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A VOLUME COMPARISON THEOREM AND NUMBER OF ENDS FOR MANIFOLDS WITH ASYMPTOTICALLY NONNEGATIVE RICCI CURVATURE

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Abstract

In this paper we establish a volume comparison theorem for cocentric metric balls at arbitrary point for manifolds with asymptotically nonnegative Ricci curvature, which will allow us to prove the finiteness of the number of ends

1 Introduction

In (CG1) and (CG2) J.Cheeger and D.Gromoll studied noncompact manifolds with nonnegative (sectional and Ricci) curvature and showed they have finite topology.
In 1985 U.Abresch in [Ab1]and [Ab2] introduced a new concept: manifolds with asymptotically nonnegative sectional curvature. A noncompact manifold of dimension m is said to be with asymptotically nonnegative sectional curvature (Ricci curvature) if there exists a point p, called base point and a positive, nonincreasing function λ so that:

$$\int_0^{+\infty} t\lambda(t)dt = b_0 < +\infty$$

and for all $x \in M$ and u in T_xM

$$K(u) \geq -\lambda(d(p,x))\|u\|^2 \qquad (Ricci(u) \geq -(m-1)\lambda(d(p,x))\|u\|^2).$$

One of the remaining questions is if asymptotically nonnegative curvature implies finiteness of number of ends. U.Abresch and D.Gromoll in

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[AG] showed that if M has asymptotically nonnegative curvature and has diameter growth of order $o(r^{\frac{1}{n}})$ then it is homotopically equivalent to an interior of compact manifold with boundary, provided the sectional curvature is bounded from below. M.I.Cai in [Ca] established a bound on number of ends of manifolds with nonnegative Ricci curvature outside a compact subset. S.H.Zhu in [Zh] proved a volume comparison theorem for balls with center the base point in manifolds with asymptotically nonnegative Ricci curvature.

The purpose of this paper is to establish a comparison theorem at every point, not necessary the base point and deduce a finite bound on the number of ends. In the second theorem we show that manifolds with asymptotically nonnegative Ricci curvature have infinite volume.

Let take some definitions .

Two curves γ_1 and γ_2 on a riemannian manifold M with origin x are said to be cofinal if and only if for all ball B(x,r), there exists $t > 0$ so that $\gamma_1(t_1)$ and $\gamma_2(t_2)$ lie in the same connected component of $M/\overline{B(x,r)}$ for all $t_1, t_2 \geq t$ where $\overline{B(x,r)}$ denotes the closed ball of center x and radius r.

In this way, an equivalence relation is defined and every class is called an end of M.

Let $r > 0$ and $B(p,r)$ the ball of center p and radius r ; we note $C(p,r)$ the union of all connected unbounded components of $M/\overline{B(p,r)}$. Let ζ be a real, $\frac{1}{2} < \zeta < 1$, Ω an open set in M and Σ a connected subset of Ω ; for $x, y \in \Sigma \cap S(p,r)$ let:

$$d_r(x,y) = inf_\gamma L(\gamma),$$

where infinum is taken over all smooth curves γ in Σ from x to y. We set

$$diam(\Sigma \cap S(p,r), \Omega) = sup_{(x,y)\in\Sigma} d_r(x,y).$$

and

$$Diam(p,r) = Supdiam\left(\Sigma \cap S(p,r), C(p,\zeta r)\right),$$

where suprenum is taken over all unbounded connected components Σ of $\partial C(p,r)$. The function $Diam(p,r)$ measures the diameter of ends.

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2 Main Results

Theorem 1. *Let M be a complete open Riemannian manifold with dimension m and asymptotically nonnegative Ricci curvature. Then, for all point x in M and for all $0 < r \leq R$*

$$\frac{Vol\,(B(x,R))}{Vol\,(B(x,r))} \leq e^{(m-1)b_0} \left(\frac{R}{r}\right)^m \text{ if } 0 \leq R \leq l$$

$$\frac{Vol\,(B(x,R))}{Vol\,(B(x,r))} \leq e^{(m-1)b_0} \left(\frac{R+l}{r}\right)^m \text{ if } R \geq l$$

where $B(x,r)$ denotes the ball of radius r with center x in M.

Corollary. *Let M be a manifold with asymptotically nonnegative Ricci curvature with base point p. Then,*

$$Diam(p,r) \leq 4\xi e^{(m-1)b_0} \left(1 + \frac{3}{\xi}\right) r.$$

where $\xi = \frac{1}{2}(1-\zeta)$ and consequently M has finite number of ends. This corollary shows that the diameter of ends of an asymptotically nonnegative curved manifold grown at most lineary as on nonnegative curved manifolds.

Theorem 2. *Let M be a manifold with asymptotically nonnegative Ricci curvature with base point p. Then,there exist two positive constants C and ρ such that,for all $R > 0$*

$$Vol(B(p,R)) \geq C(\ln R)^\rho.$$

For all $t \geq 0$, let $\alpha(t) = \lambda(|t-l|)$ where $l = d(p,x)$. If $z \in M$ and $u \in T_zM$ then, by triangle inequality, we have:

$$|d(x,z) - d(x,p)| \leq d(p,z)$$

and since λ is nonincreasing ,

$$Ric_z(u) \geq -(m-1)\lambda(d(p,z)) \geq -(m-1)\alpha(d(x,z))$$

3 Proofs

To prove our results we need three lemmas:

Lemma1. *Let $y(t)$ be the unique solution of equation:*

$$(*)\begin{cases} y''(t) - \alpha(t)y(t) = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}$$

then for all $t \geq 0$, $t \leq y(t) \leq ce^{b_0}.t$ where

$$c = \begin{cases} 1 & if\, t \leq l \\ (1 + 2l\lambda_1(0))e^{b_0} & if\, t \geq l \end{cases}$$

where $\lambda_1(0) = \int_0^\infty \lambda(x)dx$

Proof. Since $y''(t) = \alpha(t)y(t) \geq 0$, y' is increasing in a neighbourhood of zero and one easily shows like in [Zh] that $y(t) \geq t$ for all $t \geq 0$ and that y is increasing.
By initial conditions, we have:

$$y'(t) = 1 + \int_0^t \alpha(s)y(s)ds \Longrightarrow$$

$$y(t) = t + \int_0^t \left(\int_0^s \alpha(x)y(x)dx\right) ds = t + \int_0^t \left(\int_x^t \alpha(x)y(x)ds\right) dx$$

by Fubini.

$$y(t) = t + \int_0^t (t-x)\alpha(x)y(x)dx = t\left(1 + \int_0^t \alpha(x)y(x)dx\right) - \int_0^t x\alpha(x)y(x)dx$$

$$\Longrightarrow y(t) \geq t\left(1 + \int_0^t \alpha(x)y(x)dx\right) - ty(t)\int_0^t \alpha(x)dx$$

$$\Longrightarrow 1 + t\int_0^t \alpha(x)dx \geq \frac{ty'(t)}{y(t)} (**)$$

If $l = 0$ inequality holds by theorem2.1 in [Zh]
Suppose $l > 0$.
If $t \leq l$ then $(**)$ implies

$$\frac{1}{t} + \int_0^t \alpha(x)dx \geq \frac{y'(t)}{y(t)}.$$

After integrating this expression we have

$$\Longrightarrow \ln\frac{t}{\epsilon}+\int_\epsilon^t\int_0^x \lambda(l-u)dudx \geq \ln\frac{y(t)}{y(\epsilon)}$$

$$\Longrightarrow \ln y(t) \leq \ln\frac{ty(\epsilon)}{\epsilon}+\int_\epsilon^t\int_0^x \lambda(l-u)dudx$$

By the initial conditions, we have

$$\ln y(t) \leq \ln t+\int_\epsilon^t\int_0^x \lambda(l-u)dudx \leq \ln y(t) \leq \ln t+\int_0^t\int_0^x \lambda(l-u)dudx$$

$$\Longrightarrow \ln y(t) \leq \ln t+\int_0^t\int_u^t dx\lambda(l-u)du$$

$$\Longrightarrow \ln y(t) \leq \ln t+\int_0^t (t-u)\lambda(l-u)du \leq \ln t+\int_0^l u\lambda(u)du$$
$$\leq \ln t+b_0$$

that is

$$y(t) \leq e^{b_0}.t$$

If $t \geq l$ we have:

$$y'(t) = y'(l)+\int_l^t \alpha(x)y(x)dx$$

$$\Longrightarrow y(t)-y(l) = (t-l)y'(l)+\int_l^t\int_l^x y(u)\lambda(u-l)dudx$$

$$= (t-l)y'(l)+\int_l^t (t-u)\lambda(u-l)y(u)du$$

$$\leq (t-l)\left(y'(l)+\int_l^t \lambda(u-l)y(u)du\right) = (t-l)y'(t).$$

then

$$\frac{\alpha(t)y(t)}{y'(t)} \leq (t-l)\alpha(t)+\frac{\alpha(t)y(l)}{y'(t)}$$

hence

$$\ln\frac{y'(t)}{y'(l)} \leq \int_l^t (x-l)\alpha(x)+\int_l^t \frac{\alpha(x)y(l)dx}{y'(l)+\int_l^x \alpha(u)y(u)du}$$

$$\ln \frac{y'(t)}{y'(l)} \leq b_0 + \int_l^t \frac{\alpha(x)y(l)dx}{y'(l) + y(l)\int_l^x \alpha(u)du}$$

$$\ln\left(\frac{y'(t)}{y'(l)}\right) \leq b_0 + \ln\left(\frac{y'(l) + y(l)\lambda_1(0)}{y'(l)}\right)$$

From $(**)$ we have

$$y'(l) \leq y(l)(\frac{1}{l} + \lambda_1(0)) \leq e^{b_0}\,(1 + l\lambda_1(0))$$

$$\Longrightarrow y'(t) \leq e^{2b_0}.\,(1 + 2l\lambda_1(0))$$

hence

$$y(t) \leq e^{2b_0}(1 + 2l\lambda_1(0))t$$

and the lemma follows. The lemmal is far from giving a sharp comparison for volume of balls with center $x \neq p$ and a sufficient large radius. The basic fact on this is related to the choice of the function α. So, we state the following lemma to prove our comparison theorem:

Lemma 2. *Let $\overline{M}$ be a noncompact simply connected manifold with dimension m. Suppose there exists a point $\overline{p}$ so that*

$$K(\overline{x}) = -\lambda(d(\overline{p}, \overline{x}))$$

for all $\overline{x} \in \overline{M}$. Then, for all $R > 0$ and $\overline{x}$, we have 0

$$(1) \qquad Vol(B(\overline{p}, R) \leq \omega_m R^m e^{(m-1)b_0}$$

$$(2) \qquad Vol(B(\overline{x}, r)) \geq \omega_m r^m$$

and consequently

$$(3) \qquad \frac{Vol(B(\overline{x}, R)}{Vol(B(\overline{x}, r)} \leq \begin{cases} e^{(n-1)b_0}\left(\frac{R}{r}\right)^m & \text{if } r \leq R \leq l \\ e^{(n-1)b_0}\left(\frac{R+l}{r}\right)^m & \text{if } R \geq l \end{cases}$$

where ω_m is the volume of the unit ball in euclidian space.

Proof

For all $R > 0$ we have

$$Vol(B(\overline{p}, R) = \int_{S^{n-1}} \int_0^R y^{m-1}(t)dtd\theta$$

By lemma 1 we have $y(t) \leq e^{b_0}t(l = 0)$ and the conclusion follows. Since $K(\overline{x}) \leq 0$ for all $\overline{x}$, the inequality (2) follows from Rauch inequality and the fact that $\overline{M}$ is simply connected. .

If $R \geq l$ we have

$$B(\overline{x}, R) \subset B(\overline{p}, R + l)$$

and inequality (3) follows from (1) and (2).

The follwing lemma was proved in [Zh] (lemma2.2)

Lemma 3. *Let f and g be two positive functions defined over $[0, +\infty[$. If f/g is nonincreasing, then for $R \geq r > 0$ we have:*

$$\frac{\int_0^R f(t)dt}{\int_0^r f(t)dt} \leq \frac{\int_0^R g(t)dt}{\int_0^r g(t)dt}.$$

Proof of theorem 1

Let J denote the Jacobian of exponential application in polar coordinates and the nonnegative function z so that $z^{m-1} = J$; J. Cheeger showed in [Ch] that z satisfies the inequation:

$$\begin{cases} z''(t) - \alpha(t)z(t) \leq 0 \\ z(0) = 0, z'(0) = 1 \end{cases}$$

from which it follows that $\frac{z}{y}$ is nonincreasing where y is the solution of $(*)$ and by the lemma2 we conclude that

$$\frac{Vol\,(B(x,R))}{Vol\,(B(x,r))} = \frac{\int_0^{min\{cut(\theta),R\}} J(t)dt}{\int_0^{min\{cut(\theta),r\}} J(t)dt} \leq \frac{\int_0^{min\{cut(\theta),R\}} y^{m-1}(t)dt}{\int_0^{min\{cut(\theta),r\}} y^{m-1}(t)dt}$$

$$\leq \frac{\int_0^R y^{m-1}(t)dt}{\int_0^r y^{m-1}(t)dt} = \frac{Vol(B(\overline{x}, R)}{Vol(B(\overline{x}, r))}$$

$$\leq e^{(m-1)b_0} \left(\frac{R+l}{r} \right)^m$$

where $cut(\theta)$ means the cut point of θ

Proof of corollary

If $r > 0$ and $\{q_j\}$ is the maximal set of points on $S(p,r)$ so that the balls $B(q_j, \xi r)$ are disjoint and are contain in $M/B(p, \zeta r)$. We have :

$$B(q_j, \xi.r) \subset B(p, (1+\xi).r) \subset B(q_j, (2+\xi).r)$$

then

$$\frac{Vol\,(B\,(p,(1+\xi).r))}{Vol\,(B\,(q_j,\xi.r))} \leq \frac{Vol\,(B\,(q_j,(2+\xi).r))}{Vol\,(B\,(q_j,\xi.r))} \leq e^{(m-1)b_0} \left(\frac{3+\xi}{\xi} \right)^m,$$

therefore, the number of balls $B(q_j, \xi r)$ is no more than $e^{(m-1)b_0} \left(1 + \frac{3}{\xi}\right)^m$. The balls $B(q_j, 2\xi.r)$ cover $S(p,r)$ and if γ is a geodesic in $S(p,r)$ joining two points q_i and q_j so that $B(q_j, 2\xi.r) \cap B(q_i, 2\xi.r)$ is not empty then $L(\gamma) \leq 4\xi r$, hence

$$Diam(p,r) \leq 4\xi e^{(m-1)b_0} \left(1 + \frac{3}{\xi}\right)^m r.$$

Let $\{\gamma_i\}$ be the set of all geodesics from the base point p; if $x_1 = \gamma_1((1+\xi)r), x_2 = (\gamma_2((1+\xi)r)$ are in two different connected components of $M/\overline{B(p,r)}$ and if θ is a short geodesic joining x_1 to x_2, then θ meets $\overline{B(p,r)}$ and

$$d(\gamma_1(1+\xi)r, \gamma_2(1+\xi)r) \geq d\,(\gamma_1(1+\xi)r, B(p,r)) + d\,(B(p,r), \gamma_2(1+\xi)r)$$

$$\geq 2\xi.r,$$

hence the ball $\overline{B(q_j, \xi.r)}$ contains at most one $\gamma_k(1+\xi r)$ which means that the number of ends is less or equal to $e^{(m-1)b_0} \left(\frac{3+\xi}{\xi}\right)^m$.

Theorem2 states that, as for open complete manifolds with nonnegative Ricci curvature, the open complete manifolds with asymptotically nonnegative Ricci curvature have no finite volume.

Proof of theorem 2

Since M is noncompact and complete, for all $t > 0$ there exists a point x in M so that $d(p,x) \geq t$. Let γ be geodesic arc joining p to x, the function $s \longmapsto d(p,\gamma(s))$ is continuous and takes the value t. Let $b > 1$, $R_i = \sum_{j=0}^{i} 2r_j$ and $r_i = 2^{b^i}$. Take a point x_i on γ so that $d(p,x_i) = r_i + R_{i-1}$ and $x_0 = p$; by construction the balls $B(x_i,r_i)$ are disjoint and

$$\bigcup_{j=0}^{i} B(x_j,R_j) \subset B(x_i,r_i+R_{i-1}).$$

Let

$$\theta_i = \sum_{j=0}^{i} volB(x_j,r_j);$$

then

$$\theta_i \leq volB(x_i,r_i+R_{i-1})$$

$$\frac{\theta_i}{\theta_i-\theta_{i-1}} \leq \frac{volB(x_i,r_i+R_{i-1})}{volB(x_i,r_i)} \leq e^{(m-1)b_0}\left(1+\frac{R_{i-1}}{r_i}\right)^m$$

since $d(p,x_i) = r_i + R_{i-1} = l$ hence

$$\theta_i \geq \frac{e^{(m-1)b_0}\left(1+\frac{R_{i-1}}{r_i}\right)^m}{e^{(m-1)b_0}\left(1+\frac{R_{i-1}}{r_i}\right)^m - 1}\theta_{i-1}$$

$R_{i-1} = \sum_{j=0}^{i-1} 2.2^{b^j} \leq 2\sum j = 0^{i-1}2^{[b^j]+1} \leq 2.2^{[b^{i-1}]+2}$ where [ ] denotes the integer party; this implies that $\frac{R_{i-1}}{r_i} \leq 1$ and goes to zero at infinity.

Since the function

$$f(x) = \frac{e^{(m-1)b_0}(1+x)^m}{e^{(m-1)b_0}(1+x)^m - 1}$$

is nonincreasing on $[0,1]$ we have:

$$\frac{2^m e^{(m-1)b_0}}{2^m e^{(m-1)b_0} - 1} \leq f(x) \leq \frac{e^{(m-1)b_0}}{e^{(m-1)b_0} - 1}$$

hence

$$\theta_i \geq \frac{2^m e^{(m-1)b_0}}{2^m e^{(m-1)b_0} - 1}\theta_{i-1}.$$

which means :

$$\theta_i \geq a^i \theta_0$$

where

$$a = \frac{2^m e^{(m-1)b_0}}{2^m e^{(m-1)b_0} - 1}$$

and $\theta_0 = volB(p, 2)$
Let R be a sufficient large positive number; there exists i such that $r_{i-1} \leq R \leq r_i$, otherwise

$$r_i = 2^{b^i} \Rightarrow i = \frac{\ln\left(\frac{\ln r_i}{\ln 2}\right)}{\ln b} \geq \frac{\ln(\ln R)}{\ln b}$$

$$\theta_i \geq C.a^{\frac{\ln\left(\frac{\ln r_i}{\ln 2}\right)}{\ln b}} \geq C.a^{\frac{\ln(\ln R)}{\ln b}} = C\left(\ln(R)\right)^{\rho}$$

Where C, C_1 are positive constants and $\rho = \frac{\ln a}{\ln b} > 0$

Those results give a hope for showing the following conjecture due to S.H.Zhu and which is a version of Grove and Peterson conjecture's:

Conjecture: Given $c > 0$, do there exist constants $\epsilon(m, c)$ and $R(m, c)$ such that if

$$K(x) \geq -\lambda\left(d(p, x)\right), Vol\left(B(p, r)\right) \geq cr^m,$$

and

$$\int_0^{\infty} t\lambda(t)dt \leq \epsilon,$$

then any metric ball of radius r is contractible in the cocentric ball of radius $R.r$? Is M diffeomorphic to $\mathbf{R}^m$?

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