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ON THE UNIQUENESS OF MAXIMAL OPERATORS FOR ERGODIC FLOWS

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Abstract

The uniqueness theorem for the ergodic maximal operator is proved in the continuous case.

Let $(X, \mathbb{S}, \mu)$ be a finite measure space,

$$\mu(X) < \infty, \tag{1}$$

and let $(T_t)_{t\geq 0}$ be an ergodic semigroup of measure-preserving transformations of $(X, \mathbb{S}, \mu)$. As usual the map $(x,t) \to T_t x$ is assumed to be jointly measurable. For an integrable function f, $f \in L(X)$, the ergodic maximal function f^* is defined by equation

$$f^*(x) = \sup_{t>0} \frac{1}{t} \int_0^t f(T_\tau x) d\tau, \quad x \in X.$$

We claim that the following uniqueness theorem is valid for the maximal operator $f \to f^*$:

Theorem. *Let $f, g \in L(X)$ and*

$$f^* = g^* \tag{2}$$

almost everywhere. Then

$$f(x) = g(x)$$

for a.a. $x \in X$ (with respect to measure μ).

A slightly weaker version of the theorem is formulated without proof in [3]. The analogous theorem in the discrete case is proved in [4].

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Remark. Condition (1) is necessary for the validity of the theorem. If $\mu(X) = \infty$, then $f^* = 0$ a.e. for every negative integrable f, since

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(T_\tau x) d\tau = 0$$

for a.a. $x \in X$ because of the Ergodic Theorem (see [1]).

First we need several lemmas.

Lemma 1. *Let $f \in L(X)$. Then*

$$\operatorname{ess\,inf} f^* = \frac{1}{\mu(X)} \int_X f d\mu \equiv \lambda_0.$$

Proof. That $f^* \geq \lambda_0$ a.e. follows from the Ergodic Theorem:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(T_\tau x) d\tau = \lambda_0 \quad \text{for a.a.} \quad x \in X \tag{3}$$

(see [1], [6]). The Maximal Ergodic Equality asserts that

$$\mu(f^* > \lambda) = \frac{1}{\lambda} \int_{(f^*>\lambda)} f d\mu, \quad \lambda \geq \lambda_0 \tag{4}$$

(see [6], [2]), and if $\mu(f^* > \lambda) = \mu(X)$ for some $\lambda > \lambda_0$, we would get from (4) that $\mu(X) = \lambda^{-1} \int_X f d\mu$. This implies $\lambda = \lambda_0$, which is a contradiction. ∎

Lemma 2. *Let $(T_t)_{t\geq 0}$ be an ergodic semigroup of measure-preserving transformations on a finite measure space $(X, \mathbb{S}, \mu)$ and let $f \in L(X)$. Then*

$$f(x) = \lambda_0 \quad \textit{for a.a. } x \in (f^* = \lambda_0). \tag{5}$$

Proof. The Local Ergodic Theorem,

$$\lim_{t\to 0+} \frac{1}{t} \int_0^t f(T_\tau x) d\tau = f(x)$$

(see [6]), implies that

$$f \leq \lambda_0 \text{ a.e. on } (f^* = \lambda_0). \tag{6}$$

On the other hand we have

$$\lambda_0\mu(X) = \lambda_0(\mu(f^* > \lambda_0) + \mu(f^* = \lambda_0)) = \int_{(f^*>\lambda_0)} f d\mu + \int_{(f^*=\lambda_0)} f d\mu.$$

Thus

$$\lambda_0\mu(f^* = \lambda_0) = \int_{(f^*=\lambda_0)} f d\mu \tag{7}$$

because of Maximal Ergodic Equality (see (4)). It follows from (6) and (7) that (5) holds. ■

For a locally integrable function ξ on $\mathbb{R}_0^+ = \{t \in \mathbb{R} : t \geq 0\}$, $\xi \in L_{\text{loc}}(\mathbb{R}_0^+)$, the maximal operator M is defined by

$$M\xi(t) = \sup_{\tau > t} \frac{1}{\tau - t} \int_t^\tau \xi dm$$

(m is the Lebesgue measure on $\mathbb{R}$). Hence, if $\xi(t) = f(T_t x)$, then

$$M\xi(t) = f^*(T_t x). \tag{8}$$

Obviously, for each λ the set $(M\xi > \lambda) = \{t \in \mathbb{R}_0^+ : M\xi(t) > \lambda\}$ is open (in $\mathbb{R}_0^+$). We shall use the following well-known facts about the connected components of this set (see [5], p.58):

If $\langle a, b)$, $0 \leq a < b < \infty$, (the sign $\langle$ before a indicates that a belongs or does not belong to the interval, i.e. $\langle a, b) = (a, b)$ or $\langle a, b) = [a, b)$) is a finite connected component of $(M\xi > \lambda)$, then

$$\frac{1}{b - t} \int_t^b \xi dm > \lambda \tag{9}$$

for each $t \in \langle a, b)$. If, in addition, $a \notin (M\xi > \lambda)$ i.e. $\langle a, b) = (a, b)$, then

$$\frac{1}{b - a} \int_a^b \xi dm = \lambda. \tag{10}$$

Lemma 3. *If $\xi, \eta \in L_{\text{loc}}(\mathbb{R}_0^+)$ and $M\xi = M\eta$ almost everywhere, then $M\xi(t) = M\eta(t)$ for all $t \geq 0$.*

Proof. Let us show that for each $\xi \in L_{\text{loc}}(\mathbb{R}_0^+)$ we have

$$M\xi(t) = \lim_{\delta \to 0+} \operatorname*{ess\,inf}_{\tau \in (t, t+\delta)} M\xi(\tau), \quad t \geq 0,$$

which obviously implies the validity of the lemma.

If $M\xi(t) > \lambda$, then there exists $\delta > 0$ such that $M\xi(\tau) > \lambda$ for each $\tau \in (t, t+\delta)$. Thus

$$M\xi(t) \le \lim_{\delta\to 0+} \operatorname*{ess\,inf}_{\tau\in(t,t+\delta)} M\xi(\tau).$$

Conversely, if $M\xi > \lambda$ a.e. on $(t, t+\delta)$, then let us show that

$$M\xi(t) \ge \lambda, \tag{11}$$

which finishes the proof.

Indeed, if $(t, t+\delta) \subset (M\xi > \lambda)$, then for each $\tau \in (t, t+\delta)$ we have $\sup\{\tau' > \tau : \frac{1}{\tau'-\tau}\int_\tau^{\tau'} \xi dm \ge \lambda\} \ge t+\delta$ (see [5], p.58). Consequently, there exists $\tau' \ge t+\delta$ such that

$$\frac{1}{\tau'-\tau}\int_\tau^{\tau'} \xi dm \ge \lambda.$$

Set $\tau_n \searrow t$ and let

$$\tau_n' > t+\delta \tag{12}$$

be such that

$$\frac{1}{\tau_n'-\tau_n}\int_{\tau_n}^{\tau_n'} \xi dm \ge \lambda,$$

$n = 1, 2, \ldots$. Then

$$M\xi(t) \ge \frac{1}{\tau_n'-t}\int_t^{\tau_n'} \xi dm \ge$$

$$\left(\frac{1}{\tau_n'-\tau_n}\int_{\tau_n}^{\tau_n'} \xi dm - \frac{1}{\tau_n'-\tau}|\int_t^{\tau_n} \xi dm|\right)\frac{\tau_n'-\tau_n}{\tau_n'-t}$$

and taking into account that $\tau_n \to t$, $\tau_n' - \tau \not\to 0$ (because of (12)) and $(\tau_n'-\tau_n)/(\tau_n'-t) \to 1$ as $n \to \infty$, we shall get (11).

If $\tau \notin (M\xi > \lambda)$ for some $\tau \in (t, t+\delta)$, then (t, τ) is covered up to a set of measure 0 with the connected components of $(M\xi > \lambda)$. In other words, there exist connected components Δ_i, $i = 1, 2, \ldots$ such that $\Delta_i \subset (t, \tau)$ and $m((t,\tau) \setminus (\cup_{i=1}\Delta_i)) = 0$. Since

$$\frac{1}{m(\Delta_i)}\int_{\Delta_i} \xi dm = \lambda$$

for each i (see (10)), we have

$$\int_t^\tau \xi dm = \lambda(\tau - t)$$

and (11) holds. ■

The lemma below is actually proved in [3]. It is given here for the sake of completeness.

Lemma 4. *Let $\xi \in L_{\text{loc}}(\mathbb{R}_0^+)$, and let $\langle a, b)$ be a finite connected component of $(M\xi > \lambda)$ for some λ. Then the values $M\xi(t)$, $t \in \langle a, b)$, uniquely define the values $\xi(t)$ for a.a. $t \in \langle a, b)$.*

Hence, if another function $\eta \in L_{\text{loc}}(\mathbb{R}_0^+)$ is given such that $M\xi(t) = M\eta(t)$, $t \geq 0$, then $\xi(t) = \eta(t)$ for a.a. $t \in \langle a, b)$.

Proof. We shall show that the values $M\xi(t)$, $t \in \langle a, b)$, uniquely define the function

$$h(t) = \int_t^b \xi dm, \quad t \in \langle a, b). \tag{13}$$

Assume t fixed and let $\lambda_t = M\xi(t)$. For each $\gamma \in [\lambda, \lambda_t)$ suppose $\langle a_\gamma, b_\gamma)$ to be the connected component of $(M\xi > \lambda)$ which contains t and suppose $b_\gamma = t$ whenever $\gamma = \lambda_t$ (note that $b_\lambda = b$, by hypothesis). Obviously, $\langle a_\gamma, b_\gamma) \subset \langle a_{\gamma'}, b_{\gamma'})$, $\lambda_t > \gamma > \gamma' \geq \lambda$, and

$$\cup_{\gamma' > \gamma} \langle a_{\gamma'}, b_{\gamma'}) = \langle a_\gamma, b_\gamma), \quad \lambda_t > \gamma \geq \lambda.$$

It is easy to show that $\Psi : \gamma \to b_\gamma$ is a non-increasing function on $[\lambda, \lambda_t]$ continuous from the right. Observe also that Ψ is uniquely defined by the values $M\xi(t)$, $t \geq 0$.

Let D be the set of points of discontinuity of this function, set

$$b'_\gamma = \lim_{\gamma' \to \gamma-} b_{\gamma'} \tag{14}$$

for $\gamma \in D$, and let

$$C = \{\gamma \in [\lambda, \lambda_t] : b_{\gamma'} = b_\gamma \ \text{ for some } \ \gamma' > \gamma\}.$$

Then the interval $[t, b]$, as a range of the non-increasing continuous from the right function Ψ, can be divided into pairwise disjoint parts:

$$[t, b] = E_1 \cup E_2 \cup E_3, \tag{15}$$

where

$$E_1 = \{b_\gamma = \Psi(\gamma) : \gamma \in [\lambda, \lambda_t] \setminus (D \cup C)\}, \tag{16}$$

$$E_2 = \cup_{\gamma \in D}[b_\gamma, b'_\gamma] \tag{17}$$

and $E_3 = \{b_\gamma = \Psi(\gamma) : \gamma \in C)\}$. Note that E_3 is a countable set and the intervals $(b_\gamma, b'_\gamma)_{\gamma \in D}$ are disjoint.

Observe also that for each $e \in E_1$ there exists unique $\gamma \in [\lambda, \lambda_t]$ such that $e = b_\gamma = \Psi(\gamma)$. Hence, Ψ^{-1} exists on E_1.

If $\gamma \in [\lambda, \lambda_t) \setminus (D \cup C)$ and $b_\gamma \in E_1$ is a Lebesgue point of ξ then

$$\xi(b_\gamma) \leq \gamma \tag{18}$$

(since $M\xi(b_\gamma) \leq \gamma$). On the other hand, for each $\gamma' \in (\gamma, \lambda_t)$ we have

$$\frac{1}{b_\gamma - b_{\gamma'}} \int_{b_{\gamma'}}^{b_\gamma} \xi dm > \gamma$$

since $\langle a_\gamma, b_\gamma)$ is a connected component of $(M\xi > \gamma)$ and $b_{\gamma'} \in \langle a_\gamma, b_\gamma)$ (see (9)). Hence, taking into account that $b_{\gamma'} \to b_\gamma$ when $\gamma' \to \gamma$, we can conclude that $\xi(b_\gamma) \geq \gamma$, which together with (18) implies that

$$\xi(b_\gamma) = \gamma.$$

Thus $\xi = \Psi^{-1}$ a.e. on E_1 (see (16)) and consequently

$$\int_{E_1} \xi dm = \int_{E_1} \Psi^{-1} dm. \tag{19}$$

If $\gamma \in D$, then

$$\frac{1}{b'_\gamma - b_\gamma} \int_{b_\gamma}^{b'_\gamma} \xi dm \leq \gamma \tag{20}$$

(since $M\xi(b_\gamma) \leq \gamma$) and for each $\gamma' \in (\lambda, \gamma)$ we have

$$\frac{1}{b_{\gamma'} - b_\gamma} \int_{b_\gamma}^{b_{\gamma'}} \xi dm > \gamma'$$

since $\langle a_{\gamma'}, b_{\gamma'})$ is a connected component of $(M\xi > \gamma')$ and $b_\gamma \in \langle a_{\gamma'}, b_{\gamma'})$ (see (9)). Hence, letting γ' converge to γ from the left and taking into account (14), we get

$$\frac{1}{b'_\gamma - b_\gamma} \int_{b_\gamma}^{b'_\gamma} \xi dm \geq \gamma.$$

This together with (20) implies that

$$\int_{b_\gamma}^{b'_\gamma} \xi dm = \gamma(b'_\gamma - b_\gamma).$$

Hence

$$\int_{E_2} \xi dm = \sum_{\gamma\in D} \gamma(b'_\gamma - b_\gamma) \tag{21}$$

(see (17)). It follows from (13), (15), (19) and (21) that

$$h(t) = \int_{E_1} \Psi^{-1} dm + \sum_{\gamma\in D} \gamma(b'_\gamma - b_\gamma).$$

Thus $h(t)$ is uniquely defined by the function Ψ. ■

Corollary. *Let $\xi, \eta \in L_{\text{loc}}(\mathbb{R}_0^+)$ be such that*

$$M\xi(t) = M\eta(t), \quad t \geq 0.$$

If $0 \leq t < t'$ and

$$M\xi(t) = M\eta(t) > M\xi(t') = M\eta(t'),$$

then

$$\xi(\tau) = \eta(\tau) \tag{22}$$

for a.a. τ from some neighbourhood of t.

Proof. If we take $\lambda \in (M\xi(t'), M\xi(t))$, then $t' \notin (M\xi > \lambda)$ and some finite connected component of $(M\xi > \lambda)$ includes t. For a.a. τ from this interval (22) holds by virtue of the lemma. ■

Proof of Theorem. Equality (2) implies that

$$\operatorname{ess\,inf} f^* = \operatorname{ess\,inf} g^* \equiv \lambda_0.$$

Consequently,

$$\mu(f^* < \lambda) = \mu(g^* < \lambda) > 0 \text{ for all } \lambda > \lambda_0 \tag{23}$$

and

$$\mu(f^* < \lambda_0) = \mu(g^* < \lambda_0) = 0. \tag{24}$$

Define

$$\xi_x(t) = f(T_t x) \quad \text{and} \quad \eta_x(t) = g(T_t x), \quad x \in X,\ t \geq 0.$$

We shall prove that for a.a. $x \in X$

$$m\{t \geq 0 : \xi_x(t) \neq \eta_x(t)\} = 0. \tag{25}$$

Obviously, this implies that

$$\mu(f \neq g) = 0.$$

(If $X_1 \subset X$ and $\mu(X_1) > 0$ then, by the Ergodic Theorem, see (3),

$$m\{t \geq 0 : T_t x \in X_1\} = \lim_{t\to\infty} \int_0^t \mathbb{I}_{X_1}(T_\tau x)d\tau = \infty \tag{26}$$

for a.a. $x \in X$, while

$$\{t \geq 0 : \xi_x(t) \neq \eta_x(t)\} = \{t \geq 0 : T_t x \in (f \neq g)\}, \ \ x \in X.\ )$$

If $X_0 \subset X$ and $\mu(X_0) = 0$, then by standard application of Fubini's theorem we have

$$m\{t \geq 0 : T_t x \in X_0\} = 0 \tag{27}$$

for a.a. $x \in X$. Hence

$$m\{t \geq 0 : M\xi_x(t) \neq M\eta_x(t)\} = m\{t \geq 0 : T_t x \in (f^* \neq g^*)\} = 0$$

for a.a. $x \in X$ (see (2), (8)) and Lemma 3 implies that

$$M\xi_x(t) = M\eta_x(t), \quad t \geq 0, \tag{28}$$

for a.a. $x \in X$. We also have

$$m\{t \geq 0 : M\xi_x(t) = M\eta_x(t) < \lambda_0\} = 0 \tag{29}$$

(see (24)) and

$$m\{t \geq 0 : M\xi_x(t) = M\eta_x(t) = \lambda_0,\ \ \xi_x(t) \neq \lambda_0 \ \text{ or } \ \eta_x(t) \neq \lambda_0\} = 0 \tag{30}$$

for a.a. $x \in X$ (see (5)).

We consider two cases:

(i) $\mu(f^* = \lambda_0) = \mu(g^* = \lambda_0) > 0$. Then

$$m\{t \geq 0 : M\xi_x(t) = M\eta_x(t) = \lambda_0\} = \infty \tag{31}$$

for a.a. $x \in X$ (see (26)). Take $x \in X$ for which (28), (29), (30) and (31) hold (note that almost all x have this property). Let $E = \{t \geq 0 : M\xi_x(t) = M\eta_x(t) > \lambda_0\}$. Then for each $t \in E$ there exists $t' > t$ such that $M\xi_x(t') = M\eta_x(t') = \lambda_0$, because of (31). Thus the corollary of Lemma 4 implies that

$$\xi_x(t) = \eta_x(t) \tag{32}$$

for a.a. $t \in E$.

It follows from (29) and (30) that $\xi_x(t) = \eta_x(t) = \lambda_0$ for a.a. $t \in \mathbb{R}_0^+ \setminus E$. Thus (32) holds for a.a. $t \geq 0$ and (25) is valid.

(ii) $\mu(f^* = \lambda_0) = \mu(g^* = \lambda_0) = 0$. Then

$$m\{t \geq 0 : M\xi_x(t) = M\eta_x(t) \leq \lambda_0\} = 0 \tag{33}$$

for a.a. $x \in X$ (see (8), (24) and (27))

If λ_i is any decreasing sequence convergent to λ_0, $\lambda_i \searrow \lambda_0$, then

$$\mu(f^* < \lambda_i) = \mu(g^* < \lambda_i) > 0, \quad i = 1, 2, \dots$$

(see (23)) and consequently for a.a. $x \in X$ we have

$$\begin{aligned} m\{t \geq 0 :\ M\xi_x(t) = M\eta_x(t) < \lambda_i\} = \\ m\{t \geq 0 : f^*(T_t x) = g^*(T_t x) < \lambda_i\} = \infty, \quad i = 1, 2, \dots, \end{aligned} \tag{34}$$

(see (26)). Take $x \in X$ for which (28), (33) and (34) hold (note that almost all x have this property). It follows from (33) and (34) that for a.a. $t \geq 0$ there exists $t' > t$ such that

$$M\xi_x(t) = M\eta_x(t) > M\xi_x(t') = M\eta_x(t').$$

Thus, by virtue of the corollary of Lemma 4, (32) holds for a.a. $t \geq 0$ and (25) is valid. ■

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