

# LOW-DIMENSIONAL FILIFORM LIE SUPERALGEBRAS

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## Abstract

The aim of this paper is to give a classification up to isomorphism of low dimension filiform Lie superalgebras.

## 1 Introduction

There exists a lot of work concerning Lie superalgebras. But less of them are interested in nilpotent Lie superalgebras, although, the case of nilpotent Lie algebras has been well studied, see for example [1]. This fact is a consequence of the development of deformation theory. In this paper, we will focus a particular class of nilpotent Lie superalgebras. We recall some definitions from [3]. We point out the definition of filiform Lie superalgebras. This definition is analogue to the definition of filiform Lie algebras given by Vranceanu [10] and Vergne [9]. The classification of this super-algebras is still a problem, as the classification of filiform Lie algebras over an algebraically closed field of characteristic zero is only done up to dimension 11 [4]. In this paper, we will give a first classification for filiform Lie superalgebras in low dimensions.

## 2 Filiform Lie superalgebras

A  $\mathbb{Z}_2$ -graded vector space  $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$  over an algebraically closed field is a Lie superalgebra if there exists a bilinear product  $[\cdot, \cdot]$  over  $\mathcal{G}$  such that

$$\begin{aligned} [G_\alpha, G_\beta] &\subset \mathcal{G}_{\alpha+\beta \pmod 2}, \\ [g_\alpha, g_\beta] &= -(-1)^{\alpha\beta} [g_\beta, g_\alpha] \end{aligned}$$

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for all  $g_\alpha \in \mathcal{G}_\alpha$  and  $g_\beta \in \mathcal{G}_\beta$  and satisfying Jacobi's super-relation:

$$(-1)^{\gamma\alpha}[A, [B, C]] + (-1)^{\alpha\beta}[B, [C, A]] + (-1)^{\beta\gamma}[C, [A, B]] = 0$$

for all  $A \in \mathcal{G}_\alpha$ ,  $B \in \mathcal{G}_\beta$  and  $C \in \mathcal{G}_\gamma$ .

For such a Lie superalgebra we define the lower central series by

$$\begin{cases} C^0(\mathcal{G}) = \mathcal{G}, \\ C^{i+1}(\mathcal{G}) = [\mathcal{G}, C^i(\mathcal{G})] \end{cases}$$

A Lie superalgebra  $\mathcal{G}$  is nilpotent if there exists an integer  $n$  such that  $C^n(\mathcal{G}) = \{0\}$ .

This definition is not easy to use. Therefore we define the following sequences for a Lie superalgebra  $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ :

$$C^0(\mathcal{G}_0) = \mathcal{G}_0, \quad C^{i+1}(\mathcal{G}_0) = [\mathcal{G}_0, C^i(\mathcal{G}_0)]$$

and

$$C^0(\mathcal{G}_1) = \mathcal{G}_1, \quad C^{i+1}(\mathcal{G}_1) = [\mathcal{G}_0, C^i(\mathcal{G}_1)]$$

**Theorem 2.1.** *Let  $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$  be a Lie superalgebras. Then  $\mathcal{G}$  is nilpotent if and only if there exist  $(p, q)$  such that  $C^p(\mathcal{G}_0) = \{0\}$  and  $C^q(\mathcal{G}_1) = \{0\}$ .*

**Proof.** If the Lie superalgebra  $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$  is nilpotent it is easy to prove that there exist  $(p, q)$  such that  $C^p(\mathcal{G}_0) = \{0\}$  and  $C^q(\mathcal{G}_1) = \{0\}$ .

For the converse, we use the classical Engel's theorem. Let  $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$  be a Lie superalgebra. Assume that there exist  $(p, q)$  such that  $C^p(\mathcal{G}_0) = \{0\}$  and  $C^q(\mathcal{G}_1) = \{0\}$ , then  $ad(X)$  with  $X \in \mathcal{G}_0$  is nilpotent. We have for  $Y \in \mathcal{G}_1$ :

$$ad(Y) \circ ad(Y) = \frac{1}{2}ad([Y, Y])$$

As  $[Y, Y]$  is an element of  $\mathcal{G}_0$ , then  $ad([Y, Y])$  is nilpotent. This implies that  $ad(Y)$  is nilpotent for every  $Y \in \mathcal{G}_1$ . Hence  $ad(X)$  and  $ad(Y)$  is nilpotent, using Engel's theorem given in [6],  $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$  is a nilpotent Lie superalgebra. ■

**Definition 2.1.** *Let  $\mathcal{G}$  be a nilpotent Lie superalgebra, the super-nilindex of  $\mathcal{G}$  is the pair  $(p, q)$  such that:  $C^p(\mathcal{G}_0) = \{0\}$ ,  $C^{p-1}(\mathcal{G}_0) \neq \{0\}$  and  $C^q(\mathcal{G}_1) = \{0\}$ ,  $C^{q-1}(\mathcal{G}_1) \neq \{0\}$ . It is and invariant up to isomorphism.*

**Definition 2.2 (Filiform Lie superalgebras).** *Let  $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$  be a nilpotent Lie superalgebra with  $\dim \mathcal{G}_0 = n + 1$  and  $\dim \mathcal{G}_1 = m$ .  $\mathcal{G}$  is called filiform if its super-nilindex is  $(n, m)$ . We will note  $\mathcal{F}_{n,m}$  the set of filiform Lie superalgebras.*

Let's define the set  $\mathcal{N}_{n,m}^{p,q}$  of Lie superalgebras with  $\dim \mathcal{G}_0 = n + 1$ ,  $\dim \mathcal{G}_1 = m$  and with super-nilindex  $(k_0, k_1)$  such that  $k_0 \leq p$  and  $k_1 \leq q$ . It is obvious that this set can be define by polynomial relations given by the Jacobi relations and the nilpotency relations. This prove that it is a Zariski-closed set of the algebraic variety of nilpotent Lie superalgebras denoted by  $\mathcal{N}_{n,m}$ .

The set  $\mathcal{F}_{n,m}$  of filiform Lie superalgebras can be written as the complementary of a close set:

$$\mathcal{F}_{n,m} = \mathcal{N}_{n,m} \setminus (\mathcal{N}_{n,m}^{n-1,m} \cup \mathcal{N}_{n,m}^{n,m-1})$$

Hence the set of filiform Lie superalgebras is an open set of the variety of nilpotent Lie superalgebras (see [5]).

### 3 Classifications of filiforms over $\mathbb{C}$ in low dimensions

#### 3.1 Adapted basis

Like for the filiform Lie algebras [9], there exists an adapted bases of a filiform Lie superalgebra:

**Proposition 3.1.** *Let  $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$  be a filiform Lie superalgebra with  $\dim \mathcal{G}_0 = n + 1$  and  $\dim \mathcal{G}_1 = m$ . Then there exists a bases  $\{X_0, X_1, \dots, X_n, Y_1, Y_2, \dots, Y_m\}$  of  $\mathcal{G}$  with  $X_i \in \mathcal{G}_0$  and  $Y_i \in \mathcal{G}_1$  such that:*

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n - 1, & [X_0, X_n] = 0; \\ [X_1, X_2] \in \mathbb{C}.X_4 + \mathbb{C}.X_5 + \dots + \mathbb{C}.X_n; \\ [X_0, Y_i] = Y_{i+1} & 1 \leq i \leq m - 1, & [X_0, Y_m] = 0. \end{cases}$$

*Such basis are called adapted.*

**Proof.** Let  $gr\mathcal{G} \in \mathcal{F}_{n,m}$  be a graded filiform lie superalgebra. The lower central series implies the following graduations:

$$\begin{aligned} gr\mathcal{G}_0 &= W_1^0 \oplus W_2^0 \oplus \dots \oplus W_{n-1}^0 \\ gr\mathcal{G}_1 &= W_1^1 \oplus W_2^1 \oplus \dots \oplus W_{m-1}^1 \end{aligned}$$

with  $\dim W_1^0 = 2$ ,  $\dim W_i^0 = 1$  for  $2 \leq i \leq n - 1$  and  $\dim W_j^1 = 1$  for  $1 \leq j \leq m - 1$ . Then we have  $[W_1^0, W_i^0] = W_{i+1}^0$  and  $[W_1^0, W_j^1] = W_{j+1}^1$ .

Let  $x_i \in W_i^0$  and  $y_j \in W_j^1$  be non zero elements, then if  $w \in W_1^0$ ,

$$\begin{aligned} [w, x_i] &= \lambda_i^0(w) x_{i+1} \\ [w, y_j] &= \lambda_j^1(w) y_{j+1} \end{aligned}$$

the maps  $w \rightarrow \lambda_i^0(w)$  and  $w \rightarrow \lambda_j^1(w)$  are non zero linear forms on  $W_1^0$ , because  $[W_1^0, W_i^0] = W_{i+1}^0$  and  $[W_1^0, W_j^1] = W_{j+1}^1$ .

As the field is  $\mathbb{C}$ , there exists  $x_0$  in  $W_1^0$  such that  $\lambda_i^0(w) \neq 0$  for  $1 \leq i \leq n - 1$  and  $\lambda_j^1(w) \neq 0$  for  $1 \leq j \leq m - 1$ . Hence it exist a bases  $X_0, X_1, \dots, X_n$  and  $Y_1, Y_2, \dots, Y_m$  with  $X_i$  ( resp.  $Y_j$ ) multiple of  $x_i$  (resp.  $y_j$ ) such that:

$$\begin{aligned} [X_0, X_i] &= X_{i+1} \text{ for } 1 \leq i \leq n - 1 \\ [X_0, Y_j] &= Y_{j+1} \text{ for } 1 \leq j \leq m - 1 \\ [X_1, X_2] &= a X_3 \text{ for } a \in \mathbb{C} \end{aligned}$$

Substitute  $X_1$  by  $X_1 - a X_0$  we get the adapted bases of  $gr\mathcal{G}$ :

$$\begin{aligned} [X_0, X_i] &= X_{i+1} \text{ for } 1 \leq i \leq n - 1 \\ [X_0, Y_j] &= Y_{j+1} \text{ for } 1 \leq j \leq m - 1 \\ [X_1, X_2] &= 0 \end{aligned}$$

If the Lie superalgebras is not graded, then we introduce a graded Lie superalgebra, as it was done for Lie algebras in [9]. ■

### 3.2 Adapted changes of basis

**Definition 3.1.** *Let  $f$  be a graded change of bases of a filiform Lie superalgebra  $\mathcal{G}$ .  $f$  is an adapted changes of bases if  $f$  is Lie superalgebra homomorphism and if the image of an adapted bases of  $\mathcal{G}$  is an adapted bases.*

**Proposition 3.2.** *Let  $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$  be a filiform Lie superalgebra. Let  $f = f_0 + f_1$  be an adapted change of basis of  $\mathcal{G}$ . Then  $f_0$  is an adapted change of bases of the filiform Lie algebra  $\mathcal{G}_0$  and  $f_1$  satisfies:*

$$\begin{cases} f_1(Y_1) = d_1 Y_1 + d_2 Y_2 + \dots + d_m Y_m \\ f_1(Y_i) = [f_0(X_0), f_1(Y_{i-1})] \quad 2 \leq i \leq m \end{cases}$$

with the condition:

$$d_1 \prod_{j=1}^{m-1} A_j \neq 0$$

where the  $A_j$ 's are defined by:

$$A_j = a_0 + \sum_{k=0}^{j-1} (-1)^k C_{j-1}^k \left( \sum_{i=1}^{n-k} a_i \cdot r_{i+k,2+k} \right)$$

and the product of  $\mathcal{G}$  is given in the adapted bases by

$$[X_i, Y_1] = \sum_{s=2}^m r_{i,s} Y_s \quad 1 \leq i \leq n$$

**Proof.** The proof can be found in [3]. We give a sketch of it.

First, we establish that  $f_0$  is an adapted change of bases for Lie algebras as in [4]. Then we set:

$$\begin{cases} f_0(X_0) = a_0 X_0 + a_1 X_1 + \dots + a_n X_n \\ f_1(Y_1) = d_1 Y_1 + d_2 Y_2 + \dots + d_m Y_m \end{cases}$$

and assuming  $f(Y_t) = [f(X_0), f(Y_{t-1})]$  for  $2 \leq t \leq m$  we prove by induction on  $t$  that:

$$f(Y_t) = d_1 \prod_{p=1}^{t-1} A_p \cdot Y_t + \sum_{p \geq t+1} d_p \cdot Y_p$$

This implies that  $f_1(Y_m) = d_1 \prod_{p=1}^{m-1} A_p Y_m$ . For this vector to be non zero, we must have  $d_1 \prod_{p=1}^{m-1} A_p \neq 0$ . This proves that

$$f(Y_t) = d_1 \prod_{p=1}^{t-1} A_p \cdot Y_t + \sum_{p \geq t+1} d_p \cdot Y_p$$

start with a non zero component on  $Y_t$ , and then the images of the vectors  $Y_t$  by  $f$  form a bases.

Conversely, let  $f_0$  be an adapted change of bases of filiform Lie algebras. Then the map  $f = f_0 + f_1$ , with  $f_1$  given by:

$$\begin{cases} f_1(Y_1) = d_1 Y_1 + d_2 Y_2 + \dots + d_m Y_m \\ f_1(Y_i) = [f_0(X_0), f_1(Y_{i-1})] \quad 2 \leq i \leq m \end{cases}$$

and the condition:

$$d_1 \prod_{j=1}^{m-1} A_j \neq 0$$

is an adapted change of basis. ■

### 3.3 Classification in low dimensions

We have a description of the products of the filiform Lie superalgebras on  $\mathcal{F}_{1,m}$ . There exists two types:

(1)

$$\begin{cases} [X_0, Y_i] = Y_{i+1}, & 1 \leq i \leq m - 1 \\ [Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}} X_1, & \text{if } i + j \text{ even and } 2 \leq i + j \leq m + 1 \end{cases}$$

the other products vanish.

(2)

$$\begin{cases} [X_0, Y_i] = Y_{i+1}, & 1 \leq i \leq m - 1 \\ [X_1, Y_r] = \sum_{s=2}^m r_s Y_{s+r-1} & \text{with } s + r - 1 \leq m \end{cases}$$

the other products vanish.

Using adapted changes of bases, like it was done for Lie algebras in [4], we can eliminate the parameters  $a_{\frac{i+j}{2}}$ .

**Theorem 3.1.** *Every filiform Lie superalgebra of  $\mathcal{F}_{1,m}$  which is not a Lie algebra is isomorphic to one of the following filiform Lie superalgebras:*

$$\begin{aligned} [X_0, Y_i] &= Y_{i+1}, & 1 \leq i \leq m - 1 \\ [Y_i, Y_{2k-i}] &= (-1)^{k-i} X_1 & 1 \leq i \leq k \end{aligned}$$

with  $1 \leq k \leq z + 1$  if  $m = 2z + 1$  and  $1 \leq k \leq z$  if  $m = 2z$ .

**Proof.** We can assume that every non trivial filiform Lie superalgebra with  $1 \leq p \leq k - 1$  is isomorphic to:

$$\begin{aligned} [X_0, Y_i] &= Y_{i+1}, & 1 \leq i \leq m - 1 \\ [Y_i, Y_{2k-i}] &= (-1)^{k-i} X_1 & 1 \leq i \leq k \\ [Y_i, Y_j] &= (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}} X_1, & \text{if } i + j \text{ even and } 2 \leq i + j \leq 2(k - p) \end{aligned}$$

for  $1 \leq p \leq k - 1$ . The other products vanish.

Using this change of adapted bases:

$$\begin{cases} X_0^1 = X_0 \\ X_1^1 = X_1 \\ Y_1^1 = Y_1 - (-1)^p \frac{a_{k-1}}{2} Y_{2p+1} \\ \dots \\ Y_{k-p}^1 = Y_{k-p} - (-1)^p \frac{a_{k-1}}{2} Y_{k+p} \\ Y_{k-p+s}^1 = Y_{k-p+s} - (-1)^p \frac{a_{k-1}}{2} Y_{k+p+s} \end{cases}$$

we can reduce to

$$\begin{cases} [X_0, Y_i] = Y_{i+1}, & 1 \leq i \leq m - 1 \\ [Y_i, Y_{2k-i}] = (-1)^{k-i} X_1 & 1 \leq i \leq k \\ [Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}} X_1, & \text{if } i + j \text{ even and } 2 \leq i + j \leq 2(k - p - 1) \end{cases}$$

The other products vanish.

By induction on  $p$ , we have a complete classification. ■

By using adapted changes of bases, we established the following classifications:

$\mathcal{F}_{1,2}$ :

(1)  $\{ [X_0, Y_1] = Y_2$

(2)  $\begin{cases} [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$

(3)  $\begin{cases} [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_1 \end{cases}$

$\mathcal{F}_{1,3}$ :

(1)  $\{ [X_0, Y_1] = Y_2, [X_0, Y_2] = Y_3$

(2)  $\begin{cases} [X_0, Y_1] = Y_2, [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_1 \end{cases}$

$$(3) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [Y_2, Y_2] = X_1, & [Y_1, Y_3] = -X_1 \end{cases}$$

$$(4) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, & [X_1, Y_2] = Y_3 \end{cases}$$

$$(5) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_3 \end{cases}$$

$$(6) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2 + Y_3, & [X_1, Y_2] = Y_3 \end{cases}$$

$\mathcal{F}_{1,4}$ :

$$(1) \{ [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = Y_4$$

$$(2) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, & [X_0, Y_3] = Y_4 \\ [Y_1, Y_1] = X_1 \end{cases}$$

$$(3) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, & [X_0, Y_3] = Y_4 \\ [Y_2, Y_2] = X_1, & [Y_1, Y_3] = -X_1 \end{cases}$$

$$(4) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, & [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_2, & [X_1, Y_2] = Y_3, & [X_1, Y_3] = Y_4 \end{cases}$$

$$(5) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, & [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_3, & [X_1, Y_2] = Y_4 \end{cases}$$

$$(6) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, & [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_4 \end{cases}$$

$$(7) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, & [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_2 + Y_3, & [X_1, Y_2] = Y_3 + Y_4, & [X_1, Y_3] = Y_4 \end{cases}$$

$$(8) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, & [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_2 + Y_4, & [X_1, Y_2] = Y_3, & [X_1, Y_3] = Y_4 \end{cases}$$

$$(9) \begin{cases} [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3, & [X_0, Y_3] = Y_4 \\ [X_1, Y_1] = Y_2 + Y_3 + 2 Y_4, & [X_1, Y_2] = Y_3 + Y_4, & [X_1, Y_3] = Y_4 \end{cases}$$

$\mathcal{F}_{2,2}$ :

$$(1) \{ [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2$$



$$(2) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_1, & [Y_1, Y_2] = \frac{1}{2}X_2 \end{cases}$$

$$(3) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_2 \end{cases}$$

$$(4) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$$

$$(5) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, & [Y_1, Y_1] = X_2 \end{cases}$$

$\mathcal{F}_{2,3}$ :

$$(1) \{ [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3$$

$$(2) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, & [X_1, Y_2] = Y_3 \end{cases}$$

$$(3) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_3 \end{cases}$$

$$(4) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_2] = -Y_3, & [X_2, Y_1] = Y_3 \end{cases}$$

$$(5) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2 + Y_3, & [X_1, Y_2] = Y_3 \end{cases}$$

$$(6) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, & [X_2, Y_1] = Y_3 \end{cases}$$

$$(7) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = 2Y_2, & [X_1, Y_2] = Y_3, & [X_2, Y_1] = Y_3 \end{cases}$$

$$(8) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_2 \end{cases}$$

$$(9) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, & [X_1, Y_2] = Y_3 \\ [Y_1, Y_1] = X_2 \end{cases}$$

$$(10) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_3 \\ [Y_1, Y_1] = X_2 \end{cases}$$

$$(11) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2 + Y_3, & [X_1, Y_2] = Y_3 \\ [Y_1, Y_1] = X_2 \end{cases}$$

$$(12) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [Y_1, Y_3] = -X_2, & [Y_2, Y_2] = X_2 \end{cases}$$

$$(13) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, & [X_1, Y_2] = Y_3 \\ [Y_1, Y_3] = -X_2, & [Y_2, Y_2] = X_2 \end{cases}$$

$$(14) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_1, & [Y_1, Y_2] = \frac{1}{2}X_2 \end{cases}$$

$$(15) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [X_1, Y_2] = -Y_3, & [X_2, Y_1] = Y_3 \\ [Y_1, Y_1] = X_1, & [Y_1, Y_2] = \frac{1}{2}X_2 \end{cases}$$

$$(16) \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2, & [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_1, & [Y_1, Y_2] = \frac{1}{2}X_2, & [Y_1, Y_3] = -X_2, & [Y_2, Y_2] = X_2 \end{cases}$$

$\mathcal{F}_{3,2}$ :

$$(1) \{ [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, Y_1] = Y_2$$

$$(2) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_1, & [Y_1, Y_2] = \frac{1}{2}X_2, & [Y_2, Y_2] = \frac{1}{2}X_3 \end{cases}$$

$$(3) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_2, & [Y_1, Y_2] = \frac{1}{2}X_3 \end{cases}$$

$$(4) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_3 \end{cases}$$

$$(5) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$$

$$(6) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, & [Y_1, Y_1] = X_3 \end{cases}$$

$\mathcal{F}_{4,2}$ :

$$(1) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \end{cases}$$

$$(2) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_2, & [Y_1, Y_2] = \frac{1}{2}X_3, & [Y_2, Y_2] = \frac{1}{2}X_4 \end{cases}$$

$$(3) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_3, & [Y_1, Y_2] = \frac{1}{2}X_4 \end{cases}$$

$$(4) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_4 \end{cases}$$

$$(5) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$$

$$(6) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, & [Y_1, Y_1] = X_4 \end{cases}$$

$$(7) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 \end{cases}$$

$$(8) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, & [Y_1, Y_1] = X_3, & [Y_1, Y_2] = \frac{1}{2}X_4 \end{cases}$$

$$(9) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, & [Y_1, Y_1] = X_4 \end{cases}$$

$$(10) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, & [X_1, Y_1] = Y_2 \end{cases}$$

$$(11) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, & [X_1, Y_1] = Y_2, & [Y_1, Y_1] = X_4 \end{cases}$$

$\mathcal{F}_{5,2}$ :

$$(1) \begin{cases} [X_0, X_1] = X_2, & [X_0, X_2] = X_3, & [X_0, X_3] = X_4, & [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \end{cases}$$

$$\begin{aligned}
(2) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_3, \quad [Y_1, Y_2] = \frac{1}{2}X_4, \quad [Y_2, Y_2] = \frac{1}{2}X_5 \end{array} \right. \\
(3) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \frac{1}{2}X_5 \end{array} \right. \\
(4) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_5 \end{array} \right. \\
(5) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{array} \right. \\
(6) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_5 \end{array} \right. \\
(7) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5 \end{array} \right. \\
(8) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [Y_1, Y_1] = X_3, \quad [Y_1, Y_2] = \frac{1}{2}X_4, \quad [Y_2, Y_2] = \frac{1}{2}X_5 \end{array} \right. \\
(9) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \frac{1}{2}X_5 \end{array} \right. \\
(10) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [Y_1, Y_1] = X_5 \end{array} \right. \\
(11) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, Y_1] = Y_2 \end{array} \right.
\end{aligned}$$

$$\begin{aligned}
(12) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_5 \end{array} \right. \\
(13) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \end{array} \right. \\
(14) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [Y_1, Y_1] = X_5 \end{array} \right. \\
(15) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = Y_2 \end{array} \right. \\
(16) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_5 \end{array} \right. \\
(17) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \end{array} \right. \\
(18) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5, \quad [Y_1, Y_1] = X_5 \end{array} \right. \\
(19) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5, \quad [X_1, Y_1] = Y_2 \end{array} \right. \\
(20) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5, \quad [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_5 \end{array} \right. \\
(21) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5, \quad [X_1, Y_1] = -Y_2 \end{array} \right.
\end{aligned}$$

$$(22) \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2, \quad [X_1, X_4] = -X_5 \\ [X_2, X_3] = X_5, \quad [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \frac{1}{2}X_5 \end{array} \right.$$

$$(23) \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5, \quad [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_5 \end{array} \right.$$

$$(24) \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \end{array} \right.$$

$$(25) \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [Y_1, Y_1] = X_5 \end{array} \right.$$

$$(26) \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = Y_2 \end{array} \right.$$

$$(27) \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = Y_2, \quad [Y_1, Y_1] = X_5 \end{array} \right.$$

$$(28) \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2 \end{array} \right.$$

$$(29) \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \frac{1}{2}X_5 \end{array} \right.$$

$$\begin{aligned}
(30) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_5 \end{array} \right. \\
(31) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 + X_5, \quad [X_1, X_3] = X_5, \quad [X_1, Y_1] = -2Y_2 \end{array} \right. \\
(32) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 + X_5, \quad [X_1, X_3] = X_5, \quad [X_1, Y_1] = -2Y_2, \quad [Y_1, Y_1] = X_5 \end{array} \right. \\
(33) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2 \end{array} \right. \\
(34) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_4, \quad [Y_1, Y_2] = \frac{1}{2}X_5 \end{array} \right. \\
(35) \quad & \left\{ \begin{array}{l} [X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_0, X_3] = X_4, \quad [X_0, X_4] = X_5 \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = -X_5, \quad [X_2, X_3] = X_5 \\ [X_1, Y_1] = -Y_2, \quad [Y_1, Y_1] = X_5 \end{array} \right.
\end{aligned}$$

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