

ENTIRE FUNCTIONS AND EQUICONTINUITY OF POWER MAPS IN BAIRE ALGEBRAS

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Abstract

We obtain that the power maps are equicontinuous at zero in any Baire locally convex algebra with a continuous product in which all entire functions operate; whence is m -convex in the commutative case. As a consequence, we get the same result of Mityagin, Rolewicz and Zelazko for commutative B_0 -algebras.

B. S. Mityagin, S. Rolewicz and W. Zelazko showed ([5]) that a unitary commutative B_0 -algebra in which all entire functions operate is necessarily m -convex. Their proof is long and technical. In [3], using a Baire argument and the polarization formula, we obtain that a unitary commutative Baire locally convex algebra with a continuous product in which all entire functions operate is actually m -convex. Our proof is direct and self contained. In the non commutative case, W. Zelazko exhibits in [8] an example of a non m -convex non-commutative B_0 -algebra on which all entire functions operate. In his example, the power maps are equicontinuous at zero. Using a Baire argument and the Mazur-Orlicz formula, we obtain the same result in a more general context. We show that the sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero in any unitary Baire l. c. a. with a continuous product in which all entire functions operate. Whence, as a consequence, our result of [3], in the commutative case and hence the result of Mityagin, Rolewicz and Zelazko in commutative B_0 -algebras ([5]).

A locally convex algebra (l. c. a. for short) is a Hausdorff locally convex space which is an algebra over a field K ($K = \mathbb{R}$ or $K = \mathbb{C}$) with separately continuous product. If the product is continuous in two variables, it is said to be with continuous product. Let (A, τ) be a locally

convex algebra the topology of which is given by a family of $(p_i)_{i \in I}$ of seminorms. It is said to be multiplicatively m -convex (m -convex for short) if

$$p_i(xy) \leq p_i(x)p_i(y), \text{ for all } x, y \in A, i \in I.$$

A B_0 -algebra is a *l. c. a.* whose underlying locally convex space is a completely metrisable space. An entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, $a_n \in K$, operates in a unitary *l. c. a.* (A, τ) if, for every x in A , $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, converges in (A, τ) . A topological algebra A is said to be Q -algebra if and only if the set of all invertible elements of A is open. For a detailed account of basic properties of general locally m -convex algebras and B_0 -algebras, we refer the reader to [4] and [7].

Here is the main result

Theorem 1. *Let (A, τ) be a unitary Baire l. c. a. with a continuous product. If entire functions operate in A , then the sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero. In particular, if A is commutative, then it is an m -convex algebra.*

Proof. Let U be an absolutely convex and closed neighbourhood of zero in A , and $\|\cdot\|_U$ its gauge. The product being continuous, there is another continuous seminorm $|\cdot|$ such that

$$\|xy\|_U \leq |x| |y|, \quad x, y \in A.$$

Since entire functions operate in A , one has $\sup_n |x^n|^{1/n} < +\infty$, for every

x in A . Let $r : A \rightarrow R_+$, be the map given by $r(x) = \sup_n |x^n|^{1/n}$.

It is clear that r is lower semicontinuous. For every, integer p , set $A_p = \{a \in A : r(a) \leq p\}$. It is a closed subset of A . By Baire's argument, there is an integer k such that A_k is of non void interior. Hence, there is an $x_0 \in A_k$ and an absolutely convex neighbourhood V of zero such that $x_0 + V \subset A_k$. So for every x in V , we have

$$|(x_0 + x)^n| \leq k^n, \quad n = 1, 2, \dots$$

Hence,

$$\|(x_0 + x)^n\|_U \leq k^n, \quad n = 1, 2, \dots$$

By Mazur-Orlicz formula ([2]), we have

$$\left(\frac{x}{kn}\right)^2 = \frac{1}{n!} \sum_{j=0}^n (-1)^{n-j} C_n^j \left(\frac{x_0}{k} + \frac{j}{kn}x\right)^n, \quad x \in V, n \in N^*.$$

Then

$$\left(\frac{x}{k}\right)^n \in \frac{n^n}{n!} \sum_{j=0}^n C_n^j U, \quad \text{for } U \text{ is balanced.}$$

But there exists $c > 0$ such that $\frac{(2n)^n}{n!} \leq c^n$, for every integer n . Thus

$$\left(\frac{x}{k}\right)^n \in c^n U, \quad x \in V, n \in N^*, \text{ for } U \text{ is balanced.}$$

Whence

$$x^n \in U; \text{ for every } x \in \frac{1}{ck}V, n \in N^*.$$

Now if A is commutative, consider the polarization formula

$$x_1 x_2 \cdots x_n = \frac{1}{n!} \sum_I (-1)^{n-c(I)} \left(\sum_{i \in I} x_i\right)^n,$$

where I runs over the collection of all finite subsets of $\{1, 2, \dots, n\}$, $c(I)$ the cardinal of I and x_1, x_2, \dots, x_n elements of A . For $t > 0$, if $x_i \in \frac{1}{ck}tV, 1 \leq i \leq n$, we have

$$x_1 x_2 \cdots x_n \in \frac{(2nt)^n}{n!} U.$$

Then, for t small enough, U contains an idempotent neighbourhood of zero.

As a consequence, we obtain the following results

Theorem 2. *The sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero on any unitary pseudo complete l. c. a. (A, τ) which is a Q -algebra and with continuous inverse. In particular, if A is commutative, then (A, τ) is an m -convex algebra.*

Proof. Since (A, τ) is pseudo complete l. c. a. with continuous inverse, the Q -algebra property implies the boundedness of every element. Hence entire functions operate on A . The commutative case follows from [6].

Theorem 3. *Let (A, τ) be a l. c. a. with continuous inverse. If $\text{Rad}A$ is closed, then, in this radical, the sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero. In particular, if $\text{Rad}A$ is commutative, then $\text{Rad}A$ is an m -convex algebra.*

Proof. The unitary subalgebra $(\text{Rad}A)^1 = \text{Rad}A \oplus Ce$, of A , is closed such that $\text{Rad}[(\text{Rad}A)^1] = \text{Rad}A$. It follows that $(\text{Rad}A)^1$ is a Q -algebra. But, by hypothesis, it is with continuous inverse. Then, by Theorem 2, the sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero on $(\text{Rad}A)^1$ and so on $\text{Rad}A$. If A is commutative, the conclusion follows also from Theorem 2.

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