{2}} + \left(\int_M |u|^2 dv(g)\right)^{\frac{1}{2}}$$

1.3. The point here is that the presence of symmetries allows one to improve some well known results concerning the best constant in the Sobolev embedding and the Rellich-Kondrakov theorem. More precisely, if one assume that G has at least one orbit of finite cardinality, Hebey and Vaugon proved (see [27]), that it is possible to improve the value of the best constant in the Sobolev embedding $H_{1,G}(M) \hookrightarrow L^{\frac{2n}{n-2}}(M)$ (its value has been obtained by Aubin [2]). The result is the following

Theorem A. *Let (M,g) be a compact riemannian n-manifold, $n \geq 3$, and let G be a subgroup of the isometry group of (M,g), $Isom_g(M)$, having at least one point of finite orbit. We set $k = \min_{x \in M} Card O_G(x)$. Then $\exists B \in \mathbb{R}_+^*$ such that for all $u \in H_{1,G}(M)$*

$$\left(\int_M |u|^{\frac{2n}{n-2}} dv(g)\right)^{\frac{n-2}{n}} \leq \frac{K(n,2)^2}{k^{\frac{2}{n}}} \int_M |\nabla u|^2 dv(g) + B \int_M u^2 dv(g)$$

where $K(n,2) = \sqrt{\frac{4}{n(n-2)\omega_n^{\frac{2}{n}}}}$ (ω_n being the volume of the standard n-sphere of $\mathbb{R}^{n+1}$) is the best constant in the Sobolev embedding $H_1(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n)$.

1.4. Besides, if we now assume that all the orbits under the action of G are infinite, Hebey and Vaugon (see [27]) have proved that it is possible to improve the "exponent" of the embedding. More precisely, we have the following theorem

Theorem B. *Let (M,g) be a smooth, compact, riemannian n-manifold, G a subgroup of the isometry group of (M,g), $Isom_g(M)$, and $r \geq 1$*

a real number. We assume that $\forall x \in M \quad Card O_G(x) = +\infty$. *Let* $k = \min_{x\in M} dim O_{G_0}(x)$ *where* G_0 *denotes the connected component of the identity in* $\bar{G}$ *(the closure of* G *in* $Isom_g(M)$*). Then, if*

- $n-k \leq r: \quad \forall s \geq 1 \quad H^r_{1,G}(M) \subset L^s(M)$ *with compact embedding*
- $n-k > r: \quad \forall 1 \leq s \leq \frac{(n-k)r}{n-k-r} \quad H^r_{1,G}(M) \subset L^s(M)$ *with compact embedding if* $1 \leq s < \frac{(n-k)r}{n-k-r}$

Note that $\frac{nr}{n-r} < \frac{(n-k)r}{n-k-r}$. Roughly speaking, we can say that we can increase the value of the critical Sobolev exponent when considering $H_{1,G}(M)$ with G such that all the orbits under the action of G are infinite.

2 Statements of the results

2.1. Following this distinction, this work will be divided in two sections.

2.1 The finite case

In this part, we assume that there exists at least one point of finite orbit under the action of G. First, we prove the following lemma (a kind of generic existence lemma)

Lemma 2.2. *Let* (M,g) *be a compact, smooth riemannian* n*-manifold,* $n \geq 3$. *We set* $Isom_g(M)$ *the isometry group of* M *with respect to the metric* g*, and let* G *be a subgroup of* $Isom_g(M)$ *having at least one orbit of finite cardinality. We set* $p = \frac{n+2}{n-2}$ *and let* $q \in (1,p)$*, and* f*,* a*,* h *be three* G*-invariant smooth functions on* M*. We assume that* f *is positive and that the operator* $\Delta + a$ *is coercive in* $H_{1,G}(M)$*. For* $v \in H_{1,G}(M)$*, we define*

$$\Psi(v) = \int \left\{ \frac{1}{2}|\nabla v|^2 + \frac{1}{2}av^2 - \frac{f}{p+1}|v|^{p+1} - \frac{h}{q+1}|v|^{q+1} \right\}$$

and we set $K(n,2)$ *the best constant in the Sobolev imbedding :* $H_1(M) \hookrightarrow L^{p+1}(M)$ *and* $k = \inf_{x\in M} Card O_G(x)$ *where* $O_G(x)$ *is the orbit of* x *under the action of* G*. If there exist* $v_0 \in H_{1,G}(M), \quad v_0 \geq 0$ sur $M, \quad v_0 \not\equiv$

0 such that

$$(\star) \qquad \sup_{t\geq 0} \Psi(tv_0) < \frac{k}{nK(n,2)^n(\sup_M f)^{\frac{n-2}{2}}}$$

then the problem

$$\begin{cases} \Delta_g u + au = fu^{\frac{n+2}{n-2}} + hu^q \\ u \in C^\infty(M) \quad , \quad u > 0 \quad \text{on } M \end{cases}$$

admits a G-invariant solution.

According to this lemma, the problem reduces to the the existence of some test function v_0 satisfying the condition $(\star)$. Here, we will use, as in Djadli [15], two kinds of test functions : local ones (the symmetrisation of the test functions introduced by Aubin [4]) and the test function identically equal to 1 (which is of course G-invariant). Using local test functions, we prove

Proposition 2.3. *Let (M, g) be a compact, smooth, riemannian n-manifold, $n \geq 4$, and G a subgroup of the isometry group of (M, g) having at least one orbit of finite cardinality. We consider a, f and h three smooth G-invariant functions with $f > 0$ on M, and $q \in (1; \frac{n+2}{n-2})$. Let $\max f$, be the set where f attains its maximum and we assume that $\exists P_0 \in \max f$ such that $CardO_G(P_0) = \min_{x\in M} CardO_G(x)$. We also assume that $\Delta_g + a$ is coercive on $H_{1,G}(M)$ and that*

$$\max_{\{P\in\max f \text{ such that } CardO_G(P)=\min_{x\in M} CardO_G(x)\}} h(P) > 0$$

Then the following problem

$$\begin{cases} \Delta_g u + au = fu^{\frac{n+2}{n-2}} + hu^q \\ u \in C^\infty(M) \quad , \quad u > 0 \quad \text{on } M \end{cases}$$

possesses a G-invariant solution.

2.4. One can also deal with the case where

$$\max_{\{P\in\max f \text{ such that } CardO_G(P)=\min_{x\in M} CardO_G(x)\}} h(P) = 0$$

Pushing further the expansions for the test functions, we can prove the following proposition

Proposition 2.5. *Let (M,g) be a compact, smooth, riemannian n-manifold, $n \geq 4$, and G a subgroup of the isometry group of (M,g) having at least one orbit of finite cardinality. We consider a, f and h three smooth G-invariant functions with $f > 0$ on M, and $q \in (1; \frac{n+2}{n-2})$. Let* $\max f$, *be the set where f attains its maximum and we assume that $\Delta_g + a$ is coercive on $H_{1,G}(M)$. We also assume that $\exists P_0 \in \max f$ such that*

$$(i) \quad CardO_G(P_0) = \min_{x \in M} CardO_G(x)$$

$$(ii) \quad h(P_0) = 0$$

$$(iii) \quad \begin{cases} \frac{2Scal_g(P_0)}{n-4} - \frac{8(n-1)a(P_0)}{(n-2)(n-4)} > \frac{\Delta_g f(P_0)}{f(P_0)} & \text{if } n \geq 5 \\ a(P_0) - \frac{Scal_g(P_0)}{6} < 0 & \text{if } n = 4 \end{cases}$$

Then the following problem

$$\begin{cases} \Delta_g u + au = f u^{\frac{n+2}{n-2}} + h u^q \\ u \in C^\infty(M) \quad , \quad u > 0 \quad \text{on } M \end{cases}$$

possesses a G-invariant solution.

Using the test function equal to 1, we prove the following

Proposition 2.6 *Let (M,g) be a compact, smooth, riemannian n-manifold, $n \geq 3$, and G a subgroup of the isometry group of (M,g) having at least one orbit of finite cardinality. We consider a, f and h three smooth G-invariant functions with $f > 0$ on M, and $q \in (1; \frac{n+2}{n-2})$. We assume that $\Delta_g + a$ is coercive on $H_{1,G}(M)$ and that*

$$\left(\frac{\int f}{\sup_{x \in M} f} \right)^{\frac{n-2}{2}} > \frac{(K(n,2))^n}{k} \left(\int a \right)^{\frac{n}{2}}$$

Then there exists $\varepsilon \in \mathbb{R}_+^*$ such that for all $h \in C_G^\infty(M)$ satisfying

$$\left| \int h \right| < \varepsilon$$

the following problem

$$\begin{cases} \Delta_g u + au = f u^{\frac{n+2}{n-2}} + hu^q \\ u \in C^\infty(M) \quad , \quad u > 0 \quad on\ M \end{cases}$$

possesses a G-invariant solution.

2.2 The infinite case

In this part, we assume that all the orbits under the action of G are infinite. Using theorem B of Hebey and Vaugon, we distinguish two cases, leading to the following proposition and the following lemma.

Proposition 2.7. *Let (M, g) be a smooth, compact, riemannian n-manifold, $n \geq 3$. Let $Isom_g(M)$ be the isometry group of (M, g), and let G be a subgroup of $Isom_g(M)$ such that*

$$\forall x \in M \quad CardO_G(x) = +\infty$$

We set $k = \min_{x\in M} dimO_{G_0}(x)$ where G_0 denotes the connected component of the identity in $\bar{G}$ (the closure of G in $Isom_g(M)$), and we set also

$$p^* = \frac{n-k+2}{n-k-2} \qquad if\ k < n-2$$
$$p^* = +\infty \qquad if\ k \geq n-2$$

Let $p \in (1; p^*)$, q be a real number, $1 < q < p$, and f, a, h be three smooth G-invariant functions. We assume that f is positive on M and that $\Delta_g + a$ is coercive on $H_{1,G}(M)$. Then the problem

$$\begin{cases} \Delta_g u + au = f u^p + hu^q \\ u \in C^\infty(M) \quad u > 0 \quad on\ M \end{cases}$$

possesses a G-invariant solution.

2.8. This theorem gives immediately the existence of a solution because here p is supposed to be subcritical for the embedding of $H_{1,G}(M) \hookrightarrow L^{p^*}(M)$. If we assume now that there is a critical exponent (i.e. in the case $k < n-2$) and that p is equal to this critical exponent, we have the following lemma (similar to lemma 2.2).

Lemma 2.9. *Let (M,g) be a smooth, compact, riemannian n-manifold, $n \geq 3$. Let $Isom_g(M)$ be the isometry group of (M,g), and let G be a subgroup of $Isom_g(M)$ such that*

$$\forall x \in M \quad Card O_G(x) = +\infty$$

We set $k = \min_{x \in M} dim O_{G_0}(x)$ where G_0 denotes the connected component of the identity in $\bar{G}$ (the closure of G in $Isom_g(M)$), and we assume that $k < n-2$. Let $q \in (1; \frac{n-k+2}{n-k-2})$ and f, a, and h be three smooth G-invariant functions. We assume that f is positive on M and that $\Delta_g + a$ is coercive on $H_{1,G}(M)$. For all $v \in H_{1,G}(M)$, let

$$\Psi(v) = \int \left\{ \frac{1}{2}|\nabla v|^2 + \frac{1}{2} a v^2 - \frac{f}{p+1}|v|^{p+1} - \frac{h}{q+1}|v|^{q+1} \right\}$$

where $p = \frac{n-k+2}{n-k-2}$. We denote by $\bar{K}$ the best constant in the Sobolev embedding

$$H_{1,G}(M) \hookrightarrow L^{\frac{2(n-k)}{n-k-2}}(M)$$

(see theorem B). Then, if there exists $v_0 \in H_{1,G}(M)$, $v_0 \geq 0$ on M, $v_0 \not\equiv 0$ such that

$$(\star\star) \qquad \sup_{t \geq 0} \Psi(t v_0) < \frac{1}{(n-k)\bar{K}^{n-k} (\sup_M f)^{\frac{n-k-2}{2}}}$$

the problem $\mathcal{P}_{SC}$:

$$\begin{cases} \Delta_g u + a u = f u^{\frac{n-k+2}{n-k-2}} + h u^q \\ u \in C^\infty(M), \quad u > 0 \quad \text{on } M \end{cases}$$

possesses a G-invariant solution.

2.10. Once again, using this theorem, the problem reduces to find a test function v_0 satisfying the condition $(\star\star)$. Here, using the function identically equal to 1, we prove

Proposition 2.11. *Let (M,g) be a smooth, compact, riemannian n-manifold, $n \geq 3$. Let $Isom_g(M)$ be the isometry group of (M,g), and let G be a subgroup of $Isom_g(M)$ such that*

$$\forall x \in M \quad Card O_G(x) = +\infty$$

We set $k = \min_{x\in M} dim O_{G_0}(x)$ *where* G_0 *denotes the connected component of the identity in* $\bar{G}$ *(the closure of* G *in* $Isom_g(M)$*), and we assume that* $k < n-2$. *Let* $q \in (1; \frac{n-k+2}{n-k-2})$ *and* f, a, *and* h *be three smooth* G*-invariant functions. We assume that* f *is positive on* M, *that* $\Delta_g + a$ *is coercive on* $H_{1,G}(M)$ *and that*

$$\left(\frac{\int f}{\sup_{x\in M} f}\right)^{\frac{n-k-2}{2}} > (\bar{K}(n,2))^{n-k}\left(\int a\right)^{\frac{n-k}{2}}$$

Then there exists $\varepsilon \in \mathbb{R}_+^*$ *such that for all* $h \in C^\infty(M)$, G*-invariant, satisfying*

$$\left|\int h\right| < \varepsilon$$

the problem

$$\begin{cases} \Delta_g u + au = f u^{\frac{n+k+2}{n-k-2}} + hu^q \\ u \in C^\infty(M) \quad u > 0 \quad \text{on } M \end{cases}$$

possesses a G*-invariant solution.*

3 The finite case - Proofs of lemma 2.2 and propositions 2.3-2.6

3.1. Proof of lemma 2.2: The proof of the generic existence lemma 2.2 relies on the following variant of the mountain-pass lemma of Ambrosetti and Rabinowitz [1], as used in the reference article of Brézis-Nirenberg [10].

Mountain pass lemma. *Let* Φ *be a* C^1 *function on a Banach space* E. *Suppose that there exists a neighborhood* U *of 0 in* E, $v \in E\backslash U$, *and a constant* ρ *such that*

$$\Phi(0) < \rho, \qquad \Phi(v) < \rho, \qquad \Phi(u) \geq \rho \quad \text{for all } u \in \partial U$$

Set

$$c = \inf_{P\in\mathcal{P}} \max_{w\in P} \Phi(w)$$

where $\mathcal{P}$ *denotes the class of continuous paths joining 0 to* v. *Then there exists a sequence* (u_j) *in* E *such that* $\Phi(u_j) \to c$ *and* $\Phi'(u_j) \to 0$ *in* E^*.

With such a lemma, the proof of lemma 2.2 proceeds as follows. As one will see, only minor modifications with respect to what has been done in Brézis-Nirenberg [10] are needed. First we set

$$\begin{aligned} g: \quad & M \times \mathbb{R} \to \mathbb{R} \\ & (x,t) \to g(x,t) = -a(x)t + h(x)|t|^q \end{aligned}$$

and for $s \in \mathbb{R}^+$, we set

$$G(x,s) = \int_0^s g(x,t)dt$$

with the convention that $G(x,s) = 0$ if $s \leq 0$. Let μ be large enough so that for all $x \in M$ and all $t \in \mathbb{R}_+^*$

$$g(x,t) + f(x)t^p + \mu t \geq 0$$

This implies that $\mu > a(x)$ for all x in M. For $\varphi \in H_{1,G}(M)$, we define

$$J(\varphi) = \int \left\{ \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\mu\varphi^2 - \frac{1}{p+1}f(\varphi^+)^{p+1} - G(.,\varphi^+) - \frac{1}{2}\mu(\varphi^+)^2 \right\}$$

Clearly J is C^1 on $H_{1,G}(M)$ and its differential is given by

$$J'_\psi.\varphi = \int \left\{ \nabla^i\psi\nabla_i\varphi + \mu\psi\varphi - f(\psi^+)^p\varphi + a\psi^+\varphi - h(\psi^+)^q\varphi - \mu(\psi^+)\varphi \right\}$$

Since $p > q$ and M is compact, one gets that for all $\varepsilon > 0$ there exists C_ε such that for all $\varphi \geq 0$

$$h\varphi^q \leq \varepsilon\varphi + C_\varepsilon\varphi^p$$

Then

$$G(.,\varphi^+) \leq -\frac{1}{2}a(\varphi^+)^2 + \frac{1}{2}\varepsilon(\varphi^+)^2 + \frac{C_\varepsilon}{p+1}(\varphi^+)^{p+1}$$

and it follows that for all $\varphi \in H_{1,G}(M)$

$$\begin{aligned} J(\varphi) \quad & \geq \int \left\{ \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}a(\varphi^+)^2 + \frac{1}{2}\mu(\varphi^-)^2 - \frac{1}{2}\varepsilon(\varphi^+)^2 \right. \\ & \quad \left. - \frac{C_\varepsilon}{p+1}(\varphi^+)^{p+1} - \frac{1}{p+1}f(\varphi^+)^{p+1} \right\} \\ & \geq \int \left\{ \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}a(\varphi)^2 \right\} \\ & \quad - \int \left\{ \frac{1}{2}\varepsilon(\varphi^+)^2 + \frac{C_\varepsilon}{p+1}(\varphi^+)^{p+1} + \frac{1}{p+1}f(\varphi^+)^{p+1} \right\} \end{aligned}$$

since $\mu \geq a$. Furthermore, by using the coercivity of $\Delta_g + a$, we have for ε small

$$J(\varphi) \geq k\|\varphi\|^2_{H_{1,G}} - C\int(\varphi^+)^{p+1}$$

where $k > 0$ and $C > 0$ are positive constants. Then, using the Sobolev embedding theorem

$$\begin{aligned} J(\varphi) &\geq k\|\varphi\|^2_{H_{1,G}} - C'\|\varphi\|^{p+1}_{p+1} \\ &\geq k\|\varphi\|^2_{H_{1,G}} - C''\|\varphi\|^{p+1}_{H_{1,G}} \end{aligned}$$

where $C'' > 0$. Letting $U = B_0(r)$ in $H_{1,G}(M)$, one then has that for r small enough, there exists $\rho > 0$ such that for all $u \in \partial U$, $J(u) > \rho$. In addition $J(0) = 0 < \rho$, while for $t \geq 0$ and $\varphi \in H_{1,G}(M)$, $\varphi \geq 0$, $\varphi \not\equiv 0$,

$$\lim_{t\to+\infty} J(t\varphi) = -\infty$$

(since f is positive on M and $p > q$). This proves that the assumptions of the mountain pass lemma are satisfied with $v = t\varphi$ for t large. As a consequence there exists $(u_j) \in (H_{1,G}(M))^{\mathbb{N}}$ such that

$$\begin{aligned} &J(u_j) \to c \\ &J'_{u_j} \to 0 \quad \text{strongly in } (H_{1,G}(M))' \end{aligned}$$

where

$$c = \inf_{P\in\mathcal{P}} \max_{w\in P} \Phi(w) \geq \rho\ ,$$

and $\mathcal{P}$ denotes the class of continuous path joining 0 to v. Furthermore, taking $\varphi = v_0$ (given by lemma 2.2), and according to the assumptions of this lemma, one can assume that

$$c < \frac{k}{nK(n,2)^n\,(\max_M f)^{\frac{n-2}{2}}}$$

Now we claim that (u_j) is a bounded sequence in $H_{1,G}(M)$. According to the mountain pass lemma we get that

$$\int\left\{\frac{1}{2}|\nabla u_j|^2 + \frac{1}{2}\mu u_j^2 - \frac{1}{p+1}f(u_j^+)^{p+1} - G(.,u_j^+) - \frac{1}{2}\mu(u_j^+)^2\right\} = c + o(1) \tag{3.1}$$

and that $\|J'_{u_j}\|_{H'_1} \to 0$. Since $\left|J'_{u_j}.u_j\right| \leq \|J'_{u_j}\|_{H'_1}.\|u_j\|_{H_{1,G}}$, and applying J'_{u_j} to u_j, we obtain

$$\int \left\{|\nabla u_j|^2+\mu u_j^2-f(u_j^+)^{p+1}-g(.,u_j^+)u_j-\mu(u_j^+)^2\right\} = \|u_j\|_{H_{1,G}}o(1) \quad (3.2)$$

Taking (3.1) $-\frac{1}{2}$(3.2), one then gets that

$$\int \left\{\frac{1}{2}f(u_j^+)^{p+1} - \frac{1}{p+1}f(u_j^+)^{p+1} - G(.,u_j^+) + \frac{1}{2}g(.,u_j^+)u_j\right\}$$
$$= c + o(1) + \|u_j\|_{H_{1,G}}o(1)$$

so that

$$\begin{aligned} \frac{1}{n}\int f(u_j^+)^{p+1} &= \int\left\{G(.,u_j^+) - \tfrac{1}{2}g(.,u_j^+)u_j\right\} + c + o(1) + \|u_j\|_{H_{1,G}}o(1) \\ &= \tfrac{1-q}{2(1+q)}\int h(u_j^+)^{q+1} + c + o(1) + \|u_j\|_{H_{1,G}}o(1) \end{aligned}$$

Since $f > 0$ on M, there exists $C > 0$ such that

$$\frac{C}{n}\int (u_j^+)^{p+1} \leq \frac{q-1}{2(q+1)}\int |h|(u_j^+)^{q+1} + c + o(1) + \|u_j\|_{H_{1,G}}o(1)$$

But $p+1 > q+1$. Hence, there exists $C'_\varepsilon > 0$ such that for all nonnegative t, $t^{q+1} \leq \varepsilon t^{p+1} + C'_\varepsilon$. Then

$$\frac{C}{n}\int (u_j^+)^{p+1} - \frac{(q-1)\varepsilon}{2(q+1)}\sup_{x\in M}|h(x)|\int (u_j^+)^{p+1} \leq Constant + \|u_j\|_{H_{1,G}}o(1)$$

and

$$\left(\frac{C}{n} - \frac{(q-1)\varepsilon}{2(q+1)}\sup_{x\in M}|h|\right)\int (u_j^+)^{p+1} \leq Constant + \|u_j\|_{H_{1,G}}o(1)$$

But for ε small enough

$$\frac{C}{n} - \left(\frac{(q-1)\varepsilon}{2(q+1)}\right)\sup_{x\in M}|h| > 0$$

Hence,

$$\int (u_j^+)^{p+1} \leq Constant + C_j\|u_j\|_{H_{1,G}}$$

with $\lim_{j\to\infty} C_j = 0$. Now according to (3.1)

$$\int \left\{\frac{1}{2}|\nabla u_j|^2+\frac{1}{2}\mu u_j^2\right\} = \int \left\{\frac{1}{p+1}f(u_j^+)^{p+1}+G(.,u_j^+)+\frac{1}{2}\mu(u_j^+)^2\right\}+c+o(1)$$

and clearly

$$\frac{1}{2}\inf(1,\mu)\|u_j\|^2_{H_{1,G}} \le Constant \int (u_j^+)^{p+1} \le Constant + \|u_j\|^2_{H_{1,G}} o(1)$$

Finally

$$\|u_j\|_{H_{1,G}} \le Constant$$

where the constant involved in this inequality is independent of j. The sequence $(u_j)_{j\in\mathbb{N}}$ is then bounded in $H_{1,G}(M)$, and this proves the claim. By classical arguments, we can now extract a subsequence, still denoted by (u_j), so that (for a certain u in $H_{1,G}(M)$)

$$\begin{cases} u_j \rightharpoonup u & \text{weakly in } H_{1,G}(M) \\ u_j \to u & \text{strongly in } L^r(M) \text{ for all given } r < p+1 \\ u_j \to u & a.e. \text{ on } M \end{cases}$$

Note here that

$$(u_j^+)^p \to (u^+)^p \quad a.e. \text{ on } M$$

while

$$\int ((u_j^+)^p)^{\frac{p+1}{p}} = \int (u_j^+)^{p+1} \le C$$

By a classical result of integration, one then gets that $(u_j^+)^p \to (u^+)^p$ weakly in $L^{p+1}(M)$. In addition, $g(x,u_j^+) \to g(x,u^+)$ weakly in $L^{p+1}(M)$ since (u_j) converges strongly in $L^q(M)$. Taking the limit for $j \to +\infty$ in the following equality

$$\int_M \left\{\nabla_i u_j \nabla^i \varphi + \mu u_j \varphi - f(u_j^+)^p\varphi - g(.,u_j^+)\varphi - \frac{1}{2}\mu u_j^+\varphi\right\} = J'_{u_j}.\varphi$$

we get that for all $\varphi \in H_{1,G}(M)$,

$$\int_M \left\{\nabla_i u \nabla^i \varphi + \mu u \varphi - f(u^+)^p\varphi - g(.,u^+)\varphi - \mu u^+\varphi\right\} = 0$$

According to the Hopf maximum principle, $u \geq 0$ on M.

3.2. Now, to use the classical results of regularity, we must prove that u satisfies this equation weakly in $H_1(M)$. In this aim, we consider $v \in H_1(M)$ and we set $\bar{G}$ the closure of G in $Isom_g$. Then $u \circ \sigma = u$ a.e. on M. We denote by $d\sigma$ the Haar measure on $\bar{G}$ and we set

$$\tilde{v}(x) = \int_{\bar{G}} v(\sigma(x))d\sigma$$

for all $x \in M$. One can easily see that $\tilde{v}$ is G-invariant. It follows that

$$\int_M \left\{\nabla_i u \nabla^i \tilde{v} + au\tilde{v} - fu^p\tilde{v} - hu^q\tilde{v}\right\} dv(g) = 0$$

since $\tilde{v}$ is G-invariant. Hence

$$\begin{aligned} 0 &= \int_M \left\{\nabla_i u \nabla^i \left(\frac{1}{\int_{\bar{G}} d\sigma} \int_{\bar{G}} v(\sigma(x))d\sigma\right) + au \left(\frac{1}{\int_{\bar{G}} d\sigma} \int_{\bar{G}} v(\sigma(x))d\sigma\right)\right. \\ &- \left. fu^p \left(\frac{1}{\int_{\bar{G}} d\sigma} \int_{\bar{G}} v(\sigma(x))d\sigma\right) - hu^q \left(\frac{1}{\int_{\bar{G}} d\sigma} \int_{\bar{G}} v(\sigma(x))d\sigma\right)\right\} dv(g) \\ &= \frac{1}{\int_{\bar{G}} d\sigma} \int_M \left\{\nabla_i u \nabla^i \left(\int_{\bar{G}} v(\sigma(x))d\sigma\right) + au \left(\int_{\bar{G}} v(\sigma(x))d\sigma\right)\right. \\ &- \left. fu^p \left(\int_{\bar{G}} v(\sigma(x))d\sigma\right) - hu^q \left(\int_{\bar{G}} v(\sigma(x))d\sigma\right)\right\} dv(g) \end{aligned}$$

But

$$\nabla^i \left(\int_{\bar{G}} v(\sigma(x))d\sigma\right) = \int_{\bar{G}} \nabla^i v(\sigma(x))d\sigma$$

Henceforth

$$\int_M \int_{\bar{G}} \{\nabla_i u \nabla^i (v(\sigma(x))) + au (v(\sigma(x)) - fu^p (v(\sigma(x)) \\ -hu^q (v(\sigma(x))\} d\sigma dv(g) = 0$$

Thanks to the Fubini's theorem, we get

$$\int_{\bar{G}} \int_M \left\{\nabla_i u \nabla^i (v(\sigma(x))) + au (v(\sigma(x)) - fu^p (v(\sigma(x))\right. \\ \left. -hu^q (v(\sigma(x))\right\} dv(g)d\sigma = 0$$

But the integral on M doesn't depend on $\sigma \in \bar{G}$ since a, f, h and u are G-invariant; then we have

$$\int_M \left\{\nabla_i u \nabla^i v + auv - fu^p v - hu^q v\right\} dv(g) = 0$$

for all $v \in H_1(M)$. Hence, u is a weak solution in $H_1(M)$ of the equation

$$\Delta u + au = fu^p + hu^q$$

Now, by classical regularity theorems, u is C^∞ on M and, as we said, u is a non-negative solution of the equation

$$\Delta_g u = fu^p - g(.,u)$$

Once again, by the maximum principle, either $u \equiv 0$, either $u > 0$ on M. Moreover, by construction, u is G-invariant.
Let us now prove that $u \not\equiv 0$.
For this aim, we use the following assumption of lemma 2.2 : $\exists v_0 \in H_{1,G}(M)$, $v_0 \not\equiv 0$ such that

$$\sup_{t \geq 0} \Psi(tv_0) < \frac{k}{nK(n,2)^n (\sup f)^{\frac{n-2}{2}}}$$

(we recall that $k = \inf_{x \in M} Card O_G(x)$ where $O_G(x)$ is the orbit of x under the action of G). First note that

$$\sup_{t \geq 0} J(tv_0) < \frac{k}{nK(n,2)^n (\sup f)^{\frac{n-2}{2}}}$$

since $v_0 \geq 0$. Then

$$c < \frac{k}{nK(n,2)^n (\sup f)^{\frac{n-2}{2}}}$$

where

$$c = \lim_{j \to +\infty} J(u_j)$$

Independently, assume that $u \equiv 0$. It follows that

$$\int g(.,u_j^+)u_j^+ = \int \{-a(u_j^+)^2 + h(u_j^+)^{q+1}\}$$

and then, since $u \equiv 0$ and $u_j \to u$ strongly in $L^q(M)$:

$$\int g(.,u_j^+)u_j^+ \to 0$$

Similarly

$$\int G(.,u_j^+) \to 0$$

Up to a subsequence, we can assume that $\int |\nabla u_j|^2 \to l$ since (u_j) is bounded in $H_{1,G}(M)$ and $u_j \to 0$ in $L^2(M)$. Taking the limit in (3.2), we get

$$\int f(u_j^+)^{p+1} \to l$$

and according to (3.1) :

$$\frac{1}{2}l - \frac{1}{p+1}l = c$$

that is

$$\frac{1}{n}l = c$$

Besides, thanks to theorem A, we have

$$\begin{aligned}\|f^{\frac{1}{p+1}}u_j^+\|_{p+1}^2 &\leq (\sup f)^{\frac{2}{p+1}}\|u_j\|_{p+1}^2 \\ &\leq \frac{K(n,2)^2}{k^{\frac{2}{n}}}\|u_j\|_{H_{1,G}}^2(\sup f)^{\frac{2}{p+1}} + Constant\|u_j\|_2^2\end{aligned}$$

and taking the limit, since $\|u_j\|_2 \to 0$, we get

$$\frac{K(n,2)^2}{k^{\frac{2}{n}}}l \geq \frac{1}{(\sup f)^{\frac{2}{p+1}}}l^{\frac{2}{p+1}}$$

that is

$$\frac{K(n,2)^2}{k^{\frac{2}{n}}}l^{\frac{2}{n}} \geq \frac{1}{(\sup f)^{\frac{2}{p+1}}}$$

Hence

$$c \geq \frac{k}{nK(n,2)^n(\sup f)^{\frac{n-2}{2}}}$$

which is a contradiction. This ends the proof. ∎

According to lemma 2.2, the problem reduces to the existence of some test function v_0 in $H_{1,G}(M)$, satisfying the condition $(\star)$. Let us now construct such a test function. Let $P \in M$ where f achieves its maximum. We assume that

$$CardO_G(P) = \inf_{x \in M} CardO_G(x)$$

in other words, we assume that P is a point of minimal orbit. We set

$$O_G(P) = \{P_1, \ldots, P_k\}$$

For each P_i we consider ψ_m^i defined by

$$\begin{array}{l} \psi_m^i(Q) = \left(\frac{1}{m} + \frac{1-\cos\alpha r}{\alpha^2}\right)^{1-\frac{n}{2}} - \left(\frac{1}{m} + \frac{1-\cos\alpha\delta}{\alpha^2}\right)^{1-\frac{n}{2}} \quad \forall Q \in \bar{B}_{P_i}(\delta) \\ \psi_m^i(Q) = 0 \quad \forall Q \in M \setminus \bar{B}_{P_i}(\delta) \end{array}$$

where $r = d(P_i, Q)$, $Scal_g(P_i) = n(n-1)\alpha^2$, ψ_m^i with compact support in $\overline{B_{P_i}}(\delta)$, and where δ is fixed such that $|\alpha|\delta \leq \pi$, less than the injectivity radius of M and such that $B_{P_i}(2\delta) \cap B_{P_j}(2\delta) = \emptyset$ for all $i \neq j$.
We set

$$\psi_m = \sum_{i=1}^{k} \psi_m^i$$

Clearly ψ_m is G-invariant. One easily checks that for all $t \in \mathbb{R}^+$:

$$\Psi(t\frac{\psi_m}{\|\psi_m\|_{p+1}}) = \sum_{i=1}^{k} \Psi(t\frac{\psi_m^i}{\|\psi_m\|_{p+1}})$$

But

$$\|\psi_m\|_{p+1} = k \times \|\psi_m^i\|_{p+1}$$

for all $i \in \{1, \ldots, k\}$. Then

$$\Psi(t\frac{\psi_m}{\|\psi_m\|_{p+1}}) = \sum_{i=1}^{k} \Psi(t\frac{\psi_m^i}{k\|\psi_m^i\|_{p+1}}) = k\Psi(t\frac{\psi_m^1}{k\|\psi_m^1\|_{p+1}})$$

that is

$$\Psi(t\frac{\psi_m}{\|\psi_m\|_{p+1}}) = \frac{t^2}{2}\frac{1}{k}\int |\nabla\left(\frac{\psi_m^1}{\|\psi_m^1\|_{p+1}}\right)|^2 + \frac{t^2}{2}\frac{1}{k}\int_M a\left(\frac{\psi_m^1}{\|\psi_m^1\|_{p+1}}\right)^2$$

$$-\frac{t^{p+1}}{p+1}\frac{1}{k^p}\int_M f\left(\frac{\psi_m^1}{\|\psi_m^1\|_{p+1}}\right)^{p+1} - \frac{t^{q+1}}{q+1}\frac{1}{k^q}\int_M h\left(\frac{\psi_m^1}{\|\psi_m^1\|_{p+1}}\right)^{q+1}$$

Thanks to what we said previously and the computations of Djadli [15], we can give the expansion of $\Psi(t\frac{\psi_m}{\|\psi_m\|_{p+1}})$ for $n \geq 5$

$$\Psi(t\frac{\psi_m}{\|\psi_m\|_{p+1}}) = \frac{1}{K(n,2)^2}\frac{1}{k}\frac{t^2}{2} - \frac{1}{p+1}\frac{1}{k^p}f(P)t^{p+1} - \frac{1}{m}(\frac{Scal_g(P)t^2}{n(n-4)K(n,2)^2}\frac{1}{k}$$

$$-\frac{4(n-1)a(P)t^2}{n(n-2)(n-4)K(n,2)^2}\frac{1}{k} - \frac{\Delta f(P)t^{p+1}}{2n}\frac{1}{k^p})$$

$$-C_1' h(P)m^l t^{q+1}\frac{1}{k^q} + o(\frac{1}{m})g_1(t)$$

where

$$g_1 \in C^\infty(\mathbb{R}^+) \quad g_1(0) = 0$$

$C_1' > 0$ is a constant independent of m

$$l = \frac{(n-2)(q-1)}{4} - 1 \quad l \in (-1;0)$$

In the case $n = 4$, we have

$$\Psi(t\frac{\psi_m}{\|\psi_m\|_{p+1}}) = \frac{1}{K(n,2)^2}\frac{1}{k}\frac{t^2}{2} - \frac{1}{p+1}f(P)\frac{1}{k^p}t^{p+1}+$$

$$+\frac{\log m}{m}\frac{1}{k}(\frac{a(P)}{2} - \frac{(n-2)^2 Scal_g(P)}{16(n-1)})C_1 t^2$$

$$-C_1' h(P)m^l t^{q+1}\frac{1}{k^q} + O(\frac{1}{k})g_2(t)$$

where

$$g_2 \in C^\infty(\mathbb{R}^+) \quad g_2(0) = 0$$

$C_1 > 0, C_1' > 0$ are two constant independent of m

$$l = \frac{(n-2)(q-1)}{4} - 1 \quad l \in (-1;0)$$

3.3. Before proving propositions 2.3-2.6, we prove the following technical lemma, useful in the proof of propositions 2.3 and 2.5.

Lemma 3.4. *Let* $1 < q < p = \frac{n+2}{n-2}$ *and* $A > 0$, $B > 0$ *be given real numbers. For* $m \in \mathbb{N}^*$, *let also* $A(m)$, $B(m)$ *and* $C(m)$ *be real numbers such that* $A(m) \to A$, $B(m) \to B$ *and* $C(m) \to 0$ *as* $m \to +\infty$. *We define*

$$F(t,m) = A(m)t^2 - B(m)t^{p+1} - C(m)t^{q+1}$$

Then, for m large, one has that there exists t_m *such that*

$$F(t_m, m) = \max_{t \geq 0} F(t,m)$$

with the additionnal property that if $t_0 = \left(\frac{2A}{(p+1)B}\right)^{\frac{1}{p-1}}$, *then* $t_m \to t_0$ *as* $m \to +\infty$. *Furthermore, if* $A(m) = A + O(m^s)$, $B(m) = B + O(m^s)$ *and* $C = O(m^s)$, *for some* $s < 0$, *then* $t_m = t_0 + O(m^s)$.

Proof of lemma 3.4: For m large enough such that $B(m) > 0$, one has that

$$\lim_{t \to +\infty} F(t,m) = -\infty$$

As a consequence, there exists $t_m > 0$ such that

$$F(t_m, m) = \max_{t \geq 0} F(t,m)$$

Furthermore, one clearly has that there exists $T > 0$, independent of m, such that for m large enough, $t_m < T$. In the same order of idea, one clearly checks that there exists $\varepsilon > 0$, independent of m, such that for m large enough, $t_m \geq \varepsilon$. Suppose now that a subsequence (t_{m_i}) of (t_m) converges to some $\tilde{t}$. Then we have

$$\lim_{i \to +\infty} F(t_{m_i}, m_i) = 0$$

so that

$$2A\tilde{t} = (p+1)B\tilde{t}^p$$

Hence, $\tilde{t} = t_0$, where t_0 is defined in the statement of the lemma. Clearly, this proves that $t_m \to t_0$ as $m \to +\infty$. On what concerns the second part of the lemma, let us now write that $t_m = t_0 + \theta_m$ with $\theta_m \to 0$ as $m \to +\infty$. Since $F(t_m, m) = 0$ for all m, one has that

$$\begin{aligned} 2\left(A + O(m^s)\right) &= (p+1)\left(B + O(m^s)\right) t_0^{p-1}\left(1 + \frac{\theta_m}{t_0}\right) \\ &\quad + (q+1)O(m^s)\left(t_0 + \theta_m\right)^{q-1} \end{aligned}$$

Hence,

$$2A + O(m^s) = (p+1)Bt_0^{p-1}\left(1+\frac{\theta_m}{t_0}\right)^{p-1}$$

and since $2A = (p+1)Bt_0^{p-1}$, and

$$\left(1+\frac{\theta_m}{t_0}\right)^{p-1} = 1 + \frac{p-1}{t_0}\theta_m + O(\theta_m)$$

one gets that $\theta_m = O(m^s)$. This ends the proof of the lemma.

■

3.5. Proof of Proposition 2.3: We assume that $n \geq 4$ and that

$$\max_{\{P\in Maxf \text{ such that } CardO_G(P)=\min_{x\in M} CardO_G(x)\}} h(P) > 0$$

In other words, we assume that there exists $P \in \max f$ such that $h(P) > 0$ with P of minimal orbit. We choose such a point to construct the ψ_m's, and we set

$$\Psi\left(t\frac{\psi_m}{\|\psi_m\|_{p+1}}\right) = F(t,m) = A(m)t^2 - B(m)t^{p+1} - C(m)t^{q+1}$$

There,

$$\begin{aligned} A(m) &= \tfrac{1}{2}\int\{|\nabla(\tfrac{\psi_m}{\|\psi_m\|_{p+1}})|^2 + a(x)(\tfrac{\psi_m}{\|\psi_m\|_{p+1}})^2\} > 0 \\ B(m) &= \tfrac{1}{p+1}\int f.(\tfrac{\psi_m}{\|\psi_m\|_{p+1}})^{p+1} > 0 \\ C(m) &= \int h.(\tfrac{\psi_m}{\|\psi_m\|_{p+1}})^{q+1} \end{aligned}$$

and (see Djadli [15])

$$\begin{aligned} \lim_{m\to+\infty} A(m) &= \tfrac{1}{2K(n,2)^2} = A > 0 \\ \lim_{m\to+\infty} B(m) &= \tfrac{1}{p+1}f(P) = B > 0 \\ \lim_{m\to+\infty} C(m) &= 0 \end{aligned}$$

Let t_m and t_0 be as in lemma 3.4. According to the above estimates,

$$\Psi\left(t_m\frac{\psi_m}{\|\psi_m\|_{p+1}}\right) = \frac{1}{K(n,2)^2}\frac{1}{k}\frac{t_m^2}{2} - \frac{1}{p+1}\frac{1}{k^p}f(P)t_m^{p+1} - C_1'h(P)m^l\frac{1}{k^q}t_m^{q+1} + o(m^l)$$

with $C_1' > 0$ and $h(P) > 0$. One can then write for m large

$$\begin{aligned}\Psi(t_m \frac{\psi_m}{\|\psi_m\|_{p+1}}) &< \frac{1}{K(n,2)^2}\frac{1}{k}\frac{t_m^2}{2} - \frac{1}{p+1}\frac{1}{k^p}f(P)t_m^{p+1} \\ &\leq \frac{1}{K(n,2)^2}\frac{1}{k}\frac{t_0^2}{2} - \frac{1}{p+1}\frac{1}{k^p}f(P)t_0^{p+1}\end{aligned}$$

As a consequence

$$\Psi(t_m \frac{\psi_m}{\|\psi_m\|_{p+1}}) < \frac{k}{nK(n,2)^n(\sup f)^{\frac{n-2}{2}}}$$

and condition $(\star)$ of lemma 2.2 is verified. This ends the proof of proposition 2.3.

■

3.6. Proof of Proposition 2.5: First we assume that $n \geq 5$ and that

$$\max_{\{P \in Maxf \text{ such that } CardO_G(P) = \min_{x \in M} CardO_G(x)\}} h(P) = 0$$

Let P be some point of $\max f$ of minimal orbit for which $h(P) = 0$. According to the above expansions, we have

$$\Psi(t \frac{\psi_m}{\|\psi_m\|_{p+1}}) = \frac{1}{K(n,2)^2}\frac{1}{k}\frac{t^2}{2} - \frac{1}{p+1}\frac{1}{k^p}f(P)t^{p+1} - \frac{1}{m}(\frac{Scal_g(P)t^2}{n(n-4)K(n,2)^2}\frac{1}{k}$$

$$- \frac{4(n-1)a(P)t^2}{n(n-2)(n-4)K(n,2)^2}\frac{1}{k} - \frac{\Delta_g f(P) t^{p+1}}{2n}\frac{1}{k^p}) + o(\frac{1}{m})$$

By lemma 3.4, one can write $t_m = t_0 + \varepsilon(\frac{1}{m})$ with $\varepsilon(\frac{1}{m}) = O(\frac{1}{m})$. Hence

$$\begin{aligned}\Psi(t_m \frac{\psi_m}{\|\psi_m\|_{p+1}}) &= \frac{1}{K(n,2)^2}\frac{1}{k}\frac{t_m^2}{2} \\ &- \frac{1}{p+1}\frac{1}{k^p}f(P)t_m^{p+1} - \frac{1}{m}(\frac{Scal_g(P)(t_0 + O(\frac{1}{m}))^2}{n(n-4)K(n,2)^2}\frac{1}{k}\end{aligned}$$

$$- \frac{4(n-1)a(P)(t_0 + O(\frac{1}{m}))^2}{n(n-2)(n-4)K(n,2)^2}\frac{1}{k} - \frac{\Delta_g f(P)(t_0 + O(\frac{1}{m}))^{p+1}}{2n}\frac{1}{k^p}) + o(\frac{1}{m})$$

and

$$\Psi(t_m \frac{\psi_m}{\|\psi_m\|_{p+1}}) = \frac{1}{K(n,2)^2}\frac{1}{k}\frac{t_m^2}{2} - \frac{1}{p+1}\frac{1}{k^p}f(P)t_m^{p+1} - \frac{1}{m}(\frac{Scal_g(P)t_0^2}{n(n-4)K(n,2)^2}\frac{1}{k}$$

$$-\frac{4(n-1)a(P)t_0^2}{n(n-2)(n-4)K(n,2)^2}\frac{1}{k} - \frac{\Delta_g f(P)t_0^{p+1}}{2n}\frac{1}{k^p}) + o(\frac{1}{m})$$

Now assume that

$$\frac{Scal_g(P)t_0^2}{n(n-4)K(n,2)^2} - \frac{4(n-1)a(P)t_0^2}{n(n-2)(n-4)K(n,2)^2} - \frac{\Delta_g f(P)t_0^{p+1}}{2n}\frac{1}{k^{p-1}} > 0$$

that is

$$\frac{2Scal_g(P)}{n-4} - \frac{8(n-1)a(P)}{(n-2)(n-4)} > \frac{\Delta_g f(P)}{f(P)}$$

Then, for m large,

$$\Psi(t_m \frac{\psi_m}{\|\psi_m\|_{p+1}}) < \frac{1}{K(n,2)^2}\frac{1}{k}\frac{t_m^2}{2} - \frac{1}{p+1}\frac{1}{k^p}f(P)t_m^{p+1}$$
$$\leq \frac{1}{K(n,2)^2}\frac{1}{k}\frac{t_0^2}{2} - \frac{1}{p+1}\frac{1}{k^p}f(P)t_0^{p+1}$$

so that, for m large,

$$\Psi(t_m \frac{\psi_m}{\|\psi_m\|_{p+1}}) < \frac{1}{nK(n,2)^n(\sup f)^{\frac{n-2}{2}}}$$

This is condition ($\star$) of lemma 2.2. In the case $n = 4$, the arguments to obtain such an inequality are similar to those just developed. This ends the proof of proposition 2.5.

■

3.7. Proof of Proposition 2.6: As a test function, we use here the constant function 1. For a fixed $t \geq 0$, if we note $C_t \equiv t$, we can write

$$\Psi(C_t) = \frac{t^2}{2}\int a - \frac{t^{p+1}}{p+1}\int f - \frac{t^{q+1}}{q+1}\int h$$

We set

$$\tilde{A} = \int a \ , \ \tilde{B} = \int f \ , \ \tilde{C} = \int h$$

Then

$$\Psi(C_t) = \frac{t^2}{2}\tilde{A} - \frac{t^{p+1}}{p+1}\tilde{B} - \frac{t^{q+1}}{q+1}\tilde{C}$$

One clearly has

$$\Psi(C_t) \leq \frac{\tilde{A}}{2}t^2 - \frac{\tilde{B}}{p+1}t^{p+1} + \frac{|\tilde{C}|}{q+1}t^{q+1}$$

Setting

$$F(t) = \frac{\tilde{A}}{2}t^2 - \frac{\tilde{B}}{p+1}t^{p+1} + \frac{|\tilde{C}|}{q+1}t^{q+1}$$

we compute

$$F'(t) = \tilde{A}t - \tilde{B}t^p + |\tilde{C}|t^q$$

Hence, if $2\tilde{A} \leq \tilde{B}t^{p-1}$ and $2|\tilde{C}|t^{q-1} \leq \tilde{B}t^{p-1}$ we will have $F'(t) \leq 0$. That is, if

$$t \geq (\frac{2\tilde{A}}{\tilde{B}})^{\frac{1}{p-1}} \quad \text{and} \quad t \geq (\frac{2|\tilde{C}|}{\tilde{B}})^{\frac{1}{p-q}}$$

then $F'(t) \leq 0$. We set

$$T' = \max((\frac{2\tilde{A}}{\tilde{B}})^{\frac{1}{p-1}}, (\frac{2|\tilde{C}|}{\tilde{B}})^{\frac{1}{p-q}})$$

One has that F is deacreasing in $[T'; +\infty[$ and that its maximum is attained in $]0; T']$. Now let t'_0 be such that

$$\Psi(C_{t'_0}) = \sup_{t \geq 0} \Psi(C_t)$$

One has easily

$$\begin{aligned} F(t'_0) &= \frac{\tilde{A}}{2}t'^2_0 - \frac{\tilde{B}}{p+1}t'^{p+1}_0 + \frac{|\tilde{C}|}{q+1}t'^{q+1}_0 \\ &\leq \frac{1}{n}\frac{\tilde{A}^{\frac{p+1}{p-1}}}{\tilde{B}^{\frac{2}{p-1}}} + \frac{|\tilde{C}|}{q+1}T'^{q+1} \end{aligned}$$

Assume that

$$\frac{1}{nK(n,2)^n(\sup f)^{\frac{n-2}{2}}} - \frac{1}{n}\frac{\tilde{A}^{\frac{p+1}{p-1}}}{\tilde{B}^{\frac{2}{p-1}}} > 0$$

that is

$$\left(\frac{\int f}{\sup f}\right)^{\frac{n-2}{2}} > K(n,2)^n\left(\int a\right)^{\frac{n}{2}}$$

Assume also that

$$\frac{|\tilde{C}|}{q+1}T'^{q+1} < \frac{1}{nK(n,2)^n(\sup f)^{\frac{n-2}{2}}} - \frac{1}{n}\frac{\tilde{A}^{\frac{p+1}{p-1}}}{\tilde{B}^{\frac{2}{p-1}}}$$

that is

$$|\tilde{C}| < \frac{q+1}{nK(n,2)^n\tilde{B}^{\frac{n-2}{2}}(\sup f)^{\frac{n-2}{2}}}\left\{\tilde{B}^{\frac{n-2}{2}} - \tilde{A}^{\frac{n}{2}}K(n,2)^n(\sup f)^{\frac{n-2}{2}}\right\} \times \frac{1}{T'^{q+1}}$$

One clearly gets

$$|\tilde{C}| < \frac{(q+1)\tilde{B}^{\frac{q+1}{p-1}}}{nK(n,2)^n\tilde{B}^{\frac{n-2}{2}}(\sup f)^{\frac{n-2}{2}}(2\tilde{A})^{\frac{q+1}{p-1}}}\left\{\tilde{B}^{\frac{n-2}{2}} - \tilde{A}^{\frac{n}{2}}K(n,2)^n(\sup f)^{\frac{n-2}{2}}\right\}$$

in the case where $T' = (\frac{2\tilde{A}}{\tilde{B}})^{\frac{1}{p-1}}$ and

$$|\tilde{C}| < \left(\frac{(q+1)\tilde{B}^{\frac{q+1}{p-q}-\frac{n-2}{2}}}{nK(n,2)^n(\sup f)^{\frac{n-2}{2}}2^{\frac{q+1}{p-q}}}\left\{\tilde{B}^{\frac{n-2}{2}} - \tilde{A}^{\frac{n}{2}}K(n,2)^n(\sup f)^{\frac{n-2}{2}}\right\}\right)^{\frac{p-q}{p+1}}$$

in the case where $T' = (\frac{2|\tilde{C}|}{\tilde{B}})^{\frac{1}{p-q}}$. This ends the proof of the proposition.

■

Remark 3.8. ε in the previous proposition, depends on the dimension of the manifold, of G, of $\int_M a$, of $\int_M f$, of $\max_M f$ and of q. More precisely if

$$C_1 = \frac{k(q+1)\left(\int_M f\right)^{\frac{q+1}{p-1}-\frac{n-2}{2}}}{nK(n,2)^n(\sup f)^{\frac{n-2}{2}}(2\int_M a)^{\frac{q+1}{p-1}}} \left\{\left(\int_M f\right)^{\frac{n-2}{2}} - \left(\int_M a\right)^{\frac{n}{2}}\frac{K(n,2)^n}{k}(\sup f)^{\frac{n-2}{2}}\right\}$$

$$C_2 = \left(\frac{k(q+1)\left(\int_M f\right)^{\frac{q+1}{p-q}-\frac{n-2}{2}}}{nK(n,2)^n(\sup f)^{\frac{n-2}{2}}2^{\frac{q+1}{p-q}}} \right.$$
$$\left. \left\{ \left(\int_M f\right)^{\frac{n-2}{2}} - \left(\int_M a\right)^{\frac{n}{2}} \frac{K(n,2)^n}{k}(\sup f)^{\frac{n-2}{2}} \right\} \right)^{\frac{p-q}{p+1}}$$

then we can take $\varepsilon = \min\{C_1, C_2\}$.

4 The infinite case - Proofs of lemma 2.9 and proposition 2.7 and 2.11

In this section, we assume that

$$\forall x \in M \quad Card O_G(x) = +\infty$$

Proof of proposition 2.7. The first part of the proof (to prove that there exists a positive, smooth solution of the equation), is similar to that of lemma 2.2. We omit it.

Let us now prove that $u \not\equiv 0$. Setting $v_j = u_j - u$, we have

$$\int_M |\nabla u_j|^2 = \int_M |\nabla u|^2 + \int_M |\nabla v_j|^2 + o(1)$$

and

$$\int_M (u_j^+)^{p+1} = \int_M u^{p+1} + \int_M (v_j^+)^{p+1} + o(1)$$

since the embedding $H_{1,G}(M) \hookrightarrow L^{p+1}(M)$ is compact. Then according to (3.1) and (3.2), since $u_j \to u$ strongly in $L^r(M)$ for all $2 \le r \le p+1$:

$$J(u) + \int_M \left\{ \frac{1}{2}|\nabla v_j|^2 \right\} = c + o(1)$$

and

$$\int_M \left\{ |\nabla u|^2 + au^2 - fu^{p+1} - hu^{q+1} \right\} + \int_M \left\{ |\nabla v_j|^2 \right\} = o(1)$$

It follows that

$$\int_M |\nabla v_j|^2 = o(1)$$

and consequently

$$J(u) = c$$

But $c > 0$. Then $u \not\equiv 0$ and $u > 0$ on M. This ends the proof of the proposition.

■

4.1. Proof of lemma 2.9: The proof is the same that the proof of lemma 2.2. Of course, instead of using theorem A, we use theorem B.

■

4.2. Proof of proposition 2.11: The proof is similar to that of proposition 2.6. We omit it.

■

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