REVISTA MATEMÁTICA COMPLUTENSE ISSN 1139-1138
(2000) vol. XIII, num. 2, 413-422

BREAKDOWN IN FINITE TIME OF SOLUTIONS TO A ONE-DIMENSIONAL WAVE EQUATION

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Abstract

We consider a special type of a one-dimensional quasilinear wave equation $w_{tt} - \phi(\frac{w_t}{w_x})w_{xx} = 0$ in a bounded domain with Dirichlet boundary conditions and show that classical solutions blow up in finite time even for small initial data in some norm.

1 Introduction

Consider the following hyperbolic system

$$u_t(x,t) = A(u(x,t))u_x(x,t), \tag{1.1}$$

where $u : I \times (0,T) \to \mathbb{R}^n$ is a vector-valued function, A is an $(n \times n)$ matrix, and I is an (bounded or unbounded) interval. For the Cauchy problem, results concerning global existence and finite life span have been established by many authors. The first who discussed such a problem, in its generality, was John [4] in 1975. He showed that any C^2 solution of (1.1) blows up in finite time if the initial data $u_0(x) = u(x,0)$ is of compact support and satisfies that $\max_x\{s^2|u_0''(x)|\}$ is small enough. Here s denotes the length of supp u_0. His proof makes a crucial use of the local strict hyperbolicity of the system (1.1) in the sense that the eigenvalues of $A(u)$ are real and distinct in a neighborhood of $u = 0$. Ta-Tsien et *al* [14] discussed (1.1) associated with decaying initial data. They proved a global C^1 solution for the Cauchy problem if, in addition to the local strict hyperbolicity condition, (1.1) is weakly linearly degenerate and the initial data satisfy, for $\mu > 0$, that

1991 Mathematics Subject Classification: 35L45.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2000

$\max_x \left\{(1+|x|)^{1+\mu}(|u_0'(x)|+|u_0(x)|)\right\}$ is small enough. They also established a blow up result of C^1 solutions for non weakly linearly degenerate systems. As they pointed out, their work generalizes their result of [13] to the case of initial data with no compact support but they possess certain decay properties.

For the case $n = 2$, the situation is less involved and interesting results have been obtained. For instance, the system of nonlinear elasticity

$$u_t(x,t) = \varphi(v(x,t))\, v_x(x,t), \qquad v_t(x,t) = u_x(x,t), \tag{1.2}$$

has been discussed by Lax [7] and MacCamy and Mizel [8]. In his work, Lax required, in addition to $\varphi > 0$, that $\varphi' > 0$ and showed that classical solutions break down in finite time however smooth and small the initial data are. Whereas MacCamy and Mizel allowed φ' to change sign and proved a similar result. They also showed, under appropriate conditions on φ, that there are x-intervals, for which the solution must exist for all time even though it blows up for values of x outside these intervals. Messaoudi [11] discussed the following system

$$u_t(x,t) = \alpha(x)\varphi(v(x,t))\, v_x(x,t) \qquad v_t(x,t) = u_x(x,t), \tag{1.3}$$

which models a transverse motion of a string with variable density. He showed that C^1-solutions develop singularities in finite time if the initial data are taken with large enough gradients.

For systems with dissipation, we mention the equations

$$\theta_t + c(\theta)\, q_x = 0 \qquad q_t + \sigma(\theta)\theta_x = -\lambda(\theta) q, \tag{1.4}$$

which describe heat propagation in materials that predict finite propagation speed. This phenomenon is called second sound [1], [2], [9] and [10]. Here θ is the difference temperature and q is the heat flux. The Cauchy problem was studied by Messaoudi [9] and a blow up result for classical solutions was proved. We should note here that, if λ is constant and $c(\theta) = -1$ then (1.4) reduces to a system describing steady shearing flows in nonlinear viscoelastic fluids. This problem was studied by Slemrod [12] and a blow up result for classical solutions has been established. A similar problem was also discussed by Kosinski [6] and Zheng

[15] and results concerning global existence and nonexistence have been accomplished.

For the higher-dimensional case, it is worth mentioning the result of John [5]. In his work, he considered radial solutions of the three-dimensional system of elasticity and showed that classical solutions develop singularities in finite time.

In the present paper we are concerned with a quasilinear hyperbolic system of the form

$$\begin{cases} u_t(x,t) = \varphi\left(\dfrac{u(x,t)}{v(x,t)}\right) v_x(x,t) \\ v_t(x,t) = u_x(x,t). \end{cases} \tag{1.5}$$

Note that $\varphi\left(\dfrac{u(x,t)}{v(x,t)}\right)$ has a dimension of velocity.

We will study (1.5) together with initial and boundary conditions and show that C^1-solutions blow up even for small and smooth initial data. Our result cannot be directly deduced from the results of [4] and [14] since we do not impose the same conditions regarding the size and the regularity of the initial data (See, for instance, theorem 1.2 of [14] and lemma 3.1 below).

This work is divided into two parts:
In part one we state, without proof, a local existence theorem.
In part two, our main result will be stated and proved.

2 Local Existence

We consider the following problem

$$u_t(x,t) = \varphi\left(\frac{u(x,t)}{v(x,t)}\right) v_x(x,t) \tag{2.1}$$

$$v_t(x,t) = u_x(x,t), \quad \forall x \in I = (0,\ 1), \ \ t > 0 \tag{2.2}$$

$$u(x,0) = u_0(x), \ \ v(x,0) = v_0(x), \quad \forall x \in I \tag{2.3}$$

$$u(0,t) = u(1,t) = 0, \ v_x(0,t) = v_x(1,t), \quad t \geq 0 \tag{2.4}$$

where φ is a C^2 strictly positive function; i.e

$$\varphi(\xi) \geq \kappa > 0, \quad \forall \xi \in \mathbb{R}. \tag{2.5}$$

We also require

$$u_0 \in H^2(I) \cap H_0^1(I), \;\; v_0 \in H^2(I), \;\; v_0(x) \neq 0, \;\; \forall x \in [0,1]. \tag{2.6}$$

Proposition. *Assume that φ satisfies (2.5) and let u_0 and v_0 be given and satisfying (2.6). Then the problem (2.1) - (2.4) has a unique local solution (u,v), on a maximal time interval $[0,T)$, satisfying*

$$u, v \in C\left([0,T), H^2(I)\right) \cap C^1\left([0,T), H^1(I)\right) \cap C^2\left([0,T), L^2(I)\right). \tag{2.7}$$

Remark 2.1. This proposition is a direct application to the results of Dafermos and Hrusa [3] based on the use of energy estimates.

Remark 2.2. u, v are C^1 functions by the standard Sobolev embedding theory.

Remark 2.3. The local existence can be obtained even if we have $\varphi(0) \geq \kappa > 0$ instead of (2.5). In this case, we only consider initial data satisfying (2.6) with $\left\| \frac{u_0}{v_0} \right\|_\infty$ small enough.

3 Formation of singularities

In this section, we state and prove our main result. We first begin with a lemma that gives a uniform bound on $\frac{u}{v}$ in terms of the initial data.

Lemma 3.1. *Let φ be as in the proposition. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any initial data satisfying (2.6) and*

$$||v_0||_\infty < \delta, \qquad ||u_0||_\infty < \delta, \tag{3.1}$$

the solution of (2.1) - (2.4) satisfies

$$\left| \frac{u(x,t)}{v(x,t)} \right| < \varepsilon, \qquad \forall x \in [0,1], \quad t \geq 0. \tag{3.2}$$

Proof. We define the functions and the differential operators

$$r := \ln|v| + \int_0^{\frac{u}{v}} \alpha(\xi)d\xi, \quad s := \ln|v| - \int_0^{\frac{u}{v}} \beta(\xi)d\xi \tag{3.3}$$

$$\partial_t := \frac{\partial}{\partial t} - \rho\left(\frac{u}{v}\right)\frac{\partial}{\partial x}, \quad D_t := \frac{\partial}{\partial t} + \rho\left(\frac{u}{v}\right)\frac{\partial}{\partial x},$$

where

$$\rho(\xi) = \sqrt{\varphi(\xi)}, \quad \alpha(\xi) = \frac{1}{\sqrt{\varphi(\xi)} + \xi}, \quad \beta(\xi) = \frac{1}{\sqrt{\varphi(\xi)} - \xi}.$$

We then compute

$$\partial_t r = \frac{v_t}{v} + \alpha\frac{u_t v - u v_t}{v^2} - \rho\left[\frac{v_x}{v} + \alpha\frac{u_x v - u v_x}{v^2}\right] \tag{3.4}$$

$$= \frac{1}{v}\left[\left(1 - \alpha\frac{u}{v}\right)v_t - \alpha\rho u_x\right] + \frac{1}{v}\left[\alpha u_t - \rho\left(1 - \alpha\frac{u}{v}\right)v_x\right].$$

We remind that, unless otherwise stated, α, β, ρ, and φ are functions of $\frac{u}{v}$. By noting that $\left(1 - \alpha\frac{u}{v}\right) = \alpha\rho$ and $\rho\left(1 - \alpha\frac{u}{v}\right) = \alpha\varphi$, (3.4) gives

$$\partial_t r = 0. \tag{3.5}$$

Similar calculations also yield

$$D_t s = 0.$$

Therefore as long as a smooth solution continues to exist and

$$\left|\frac{u(x,t)}{v(x,t)}\right| < \sqrt{\varphi(\frac{u(x,t)}{v(x,t)})}, \tag{3.6}$$

r and s remain constant along backward and forward characteristics, respectively; hence

$$||r||_\infty = ||r_0||_\infty, \quad ||s||_\infty = ||s_0||_\infty.$$

To establish (3.6) we note, by virtue of (3.3), that

$$r - s = \phi\left(\frac{u(x,t)}{v(x,t)}\right),$$

where $\phi(\tau) = \displaystyle\int_0^\tau \frac{2\sqrt{\varphi(\xi)}d\xi}{\varphi(\xi) - \xi^2}$ is continuous and strictly monotone, so it admits a continuous inverse ψ, at least in a neighborhood of zero. Since $g(\xi) = \varphi(\xi) - \xi^2$ is continuous and $g(0) > \kappa$, one can choose $\eta \leq \varepsilon$ such that $g(\xi) \geq \frac{\kappa}{2}$, for all $|\xi| < \eta$ and choose $\gamma > 0$ so that $|\psi(\xi)| < \eta$, for all $|\xi| < \gamma$. Therefore as long as $||r - s||_\infty < \gamma$, we have

$$\left|\frac{u}{v}\right| = |\psi\,(r - s)| < \eta \leq \varepsilon. \tag{3.7}$$

From the definition of r and s, it is easy to see that if δ, in (3.1), is chosen small enough then we get

$$||r_0||_\infty + ||s_0||_\infty < \gamma; \tag{3.8}$$

consequently $||r - s||_\infty \leq ||r_0||_\infty + ||s_0||_\infty < \gamma$. This implies (3.5); hence

$$g(\frac{u}{v}) = \varphi(\frac{u}{v}) - (\frac{u}{v})^2 \geq \frac{\kappa}{2}. \tag{3.9}$$

Therefore (3.6) is established and the proof of the lemma is completed.

Remark 3.1. By using (3.9) and the boundedness of r we conclude, from (3.3) that $\ln|v|$ remains bounded. Therefore, with the above choice of the initial data, v is never equal to zero.

Now, we state our principal result.

Theorem 3.2. *Assume that, in addition to (2.5), φ satisfies*

$$\varphi'(0) > 0,$$

and

$$\frac{v_0'}{v_0} + \alpha\left(\frac{u_0}{v_0}\right)\frac{u_0'v_0 - u_0v_0'}{v_0^2} > 0, \quad \forall x \in [0,1].$$

Then there exist initial data u_0, v_0 satisfying (3.1), for which the solution of the problem (2.1) - (2.4) blows up "pointwise" in finite time.

Proof. As usual, we have to derive an ordinary differential inequality with a quadratic nonlinearity for a combination of u and v from which the desired conclusion can be drawn.

For this, we take an x-partial derivative of (3.5) to get

$$(\partial_t r)_x = r_{xt} - \rho r_{xx} - r_x\frac{\partial}{\partial x}\rho = 0 \tag{3.10}$$

which, in turn, implies

$$\partial_t(r_x) = r_x \frac{\partial}{\partial x}\rho = \frac{\varphi'}{2\sqrt{\varphi}} r_x \frac{\partial}{\partial x}\left(\frac{u}{v}\right). \tag{3.11}$$

By using

$$r_x = \frac{v_x}{v} + \alpha \frac{\partial}{\partial x}\left(\frac{u}{v}\right), \quad s_x = \frac{v_x}{v} - \beta \frac{\partial}{\partial x}\left(\frac{u}{v}\right)$$

and substituting in (3.11), we obtain

$$\partial_t r_x = \frac{\varphi'}{4\varphi}[\varphi - \left(\frac{u}{v}\right)^2] r_x^2 - \frac{\varphi'}{4\varphi}[\varphi - \left(\frac{u}{v}\right)^2] r_x s_x. \tag{3.12}$$

To handle the last term in (3.12), we set

$$W := \lambda\left(\frac{u}{v}\right) r_x$$

and substitute in (3.12), to get

$$\partial_t W = \frac{\varphi'}{4\lambda\varphi}[\varphi - \left(\frac{u}{v}\right)^2] W^2 - \lambda \frac{\varphi'}{4\varphi}[\varphi - \left(\frac{u}{v}\right)^2] r_x s_x + r_x \lambda' \partial_t \left(\frac{u}{v}\right). \tag{3.13}$$

By using equations (2.1), (2.2), and (3.3) we easily deduce

$$\partial_t\left(\frac{u}{v}\right) = \frac{v(u_t - \sqrt{\varphi} u_x) - u(v_t - \sqrt{\varphi} v_x)}{v^2}$$

$$= \frac{v(\varphi v_x - \sqrt{\varphi} u_x) - u(u_x - \sqrt{\varphi} v_x)}{v^2} = \frac{(\sqrt{\varphi} v_x - u_x)(u + \sqrt{\varphi} v)}{v^2} \tag{3.14}$$

and

$$s_x = \frac{\beta}{v}(\sqrt{\varphi} v_x - u_x) = \frac{1}{v}\frac{1}{\sqrt{\varphi} - \left(\frac{u}{v}\right)}(\sqrt{\varphi} v_x - u_x). \tag{3.15}$$

Now we choose λ so that

$$-\lambda \frac{\varphi'}{4\varphi}[\varphi - \left(\frac{u}{v}\right)^2] r_x s_x + r_x \lambda' \partial_t\left(\frac{u}{v}\right) = 0. \tag{3.16}$$

By combining (3.14) - (3.16) we arrive, by simple computations, at

$$\frac{\lambda'\left(\frac{u}{v}\right)}{\lambda\left(\frac{u}{v}\right)} = \frac{\varphi'\left(\frac{u}{v}\right)}{4\varphi\left(\frac{u}{v}\right)},$$

which yields, by a direct integration,

$$\lambda(\xi) = \varphi^{1/4}(\xi);$$

consequently (3.13) reduces to

$$\partial_t W = \frac{\varphi'}{4\lambda\varphi}[\varphi - \left(\frac{u}{v}\right)^2] W^2. \tag{3.17}$$

If we choose δ sufficiently small then the coefficient of the quadratic term in (3.17) remains bounded away from zero by virtue of (3.10), the lemma, and the continuity of φ and λ. So there exists a constant $k > 0$ such that

$$\varphi'[\varphi - \left(\frac{u}{v}\right)^2]\frac{1}{4\lambda\varphi} \geq k,$$

consequently (3.17) yields

$$\partial_t W \geq kW^2. \tag{3.18}$$

Therefore (3.18) shows that W (hence r_x) blows up in a maximal time $t_m \leq \frac{1}{kW_0}$, if we choose initial data satisfying (3.1) with derivatives satisfying $W_0 > 0$; i.e

$$\frac{v_0'}{v_0} + \alpha\left(\frac{u_0}{v_0}\right)\frac{u_0'v_0 - u_0v_0'}{v_0^2} > 0, \qquad \forall x \in [0,1].$$

Remark 3.2. A similar result can be established for $\varphi'(0) < 0$. In this case we consider the evolution of s_x on the forward characteristics; i.e we repeat the same calculations (3.10)-(3.18) with s and $D_t s$.

Acknowledgement: The authors would like to express their sincere thanks to the referees for their useful suggestions.

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Recibido: 8 de Febrero de 1999
Revisado: 8 de Marzo de 2000