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ON JACKSON TYPE INEQUALITY IN ORLICZ CLASSES

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Abstract

It is shown that Jackson type inequality fails in the Orlicz classes $\varphi(L)$ if $\varphi(x)$ differs essentially from a power function of any order.

1 Introduction

As is known a best approximation to a given 2π -periodic function in L_p, $0 < p \leq +\infty$, by trigonometric polynomials of order at most n can be estimated by its L_p-modulus of continuity with argument $(n+1)^{-1}$. This result is called the direct theorem of Approximation Theory or Jackson type inequality in honour of D. Jackson who proved it for continuous metric $(p = +\infty)$. Afterwards it was extended to the case of an arbitrary Banach space of 2π-periodic functions, where translation is a continuous isometry (see, for instance, [1]). The quasi-normed case $0 < p < 1$ can be found in [2]. At present there is an enormous number of works dedicated to various generalizations of Jackson type inequality that deal with several variables, higher order moduli of smoothness and so forth. However, for a long time nothing was known about similar results in the setting of Orlicz classes $\varphi(L)$, where $\varphi(x)$ differs essentially from a power function of any order. As far as we know it was expected Jackson type inequality would be valid for comparatively wide set of functions $\varphi(x)$. To our surprise, it is not. In fact, the main result of this paper is that if $\varphi(x)$ decreases to 0 for $x \to +0$ or increases to $+\infty$ for $x \to +\infty$ slower than a power function of an arbitrary order, Jackson type inequality fails. Moreover, for such $\varphi(x)$ the modulus of continuity turns out in general to be unfit to estimate the rate of approaching 0 of a best trigonometric

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approximation in $\varphi(L)$ in the sense that the inequality remains false even after replacing $(n+1)^{-1}$ by an arbitrary sequence of positive numbers $\{\sigma_n\}_{n=1}^{+\infty}$ that goes to 0 for $n \to +\infty$.

We prove our result not only for the trigonometric system, but also for more general class of systems we have called *non-localized.* It will be shown that this class contains all systems of 2π-periodic analytic functions. The property of "non-locality" makes clear the difference between the trigonometric system and the system of piece-wise constant functions, for which Jackson type inequality holds in $\varphi(L)$, where $\varphi(x)$ satisfies only the natural conditions, and in particular, can have practically an arbitrary behaviour at the neighbourhood of 0 and $+\infty$ ([2]).

2 The main result

We deal with Orlicz classes $\varphi(L)$ of measurable 2π-periodic functions $f(x)$, such that the functional

$$\|f\|_\varphi = \int_0^{2\pi} \varphi(f(x))dx$$

is finite. Henceforth, $\varphi(x)$ is even, continuous, strictly monotonically increasing on $[0,+\infty)$ function, such that $\varphi(0)=0$ and

$$\varphi(2x) \le C_\varphi \cdot \varphi(x), \ x \ge 0 \tag{1}$$

for some positive constant C_φ. The condition (1) is quite natural. In particular, it provides linearity of the class $\varphi(L)$.

For $f(x) \in \varphi(L)$ we define its modulus of continuity

$$\omega(f,\delta)_\varphi = \sup_{0\le h\le\delta} \|f(x+h)-f(x)\|_\varphi, \ \delta \ge 0,$$

and its best approximation

$$E_n(f)_\varphi = \inf_{T_n} \|f-T_n\|_\varphi, \ n = 0,1,2,\ldots$$

by trigonometric polynomials T_n of order at most n. If $\varphi(x) = |x|^p$ for a certain $p > 0$, the Jackson type inequality holds, that is,

$$E_n(f)_p \leq C_p \cdot \omega\left(f, \frac{1}{n+1}\right)_p , \; n = 0, 1, \ldots , \; f \in L_p , \qquad (2)$$

where the positive constant C_p does not depend on f and n.

Before we formulate the main result of this paper we introduce a concept of non-localized system. Let e be a measurable subset of real axis with positive Lebesgue measure $\mu(e)$. As usual, we denote by symbol $L_\infty(e)$ a space of essentially bounded measurable functions equipped with the norm

$$\|f\|_{L_\infty(e)} = \operatorname{ess\,sup}|f(x)|, \; f \in L_\infty(e) .$$

Definition. *A system $\Omega = \{\omega_n\}_{n=1}^{+\infty}$ of functions in L_∞ is non-localized, if there exist $\delta \in (0, \pi)$ and a sequence of positive numbers $\{a_n\}_{n=1}^{+\infty}$, such that*

$$\|f\|_\infty \leq a_n \cdot \inf\{\|f\|_{L_\infty(e)} : e \subset [0, 2\pi), \; \mu(e) > 2(\pi - \delta)\} , \qquad (3)$$

for all $f \in \Omega_n = span\{\omega_1, \ldots, \omega_n\}$ and $n \in \mathbb{N}$.

Clearly, Haar system does not satisfy this definition. Afterwards we will prove that the trigonometric system does.

Theorem. *Let $\varphi(x)$ satisfy the conditions above and Ω be a non-localized system. If*

$$(A) \qquad x^p = O(\varphi(x)), \; x \to +0 \;\; \text{for every} \;\; p > 0$$

or

$$(B) \qquad \varphi(x) = O(x^p), \; x \to +\infty \;\; \text{for every} \;\; p > 0 ,$$

then for each sequence of positive numbers $\{\sigma_n\}_{n=1}^{+\infty}$ that converges to 0 and for each positive constant C there exist $f(x)$ in $\varphi(L)$ and $n \in \mathbb{N}$, such that

$$E_n(f)_{\varphi;\Omega} > C\omega(f, \sigma_n)_\varphi .$$

Here

$$E_n(f)_{\varphi;\Omega} = \inf_{g \in \Omega_n} \|f - g\|_{\varphi} \ , n \in \mathbb{N} \ ,$$

is a best approximation to f in $\varphi(L)$ by polynomials with respect to the system Ω and as usual, "$g(x) = O(h(x)),\ x \to +0\ (x \to +\infty)$" means that there exist a positive constant C and $X > 0$, such that $|g(x)| \leq C|h(x)|$ for $x \in (0, X)\ ((X, +\infty))$.

3 Proofs

A proof of the Theorem is based on two Lemmas. We consider the function of "getting-out a constant" given by

$$\Psi_{\varphi}(\lambda) = \sup_{x>0} \frac{\varphi(\lambda x)}{\varphi(x)},\ \lambda \in \mathbb{R}\ . \tag{4}$$

Because of (1) $\Psi_{\varphi}(\lambda)$ is well-defined. Clearly, it is monotonically increasing on $[0, \infty)$, even and $\Psi_{\varphi}(0) = 0$.

Lemma 1. *If $\varphi(x)$ satisfies (A) or (B), $\Psi_{\varphi}(\lambda) = 1$ for $\lambda \in (0, 1]$.*

Proof. First we consider the operator S defined on a set of positive on $[0, +\infty)$ functions by

$$S:\ g(x) \to (g(x^{-1}))^{-1}$$

and we notice that

$$\Psi_{S\varphi}(\lambda) = \Psi_{\varphi}(\lambda),\ \lambda \in \mathbb{R}\ .$$

Moreover, $S = S^{-1}$ and condition (A) for $\varphi(x)$ is equivalent to condition (B) for $S\varphi(x)$. Therefore, it is sufficient to prove Lemma 1 only for one of them.

Let, for example, $\varphi(x)$ satisfy (B). Then for each $p > 0$ there exist $C_p > 0$ and $X_p > 0$, such that

$$\varphi(x) \leq C_p \cdot x^{p/2},\ x \in (X_p, +\infty)$$

and

$$\frac{\ln \varphi(x)}{\ln x} \le \frac{p}{2} + \frac{\ln C_p}{\ln x},\ x \in (X_p, +\infty)\ .$$

Therefore,

$$\lim_{x\to+\infty} \varepsilon(x) = 0, \quad \left(\varepsilon(x) = \frac{\ln \varphi(x)}{\ln x}\right)\ . \tag{5}$$

As $\varepsilon(x)$ is positive and continuous on $[2, +\infty)$, we have from (5) that $\varepsilon_0 = \sup_{x\in[2,+\infty)} \varepsilon(x) \in (0, +\infty)$. We set

$$\mathcal{A}_n = \{x \in [2, +\infty) :\ \varepsilon(x) \ge n^{-1}\},\ n \ge n_1\ ,$$

where $n_1 = [\varepsilon_0^{-1}] + 1$. Clearly, $\mathcal{A}_n$ are not empty. For each $\lambda \in (0, 1)$ we consider the sequence

$$t_n \equiv t_{n;\lambda} = \lambda^{-1} \cdot \sup\mathcal{A}_n,\ n \ge n_1\ .$$

Using (5), we have $t_n < +\infty,\ n \ge n_1$. Moreover,

$$1)\ \lim_{n\to+\infty} t_n = +\infty; \quad 2)\ \ \varepsilon(\lambda t_n) > \varepsilon(t_n),\ n \ge n_1\ .$$

Indeed, if $\lim_{n\to+\infty} t_n = t_0 < +\infty$, any $x \in (t_0, +\infty)$ does not belong to $\mathcal{A}_n$ for all $n \ge n_1$ and, therefore, $\varepsilon(x) = 0$ and $\varphi(x) = 1$ in $[t_0, +\infty)$. To prove 2), we notice that $\varepsilon(\lambda t_n) = n^{-1}$ and $t_n > \lambda t_n = \sup \mathcal{A}_n$; therefore, $t_n \notin \mathcal{A}_n$ and $\varepsilon(t_n) < n^{-1} = \varepsilon(\lambda t_n)$.

Using 1), 2) and (5), we obtain

$$\begin{aligned}
\Psi_\varphi(\lambda) &= \sup_{x>0} \frac{\varphi(\lambda x)}{\varphi(x)} \ge \sup_n \frac{\varphi(\lambda t_n)}{\varphi(t_n)} = \sup_n \exp\{\ln \varphi(\lambda t_n) - \ln \varphi(t_n)\} \\
&= \sup_n (\exp\{\ln t_n \cdot (\varepsilon(\lambda t_n) - \varepsilon(t_n))\} \cdot \exp\{\ln \lambda \cdot \varepsilon(\lambda t_n)\}) \\
&\ge \sup_n \exp\{\ln \lambda \cdot \varepsilon(\lambda t_n)\} \ge \lim_{n\to+\infty} \exp\{\ln \lambda \cdot \varepsilon(\lambda t_n)\} = 1\ .
\end{aligned}$$

The upper estimate is obvious.

The proof of Lemma 1 is complete.

Lemma 2. *Let Ω be a non-localized system. If there exist a sequence of positive numbers $\{\sigma_n\}_{n=1}^{+\infty}$ that converges to 0 and a positive constant C, such that*

$$E_n(f)_{\varphi;\Omega} \leq C\omega(f, \sigma_n)_\varphi \tag{6}$$

for all $f \in \varphi(L)$ and $n \in N$, then

$$\lim_{\lambda \to +0} \Psi_\varphi(\lambda) = 0 \ .$$

Proof. Let $\Omega = \{\omega_n\}_{n=1}^{+\infty}$ be a non-localized system and $\delta > 0$ as in (3). For each $\tau > 0$ we consider a 2π-periodic function $f_\tau(x)$ that is equal to 0 for $x \in [0, 2\pi - \delta]$ and is equal to τ for $(2\pi - \delta, 2\pi)$. Let n_1 be a natural number, such that $\sigma_n < \delta$ for $n \leq n_1$. Clearly,

$$\omega(f_\tau, \sigma_n)_\varphi = 2\varphi(\tau)\sigma_n, \ n \geq n_1 \ . \tag{7}$$

Actually,

$$\begin{aligned}
\omega(f_\tau, \sigma_n)_\varphi &= \sup_{0 \leq h \leq \sigma_n} \int_0^{2\pi} \varphi(f_\tau(x+h) - f_\tau(x))dx \\
&= \sup_{0 \leq h \leq \sigma_n} \left\{ \int_0^{2\pi-\delta-h} + \int_{2\pi-\delta-h}^{2\pi-\delta} + \int_{2\pi-\delta}^{2\pi-h} + \int_{2\pi-h}^{2\pi} \right\} \\
&= \sup_{0 \leq h \leq \sigma_n} 2h\varphi(\tau) = 2\varphi(\tau)\sigma_n \ .
\end{aligned}$$

We choose $g_{n;\tau}(x) \in \Omega_n$, such that

$$\|f_\tau - g_{n;\tau}\|_\varphi \leq E_n(f_\tau)_{\varphi;\Omega} + C\omega(f, \sigma_n)_\varphi, \ n \geq n_1 \ .$$

Then we have from (6) and (7)

$$\|f_\tau - g_{n;\tau}\|_\varphi \leq 4C\sigma_n\varphi(\tau), \ n \geq n_1 \ . \tag{8}$$

We set

$$\mathcal{E} \equiv \mathcal{E}_{n;\tau} = \{x \in [0, 2\pi - \delta) : \varphi(g_{n;\tau}(x)) \leq 4C\delta^{-1}\sigma_n\varphi(\tau)\}, \ n \geq n_1, \ \tau > 0 \, .$$

To estimate their measures we use Chebyshev inequality and (8)

$$\begin{aligned}\mu(\mathcal{E}) &= 2\pi-\delta-\mu\{x\in[0,2\pi-\delta):\ \varphi(g_{n;\tau}(x))>4C\delta^{-1}\sigma_n\varphi(\tau)\}\\ &\geq 2\pi-\delta-(4C\delta^{-1}\sigma_n\varphi(\tau))^{-1}\cdot\int\limits_0^{2\pi-\delta}\varphi(g_{n;\tau}(x))dx\\ &\geq 2\pi-\delta-(4C\delta^{-1}\sigma_n\varphi(\tau))^{-1}\cdot\|f_\tau-g_{n;\tau}\|_\varphi\geq 2(\pi-\delta)\ .\end{aligned}$$

Therefore

$$\|g_{n;\tau}\|_\infty\leq a_n\cdot\|g_{n;\tau}\|_{L_\infty(\mathcal{E})}\leq\gamma_{n;\tau}\ ,\ n\geq n_1,\ \tau>0\ , \tag{9}$$

where

$$\gamma_{n;\tau}=a_n\cdot\varphi^{-1}(4C\delta^{-1}\sigma_n\varphi(\tau))$$

Without loss of generality we can assume that $\{a_n\}_{n=1}^{+\infty}$ tends to $+\infty$.

Let n_2 be a natural number, such that

$$\sigma_n^{-1}>4C\delta^{-1}\Psi(2)\geq\frac{4C\delta^{-1}\varphi(\tau)}{\varphi(\frac{\tau}{2})}$$

for all $n\geq n_2$ and $\tau>0$. Let also $n_3=\max\{n_1,n_2\}$. We will prove that

$$\tau\leq 2\gamma_{n;\tau} \tag{10}$$

for $n\geq n_3$ and $\tau>0$. Indeed, otherwise, by virtue of (9) we get for some $n\geq n_3$

$$\tau-g_{n;\tau}(x)\geq\tau-|g_{n;\tau}(x)|\geq\tau-\gamma_{n;\tau}>\frac{\tau}{2}$$

almost everywhere in $[0,2\pi)$. Furthermore,

$$\|f_\tau-g_{n;\tau}\|_\varphi\geq\int\limits_{2\pi-\delta}^{2\pi}\varphi(\tau-g_{n;\tau}(x))dx\geq\delta\cdot\varphi\left(\frac{\tau}{2}\right)>4C\sigma_n\varphi(\tau),$$

that is in contradiction with (8).

We rewrite (10) as follows:

$$\tau \le 2a_n \varphi^{-1}(4C\delta^{-1}\sigma_n \varphi(\tau));$$

$$\frac{\varphi((2a_n)^{-1}\tau)}{\varphi(\tau)} \le 4C\delta^{-1}\sigma_n,\ \tau > 0,\ n \ge n_3\ . \tag{11}$$

As the right-hand side of (11) does not depend on τ, we get

$$\Psi_\varphi((2a_n)^{-1}) \le 4C\delta^{-1}\sigma_n,\ n \ge n_3\ .$$

Thus, there exists a sequence $\lambda_n = (2a_n)^{-1},\ n \ge n_3,$ that converges to 0 and $\lim_{n\to+\infty} \Psi_\varphi(\lambda_n) = 0$. As $\Psi_\varphi(\lambda)$ increases on $[0,+\infty)$, we have finally

$$\lim_{\lambda\to+0} \Psi_\varphi(\lambda) = 0\ .$$

Lemma 2 is proved.

Proof of Theorem. Theorem follows immediately from Lemmas 1 and 2.

4 Non-locality of systems of analytic functions

Now we prove that a system of 2π-periodic analytic functions is non-localized. Without loss of generality we can assume that it is linearly independent. We consider the functions

$$\Phi(\bar\lambda) = \inf\{\|\lambda_1\omega_1 + \ldots + \lambda_n\omega_n\|_{L_\infty(e)} :\ e \subset [0,2\pi),\ \mu(e) \ge \pi\}\ ;$$

$$F(\bar\lambda) = \|\lambda_1\omega_1 + \ldots + \lambda_n\omega_n\|_\infty,\ \bar\lambda = (\lambda_1 \ldots \lambda_n) \in \mathbb{R}^n\ .$$

It is easy to see that they are continuous on the sphere $\mathcal{S}^{n-1} = \{\bar\lambda \in \mathbb{R}^n : \lambda_1^2 + \ldots + \lambda_n^2 = 1\}$.

Furthermore,

$$\Phi(\bar\lambda) \ne 0,\ \bar\lambda \in \mathcal{S}^{n-1}\ .$$

Indeed, if $\Phi(\bar{\lambda}) = 0$ for a certain $\bar{\lambda} \in \mathcal{S}^{n-1}$, there exists a sequence of measurable sets $\{e_m\}_{m=1}^{+\infty}$, such that $e_m \subset [0, 2\pi)$, $\mu(e_m) \geq \pi$ and

$$\|g\|_{L_\infty(e_m)} \leq m^{-1}, \; m \in \mathbb{N} ,$$

where $g \equiv \lambda_1 \omega_1 + \ldots + \lambda_n \omega_n$. Let

$$\mathcal{E}_m = \{x \in [0, 2\pi) : \; |g(x)| \leq m^{-1}\}, \; m \in \mathbb{N} .$$

Clearly,

$$\mathcal{E}_m \supseteq e_m, \; m \in \mathbb{N}; \; \mathcal{E}_1 \supseteq \mathcal{E}_2 \supseteq \ldots \supseteq \mathcal{E}_m \supseteq \ldots$$

Hence,

$$\mu(\mathcal{E}) = \lim_{m \to +\infty} \mu(\mathcal{E}_m) \geq \lim_{m \to +\infty} \mu(e_m) \geq \pi, \; \left(\mathcal{E} = \bigcap_{m=1}^{+\infty} \mathcal{E}_m \right) .$$

As $g(x) = 0, \; x \in \mathcal{E}$, we have by the uniqueness theorem for analytic functions that $g(x) = 0$ and, therefore, $\bar{\lambda} = 0$, that is in contradiction with the condition: $\bar{\lambda} \in S^{n-1}$.

By virtue of (12) the function $\frac{F(\bar{\lambda})}{\Phi(\bar{\lambda})}$ is continuous on $\mathcal{S}^{n-1}$. As it is homogenous, there exists a sequence of positive numbers $\{a_n\}_{n=1}^{+\infty}$, such that

$$F(\bar{\lambda}) \leq a_n \Phi(\bar{\lambda}), \quad \bar{\lambda} \in \mathbb{R}^n .$$

5 Remarks

1. As it follows from our research, the class of functions $\varphi(x)$, for which Jackson type inequality is valid, is being practically exhausted by power functions. It means that there does not exist a "bound" function as it has been believed. However, it should be noted, that the powers near 0 and $+\infty$ can be different from each other.

2. As is known, if $\varphi_1(x) = \varphi_2(x)$ for some $x_0 > 0$, then $\varphi_1(L) = \varphi_2(L)$ and convergences in these classes are equivalent to each other. It is obvious that if we change $\varphi(x)$ in any segment $[x_1, x_2]$, where $0 < x_1 < x_2 < +\infty$, this does not affect the validity of Jackson type inequality. However, in difference from the topological properties of Orlicz classes it depends essentially on behaviour of $\varphi(x)$ near 0. Besides, the contributions of 0 and $+\infty$ turn out to be similar.

3. It can be shown by the method developed in [3] that a converse result to Lemma 2 is valid, if $\varphi(x+y) \leq \varphi(x) + \varphi(y)$ for $x, y \geq 0$. However, we do not regard it to be very important, because as it follows from Lemma 1, the function of "getting-out a constant" is equal to 1, if $\varphi(x)$ is slower than any power function near 0 or $+\infty$. Moreover, if $\varphi(x)$ is a power function, $\Psi_\varphi(x) = \varphi(x)$. Thus, the situation: $\lim\limits_{\lambda \to +0} \Psi_\varphi(\lambda) = 0$ turns out to be interesting only for functions, which have "abnormal" behaviour.

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