

ON THE MODULUS OF MEASURES WITH VALUES IN TOPOLOGICAL RIESZ SPACES

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Abstract

The paper is devoted to a study of some aspects of the theory of (topological) Riesz space valued measures. The main topics considered are the following. First, the problem of existence (and, particularly, the so-called proper existence) of the modulus of an order bounded measure, and its relation to a similar problem for the induced integral operator. Second, the question of how properties of such a measure like countable additivity, exhaustivity or so-called absolute exhaustivity, or the properties of the range space, influence the properties of the modulus of the measure. Third, the problem of exhibiting (or constructing) Banach lattices that are “good” in many respects, and yet admit a countably additive measure whose modulus is not countably additive. A few applications to weakly compact operators from spaces of bounded measurable functions to Banach lattices are also presented.

0 Introduction

Let μ be a countably additive and order bounded measure from a σ -algebra Σ to a Banach lattice E . If the modulus $|\mu|$ of μ exists, it is natural to ask whether it must be countably additive as well. This problem has been recently considered by H. H. Schaefer and X. D. Zhang [hSZ], and G. Kronsbein and P. Meyer-Nieberg [KM-N]. They proved that it is so when E has order continuous norm or is a Dedekind complete AM-space with a strong unit [hSZ] (see also [kS2]), or if E contains no

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isomorphic copy of ℓ_∞ [KM-N]. (For some related earlier results, see [kS1], [kS2], and [kS3].) A particular case of this problem for the moduli of indefinite Bochner or Pettis integrals of strongly measurable functions has been dealt with by the present authors in [DW]. Therein, an example was also given showing that, in general, the above problem has a negative answer even if E is Dedekind complete (which forces the existence of $|\mu|$). We continue our investigations in the present paper. This time, however, we work in the setting of general locally solid (topological) Riesz spaces (TRS), and we include also exhaustive finitely additive measures in the area of our research. A brief description of the main points of the paper follows.

In Section 1, we collect some preliminary results concerning absolute and positive majorants and the notion of the modulus for a Riesz space valued measure. Next, in Section 2, we construct an order bounded σ -additive measure with values in the Banach lattice c that has no absolute majorant. (Another example to the same effect is given in Section 7.)

As for Section 3, assume for simplicity that E and μ are as in the beginning. Then we associate with the measure μ its *absolute semivariation* $\|\mu\|_a : \Sigma \rightarrow \overline{\mathbb{R}}_+$ defined by

$$\|\mu\|_a(A) = \sup \|\mu(B_1)| + \cdots + |\mu(B_n)|\|,$$

where the sup is taken over all Σ -partitions B_1, \dots, B_n of A . We prove that $\|\mu\|_a$ is order continuous (or exhaustive) iff μ is *absolutely exhaustive*, i.e., the series $\sum_n |\mu(C_n)|$ converges in norm for every disjoint sequence (C_n) in Σ . A condition of exactly the latter type was already shown in [DW, Th. 2] to characterize those indefinite Pettis integrals of strongly measurable functions whose modulus exists and is σ -additive.

Motivated by this, we introduce in Section 4 a new class of TRS's as follows: We say that a TRS E has Property (S) if for every subseries convergent series $\sum_n x_n$ in E with order bounded sums $\sum_{i=1}^n |x_i|$ also the series $\sum_n |x_n|$ is convergent. For example, all Lebesgue (or order continuous) Banach lattices and all AM-spaces have Property (S). This property is obviously inherited by arbitrary products and sequentially closed sublattices of TRS's having (S). We also consider Property (CS) which is a Cauchy variant of Property (S). As we have learned at a later stage of working on this paper, the class of Banach lattices satisfying the above condition has already been isolated in [G, Def. 4.3, p. 94] and called Banach lattices with *property (C)*. In a later paper [GvR], such

Banach lattices are said to be *order bounded AM*.

In Section 5, we first characterize TRS's that are σ -Dedekind complete and σ -Lebesgue as those satisfying a mild completeness condition and containing no positive copy of the Banach lattice c . We next recall from [DL] a useful characterization of TRS's containing no positive copy of the Banach lattice ℓ_∞ .

In Section 6, we use our results concerning absolute semivariations and Property (S), as well as some more direct results relating the exhaustivity of $|\mu|$ to its σ -additivity, to give far going generalizations (with much simpler proofs) of the results from [hSZ] and [KM-N] quoted above.

In Section 7, we strengthen our negative answer to the question of Schaefer and Zhang by showing that if a σ -Dedekind complete Banach lattice E is atomless and the ideal E_A of its order continuous elements is not a band, then one can find an order bounded σ -additive (and even atomless) E -valued measure whose modulus exists but is not σ -additive. We also give, following [G], an example of an order bounded σ -additive measure μ with values in a Dedekind complete AM-space for which $|\mu|$ is not σ -additive.

In Section 8, we collect some results concerning the so called proper existence of the modulus of a measure μ , and establish a close relationship between this and an analogous problem for the induced integral operator I_μ .

Finally, Section 9 contains a few results dealing with *absolute exhaustivity* and weak compactness of integral operators of type I_μ .

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1 Absolute majorants and the modulus of a measure

Here we collect some basic terminology and notation along with a few simple, mostly “folklore”, facts. We refer the reader to [AB1], [AB2], and [hS] for general Riesz spaces (or vector lattices), locally solid (topological) Riesz spaces and Banach lattices, and to [DU] and [DS] for vector measures and integrals.

Throughout, \mathcal{A} is an algebra, and Σ is a σ -algebra, of subsets of a set S . (Obviously, the definitions and results involving \mathcal{A} apply in particular to Σ .) Depending on the context, all subsets of S occurring below are tacitly assumed to be members of \mathcal{A} or Σ ; a *partition* of a set $A \subset S$ means a finite \mathcal{A} - or Σ -partition of A ; and $\Pi(A)$ stands for the family of all such partitions of A (directed upward by refinement). If $B \in \mathcal{A}$ and $\mathcal{C} \subset \mathcal{A}$, then

$$\mathcal{C}_B \equiv \mathcal{C} \cap B := \{C \cap B : C \in \mathcal{C}\}.$$

In particular, $\mathcal{A}_B = \{A \in \mathcal{A} : A \subset B\}$ is the algebra of subsets of B induced by \mathcal{A} . Likewise, for $B \in \Sigma$, Σ_B is the induced σ -algebra on B . The characteristic function of a set $A \subset S$ is denoted 1_A . Finally, $\mathbb{N} = \{1, 2, \dots\}$, and $\mathcal{P}(\mathbb{N})$ denotes the power set of \mathbb{N} .

We use the abbreviation TVS for *Hausdorff topological vector space*, and TRS for *Hausdorff locally solid (topological) Riesz space*. We follow [AB1] and [AB2] quite closely in terminology concerning TRS's. We denote by ω the Riesz space of all real sequences (with the product topology), and by ℓ_∞ , c , and c_0 the standard Banach lattices of real sequences that are, respectively, bounded, convergent, or convergent to zero (equipped with the sup norm). As usual, (e_n) will stand for the unit vectors in these (and other) sequence spaces.

By a *measure* we always mean a *finitely additive measure* (defined on \mathcal{A} or Σ), and the requirement of countable (or σ -) additivity will always be imposed explicitly when needed.

Recall that a measure μ defined on \mathcal{A} and taking values in a TVS E is said to be *exhaustive* (or strongly additive [DU]) if $\mu(D_n) \rightarrow 0$ for every disjoint sequence (D_n) in \mathcal{A} or, equivalently, if the series $\sum_n \mu(D_n)$ is (unconditionally or subseries) Cauchy for every disjoint sequence (D_n) in \mathcal{A} . Another equivalent reformulation is that the sequence $(\mu(A_n))$ has

to be Cauchy whenever $A_n \uparrow$ in \mathcal{A} . Clearly, “Cauchy” can be replaced by “convergent” in the above conditions when E is sequentially complete, or when E is a TRS with the σ -monotone completeness property for intervals (see Section 4) and the measure μ assumes only positive values ($\mu \geq 0$).

Also recall that a *submeasure* on \mathcal{A} is a function $\eta : \mathcal{A} \rightarrow \overline{\mathbb{R}}_+$ that is nondecreasing, subadditive, and vanishes at \emptyset . It is said to be *exhaustive* if $\eta(D_n) \rightarrow 0$ for every disjoint sequence (D_n) in \mathcal{A} ; *order continuous* if $\eta(A_n) \rightarrow 0$ whenever $(A_n) \subset \mathcal{A}$ and $A_n \downarrow \emptyset$. Note that an analogous property of order continuity for measures taking values in a TVS coincides with σ -additivity.

Let E be a Riesz space, and let $\mu, \nu : \mathcal{A} \rightarrow E$ be measures. We write $\mu \leq \nu$ if $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{A}$. By an *absolute* (resp., *positive*) *majorant* for μ we mean a measure $\gamma : \mathcal{A} \rightarrow E_+$ such that $|\mu(A)| \leq \gamma(A)$ (resp., $\mu(A) \leq \gamma(A)$) for all $A \in \mathcal{A}$. The smallest absolute (resp., positive) majorant for μ , if exists, is called the *modulus* (resp., *positive part*) of μ and denoted $|\mu|$ (resp., μ^+). It is not hard to see that μ admits an absolute majorant iff it admits a positive majorant, and that $|\mu|$ exists iff μ^+ exists; see Proposition 1.4 below for more details. An obvious condition that is necessary (though not sufficient, see Section 2) for the existence of an absolute or positive majorant for μ is that μ is *order bounded*, i.e., that its range $\mu(\mathcal{A})$ is order bounded. In view of the equivalencies mentioned above, in what follows we focus our attention on the existence and properties of $|\mu|$. Note that if $|\mu|$ exists, then $|\mu(A)| \leq |\mu|(A)$ for each $A \in \mathcal{A}$, whence

$$\sum_{D \in \pi} |\mu(D)| \leq |\mu|(A) \quad \text{for all } A \in \mathcal{A} \text{ and } \pi \in \Pi(A).$$

To shorten formulations, we adopt the convention that a phrase like “assume that the modulus $|\mu|$ exists” is an abbreviation for “assume that μ is order bounded and that the modulus $|\mu|$ exists”.

Proposition 1.1. *If the modulus $|\mu|$ of a measure $\mu : \mathcal{A} \rightarrow E$ exists, then for every $B \in \mathcal{A}$ the measure $\mu_B = \mu|_{\mathcal{A}_B}$ has a modulus, and $|\mu_B| = |\mu| |_{\mathcal{A}_B}$.*

Proof. Clearly, $|\mu|_B = |\mu| |_{\mathcal{A}_B}$ is an absolute majorant for μ_B . Let a measure $\gamma : \mathcal{A}_B \rightarrow E$ be an absolute majorant for μ_B . Then the measure $\nu : \mathcal{A} \rightarrow E$ defined by $\nu(A) = \gamma(A \cap B) + |\mu|(A \setminus B)$ is an absolute

majorant for μ , hence $|\mu| \leq \nu$. In consequence, $|\mu|_B \leq \nu|_{\mathcal{A}_B} = \gamma$, and it follows that $|\mu|_B$ is the smallest absolute majorant for μ_B . ■

We now recall a useful general fact, where the first (basic) equality follows by an easy induction from the identity $|x| + |y| = |x + y| \vee |x - y|$ (cf. [hS, p. 52, Cor. 1 (8)]).

Fact. For any finite family $\{x_j : j \in J\}$ in a Riesz space E ,

$$\begin{aligned} \sum_{j \in J} |x_j| &= \sup \left\{ \left| \sum_{j \in J} \varepsilon_j x_j \right| : \varepsilon_j = \pm 1 \right\} = \sup \left\{ \sum_{j \in K} x_j - \sum_{j \in J \setminus K} x_j : K \subset J \right\} \\ &= \sup \left\{ \sum_{j \in K} x_j - \sum_{j \in L} x_j : K, L \subset J \right\}. \end{aligned}$$

Hence, if $|\sum_{j \in K} x_j| \leq x$ for all $K \subset J$ and some $x \in E_+$, then $\sum_{j \in J} |x_j| \leq 2x$.

From this the following is easily obtained (cf. [DW], [G, p. 83], [kS2, 4. Remarks]).

Proposition 1.2. Let $\mu : \mathcal{A} \rightarrow E$ be a measure. Then for any finite disjoint family $\{B_j : j \in J\}$ in \mathcal{A} ,

$$\sum_{j \in J} |\mu(B_j)| = \sup \left\{ \mu \left(\bigcup_{j \in K} B_j \right) - \mu \left(\bigcup_{j \in L} B_j \right) : K, L \subset J \right\}.$$

In consequence, for every $A \in \mathcal{A}$,

(a) if one of the sups below exists, so do the other and they are all equal:

$$\begin{aligned} \sup \left\{ \sum_{D \in \pi} |\mu(D)| : \pi \in \Pi(A) \right\} &= \sup \{ \mu(B) - \mu(C) : B, C \subset A \} \\ &= \sup \{ \mu(B) - \mu(A \setminus B) : B \subset A \}. \end{aligned}$$

(b) if μ is order bounded, so is the increasing net $\{\sum_{D \in \pi} |\mu(D)| : \pi \in \Pi(A)\}$.

Corollary 1.3. Let $\mu : \mathcal{A} \rightarrow E$ be an order bounded measure. Then for every disjoint sequence (D_n) in \mathcal{A} , the series $\sum_n |\mu(D_n)|$ is order bounded; that is, the sequence of its partial sums is order bounded.

It is easily seen that if E is Dedekind complete and $\mu : \mathcal{A} \rightarrow E$ is an order bounded measure, then the modulus $|\mu|$ exists and is given by each of the equalities below:

$$\begin{aligned} |\mu|(A) &= \sup \left\{ \sum_{D \in \pi} |\mu(D)| : \pi \in \Pi(A) \right\} \\ &= \sup \{ \mu(B) - \mu(A \setminus B) : B \subset A \} \\ &= \sup \{ \mu(B) - \mu(C) : B, C \subset A \} \\ &= \sup \{ |\mu(B) - \mu(C)| : B, C \subset A \}. \end{aligned}$$

Moreover, it is also easily verified that whenever the above formulas make sense for each $A \in \mathcal{A}$, they define a positive measure, and it is precisely the modulus $|\mu|$ of μ . When this is the case, we shall say (following [DW]) that the modulus $|\mu|$ *exists properly*, or that μ has a *proper modulus*. Also, when the modulus $|\mu|$ exists, we shall say that it *exists properly on a set* $A \in \mathcal{A}$ if $|\mu|(A)$ is given by the above formulas.

It is apparently an open question whether an order bounded measure may have a non-proper modulus. At the end of the paper (Section 8), we give a few results that are relevant to this question. Here we only state the following proposition. Its parts (a) and (b) follow from the fact that the formula $\gamma = 2\nu - \mu$ establishes an order isomorphism between positive majorants ν and absolute majorants γ for the measure μ , while part (c) follows directly from the equality $\mu(B) = \frac{1}{2}(\mu(B) - \mu(A \setminus B)) + \frac{1}{2}\mu(A)$ for $B \subset A \in \mathcal{A}$.

Proposition 1.4. *Let $\mu : \mathcal{A} \rightarrow E$ be an order bounded measure.*

- (a) μ admits an absolute majorant iff μ admits a positive majorant.
- (b) $|\mu|$ exists iff μ^+ exists, and if this is the case, then $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and, consequently, $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$, where $\mu^- := (-\mu)^+$.
- (c) $|\mu|$ exists properly iff $\mu^+(A) = \sup\{\mu(B) : B \subset A\}$ in E for every $A \in \mathcal{A}$.

Before proceeding let us recall that a Fréchet-Nikodym topology on \mathcal{A} is one for which the operations of symmetric difference $(A, B) \rightarrow A \Delta B$

and intersection $(A, B) \rightarrow A \cap B$ are uniformly continuous. Equivalently, it is a topology converting (\mathcal{A}, Δ) to a topological abelian group with a base of neighborhoods of \emptyset consisting of *solid* classes $\mathcal{U} \subset \mathcal{A}$ (i.e. such that $\mathcal{A} \ni B \subset A \in \mathcal{U}$ implies $B \in \mathcal{U}$). Given a measure $\mu : \mathcal{A} \rightarrow E$, where E is a TVS, the μ -topology is the weakest Fréchet-Nikodym topology on \mathcal{A} which makes μ (uniformly) continuous. If U ranges over a base of neighborhoods of zero in E , then the classes $\mathcal{U} = \{B \in \mathcal{A} : \mu(\mathcal{A}_B) \subset U\}$ form a base of solid neighborhoods of \emptyset for the μ -topology. For more information and references, see [D2, Sec. 0].

Proposition 1.5. *Let E be a σ -Dedekind complete TRS. If $\mu : \mathcal{A} \rightarrow E$ is an order bounded measure such that \mathcal{A} is μ -separable, then $|\mu|$ exists properly. Moreover, if \mathcal{C} is a countable μ -dense subalgebra of \mathcal{A} , and $\Pi_{\mathcal{C}}(S)$ denotes the countable family of all finite \mathcal{C} -partitions of S , then*

$$|\mu|(A) = \sup \left\{ \sum_{C \in \pi} |\mu(A \cap C)| : \pi \in \Pi_{\mathcal{C}}(S) \right\} \quad \text{for all } A \in \mathcal{A}. \quad (*)$$

Proof. In view of Proposition 1.4, to verify the first statement it is enough to show that $\sup \mu(\mathcal{A} \cap A)$ exists in E for every $A \in \mathcal{A}$. Let \mathcal{C} be a countable μ -dense subclass of \mathcal{A} .

Fix $A \in \Sigma$. Since the map $B \rightarrow A \cap B$ is μ -continuous, $\mathcal{C} \cap A$ is a μ -dense subclass of $\mathcal{A} \cap A$. In consequence, $\mu(\mathcal{C} \cap A)$ is a dense subset of $\mu(\mathcal{A} \cap A)$. Now, the set $\mu(\mathcal{C} \cap A)$, being countable and order bounded, has a supremum, say z , in E , and it is easily seen that z is also a supremum of $\mu(\mathcal{A} \cap A)$.

We now verify the “moreover” part. Let \mathcal{C} be a countable μ -dense subalgebra of \mathcal{A} . Fix $A \in \mathcal{A}$ and denote the supremum in (*) by z . Using Proposition 1.2 and applying a continuity-density argument as above, we see that

$$\begin{aligned} z &= \sup \{ \mu(A \cap C_1) - \mu(A \cap C_2) : C_1, C_2 \in \mathcal{C} \} \\ &= \sup \{ \mu(A \cap B_1) - \mu(A \cap B_2) : B_1, B_2 \in \mathcal{A} \} = |\mu|(A), \end{aligned}$$

which concludes the proof. ■

For a σ -additive measure μ defined on a σ -algebra, every generating subalgebra is automatically μ -dense (see [D1, Part III, Thm. 8.2]). Hence as a consequence of the preceding result we have the following.

Corollary 1.6. *Let E be a σ -Dedekind complete TRS. If $\mu : \Sigma \rightarrow E$ is a σ -additive order bounded measure and the σ -algebra Σ is countably generated, then the modulus $|\mu|$ exists properly. Moreover, for any countable subalgebra \mathcal{C} generating Σ , the formula (*) stated above holds for all $A \in \Sigma$.*

Evidently, for a bounded measure $\mu : \mathcal{A} \rightarrow \mathbb{R}$ its modulus $|\mu|$ is the same as the (total) variation measure of μ . A reduction to variation measures is also possible when E is a solid sequence space. We first note the following, rather obvious, fact.

Proposition 1.7. *Let E be a Riesz subspace of a Riesz space F , and let $\mu : \mathcal{A} \rightarrow E$ and $\nu : \mathcal{A} \rightarrow F$ be measures with $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$.*

- (a) *If $|\nu|$ exists (resp., exists properly) and $|\nu|(\mathcal{A}) \subset E$, then also $|\mu|$ exists (resp., exists properly) and $|\mu| = |\nu|$.*
- (b) *If E is regular in F and $|\mu|$ exists properly, then so does $|\nu|$, and $|\mu| = |\nu|$.*

Recall that, in particular, a Riesz subspace E of a Riesz space F is regular if it is order dense in F or is an ideal in F .

Corollary 1.8. *Let E be a regular Riesz subspace of the Riesz space ω , $\mu = (\mu_n) : \mathcal{A} \rightarrow E$ be an order bounded measure, and define a measure $m : \mathcal{A} \rightarrow \omega$ by $m = (|\mu_n|)$. Then the modulus $|\mu|$ exists properly iff $m(\mathcal{A}) \subset E$, in which case $|\mu| = m$.*

In particular, if E is an ideal in ω , then the modulus $|\mu|$ exists properly and $|\mu| = m$.

Proof. Define a measure $\nu : \mathcal{A} \rightarrow \omega$ by $\nu(A) = \mu(A)$. Since ω is Dedekind complete, $|\nu|$ exists properly and, clearly, $|\nu| = m$. Applying Proposition 1.7 concludes the proof of the main assertion. In order to deduce the other assertion it suffices to note that if E is an ideal and $\mu(\mathcal{A}) \subset [-x, x]$ for some $x \in E_+$, then $m(\mathcal{A}) \subset [-2x, 2x] \subset E$. ■

In what follows,

$I = [0, 1]$, $(I, \mathcal{B}, \lambda)$ is the standard Borel-Lebesgue measure space,

$L_p = L_p(I, \mathcal{B}, \lambda)$ for $1 \leq p \leq \infty$, and
 (r_n) is the Rademacher sequence on I .

Recall that $\int_A r_n d\lambda \rightarrow 0$ for all $A \in \mathcal{B}$.

Example 1.9. In general, the measures that have modulus need not form a vector space. To see this, consider the measures $\mu = (\mu_n)$, $\nu = (\nu_n) : \mathcal{B} \rightarrow c$ defined by $\mu_n(A) = \int_A r_n d\lambda$ and $\nu_n(A) = (-1)^{n-1} \int_A r_n d\lambda$. (They are actually c_0 -valued.) Then $|\mu_n|(A) = |\nu_n|(A) = \lambda(A)$ so that, by Corollary 1.8, both $|\mu|$ and $|\nu|$ exist as measures from \mathcal{B} to c , and $|\mu|(A) = |\nu|(A) = (\lambda(A), \lambda(A), \dots)$ for each $A \in \mathcal{B}$. Now consider the measure $\mu + \nu = (\mu_n + \nu_n) : \mathcal{B} \rightarrow c$. Note that $\mu + \nu = 2(\mu_1, 0, \mu_3, 0, \dots)$. Hence if a measure $\xi = (\xi_n) : \mathcal{B} \rightarrow c$ is an absolute majorant for $\mu + \nu$, so is the measure $\gamma = (\gamma_n)$, where $\gamma_n = \xi_n$ when n is odd, and $\gamma_n = (1 - \frac{1}{n})\xi_n$ when n is even. Moreover, $\gamma \leq \xi$ and $\gamma \neq \xi$ (because $\gamma_n(I) < \xi_n(I)$ for large even n). Thus $\mu + \nu$ does not have a modulus.

Note: $\mu + \nu$ is of essentially the same type as the c -valued order bounded measure without a modulus given in [hSZ, p. 154].

2 An order bounded measure without an absolute majorant

In the proposition below, the first part is an immediate consequence of Dunford's characterization of relatively weakly compact subsets of $L_1(\mu)$ spaces (see [DU, Thm. III.2.15]). The verification of the second part is straightforward using Corollary 1.8.

Proposition 2.1. *The general form of a λ -continuous measure $\mu : \mathcal{B} \rightarrow \ell_\infty$ is $\mu(A) = (\int_A f_n d\lambda)$, where (f_n) is a relatively weakly compact sequence in L_1 , and the modulus of such a measure μ is given by $|\mu|(A) = (\int_A |f_n| d\lambda)$ so that also $|\mu|$ is λ -continuous.*

Remark. Obviously, a measure μ of the above form is c_0 -valued (resp. c -valued) iff the sequence (f_n) is weakly null (resp. weakly convergent).

Lemma 2.2. *Let (g_n) be a relatively weakly compact sequence in L_1 . Then there exists a sequence (ρ_n) of $\{-1, 1\}$ -valued measurable functions on I such that the sequence $(f_n) = (g_n \rho_n)$ is weakly null in L_1 .*

Proof. As easily seen, without loss of generality we may assume that the functions g_n are nonnegative. For every n denote $D_n = \{j \in \mathbb{N} : 1 \leq j \leq$

2^n and let $\{I_{nj} : j \in D_n\}$ be a partition of I into subintervals of length 2^{-n} . For every $n \in \mathbb{N}$ and $j \in D_n$ choose \mathcal{B} -partitions $I_{nj} = A_{nj} \cup B_{nj}$ so that

$$\int_{A_{nj}} g_n d\lambda = \int_{B_{nj}} g_n d\lambda,$$

and define a function ρ_n on I by setting $\rho_n(t) = -1$ if $t \in \bigcup_{j \in D_n} A_{nj}$, and $\rho_n(t) = 1$ otherwise. Denote $f_n = g_n \rho_n$ and $\varphi_n(A) = \int_A f_n d\lambda$. Clearly, $\varphi_n(I_{mj}) = 0$ whenever $n \geq m$ and $j \in D_m$. In particular,

$$\lim_{n \rightarrow \infty} \varphi_n(I_{mj}) = 0, \quad \text{for all } m \text{ and } j \in D_m. \quad (*)$$

Now, by assumption, the sequence (g_n) is relatively weakly compact in L_1 . Equivalently, the indefinite integrals $\int_{(\cdot)} g_n d\lambda$ are equi- λ -continuous. It follows that also the measures φ_n have the same property. In other words, the functions φ_n are equicontinuous on the semimetric Fréchet-Nikodym space (\mathcal{B}, d_λ) , where $d_\lambda(A, B) = \lambda(A \Delta B)$. In view of $(*)$ and since the finite unions of the intervals I_{nj} are dense in this space, it follows that $\lim_n \varphi_n(A) = 0$ for every $A \in \mathcal{B}$. Thus the sequence (f_n) is weakly null in L_1 . ■

Remark. If a sequence (g_n) in L_1 is relatively (norm) compact, and (r_n) is the usual Rademacher sequence on I , then the sequence $(f_n) = (g_n r_n)$ is weakly null in L_1 . Indeed, $(r_n) \subset L_\infty = L_1^*$ is weak* null, hence it converges uniformly to zero on every relatively compact subset of L_1 . In particular, $\int_A f_n d\lambda = \langle g_n 1_A, r_n \rangle \rightarrow 0$ for every $A \in \mathcal{B}$.

Proposition 2.3. *For every positive λ -continuous measure $\nu : \mathcal{B} \rightarrow \ell_\infty$ there exists a λ -continuous measure $\mu : \mathcal{B} \rightarrow c_0$ such that $|\mu| = \nu$.*

Proof. By Proposition 2.1, $\nu = (\int_{(\cdot)} g_n d\lambda)$, where (g_n) is a relatively weakly compact sequence in L_1 consisting of nonnegative functions. Applying Lemma 2.2, we find a sequence (ρ_n) of $\{-1, 1\}$ -valued measurable functions on I such that the sequence $(f_n) = (g_n \rho_n)$ is weakly null in L_1 . Let $\mu_n := \int_{(\cdot)} f_n d\lambda$ ($n \in \mathbb{N}$). Then $\mu := (\mu_n) : \mathcal{B} \rightarrow c_0$ is a σ -additive measure and, by the second part of Proposition 2.1, $|\mu| = (|\mu_n|) = \nu$. ■

Example 2.4. There exists a λ -continuous measure $\mu : \mathcal{B} \rightarrow c_0 \subset c$ that has no absolute majorant $\gamma : \mathcal{B} \rightarrow c$. (Though, of course, $\mu : \mathcal{B} \rightarrow c$ is order bounded.)

In view of Proposition 2.3, to prove the existence of such a measure μ , it is enough to find a positive λ -continuous measure $\nu : \mathcal{B} \rightarrow \ell_\infty$ that has no absolute majorant $\gamma : \mathcal{B} \rightarrow c$.

Fix a sequence $1 = p_1 < p_2 < \dots$ in \mathbb{N} such that

- 1) for every j , $p_{j+1} - p_j = k^2$ for some $k \in \mathbb{N}$;
- 2) for every $k \in \mathbb{N}$ the set $J_k := \{j \in \mathbb{N} : p_{j+1} - p_j = k^2\}$ is infinite.

Note that $(J_k)_{k=1}^\infty$ is a partition of \mathbb{N} .

For every $k \in \mathbb{N}$ let $\{I_{km} : m = 0, \dots, k^2 - 1\}$ be a partition of I into k^2 subintervals of length k^{-2} . Define a sequence of functions $g_n : I \rightarrow \mathbb{R}_+$ as follows: Given n , take the unique j such that $p_j \leq n < p_{j+1}$. Then $p_{j+1} - p_j = k^2$ for some $k \in \mathbb{N}$, and $n = p_j + m$, where $m \in \{0, \dots, k^2 - 1\}$. Set

$$g_n = k \cdot 1_{I_{km}}.$$

Clearly, $\|g_n\|_1 = k^{-1}$, hence the sequence (g_n) is relatively norm compact in L_1 . Define $\nu_n = \int_{(\cdot)} g_n d\lambda$ for each n , and let $\nu := (\nu_n) : \mathcal{B} \rightarrow \ell_\infty$. By Proposition 2.1, ν is a positive σ -additive measure. Suppose that ν has a (positive) majorant $\gamma = (\gamma_n) : \mathcal{B} \rightarrow c$. Then

$$\bar{\nu}(A) := \limsup_{n \rightarrow \infty} \nu_n(A) \leq \lim_{n \rightarrow \infty} \gamma_n(A) =: \bar{\gamma}(A) \quad \text{for every } A \in \mathcal{B},$$

and $\bar{\gamma} : \mathcal{B} \rightarrow \mathbb{R}_+$ is a measure.

Take any $k \in \mathbb{N}$. If $m = 0, \dots, k^2 - 1$, then $\nu_{p_j+m}(I_{km}) = k^{-1}$ for every $j \in J_k$, hence $\bar{\nu}(I_{km}) \geq k^{-1}$. Therefore,

$$k = k^2 \cdot k^{-1} \leq \sum_{m=0}^{k^2-1} \bar{\nu}(I_{km}) \leq \sum_{m=0}^{k^2-1} \bar{\gamma}(I_{km}) = \bar{\gamma}(I),$$

or $k \leq \bar{\gamma}(I) < \infty$ for every $k \in \mathbb{N}$, which is impossible.

Remarks 2.5. (a) It is worth pointing out that the essence of what we did above was a construction of a relatively norm compact nonnegative sequence (g_n) in L_1 for which there is no weakly convergent sequence (h_n) in L_1 such that $g_n \leq h_n$ for all n .

(b) Let a measure $\mu : \mathcal{B} \rightarrow c$ be as required in Example 2.4. Consider the induced integral operator $I_\mu : L_\infty(\lambda) \rightarrow c$: $I_\mu(f) = \int_I f d\mu$. It is, of course, order bounded; however, it is not regular. That is, there is

no positive linear operator $P : L_\infty(\lambda) \rightarrow c$ such that $|I_\mu(f)| \leq P(f)$ for all $0 \leq f \in L_\infty(\lambda)$. For, otherwise, the measure $\psi : \mathcal{B} \rightarrow c$ representing P would be an absolute majorant for μ , and μ does not admit such majorants. See [AB2, Example 1.11] for a different example of an order bounded nonregular operator.

(c) In Example 7.1 below, we construct a σ -additive order bounded measure on $\mathcal{P}(\mathbb{N})$, with values in a separable Banach lattice, that has no absolute majorant.

In the rest of the paper, unless stated otherwise, we assume that

$$E = (E, \tau) \text{ is a TRS.}$$

3 Absolute semivariations and absolute exhaustivity

For a measure $\mu : \mathcal{A} \rightarrow E$ and a Riesz pseudonorm ρ on E , we define the *absolute semivariation* $\mu_\rho : \mathcal{A} \rightarrow \overline{\mathbb{R}}_+$ of μ relative to ρ by

$$\mu_\rho(A) = \sup_{\pi \in \Pi(A)} \rho \left(\sum_{D \in \pi} |\mu(D)| \right).$$

It is easily verified that μ_ρ is a submeasure. In view of Proposition 1.2 (b) it is clear that if μ is order bounded, then $\mu_\rho(S) < \infty$ (so μ_ρ assumes only finite values). See also Propositions 3.2 and 3.9 below.

Recall that if ρ is a seminorm, then the (usual) semivariation of μ relative to ρ is the submeasure $\bar{\mu}_\rho : \mathcal{A} \rightarrow \overline{\mathbb{R}}_+$ that can be defined by any of the equalities below:

$$\begin{aligned} \bar{\mu}_\rho(A) &= \sup \left\{ \rho \left(\sum_{i=1}^k \alpha_i \mu(D_i) \right) : \{D_1, \dots, D_k\} \in \Pi(A), |\alpha_i| \leq 1 \right\} \\ &= \sup \{ \rho(\mu(B) - \mu(C)) : B, C \subset A \}. \end{aligned}$$

Remembering that for any set $A \in \mathcal{A}$ the net $(\sum_{B \in \pi} |\mu(B)| : \pi \in \Pi(A))$ is increasing, and making use of the Fact stated after Proposition 1.1, one easily verifies the following.

Proposition 3.1. *Let $\mu : \mathcal{A} \rightarrow E$ be a measure and ρ a Riesz pseudonorm on E .*

- (a) If γ is an absolute majorant for μ , then $\mu_\rho(A) \leq \rho(\gamma(A))$ for all $A \in \mathcal{A}$. In particular, if $|\mu|$ exists, then $\mu_\rho(A) \leq \rho(|\mu|(A))$ for all $A \in \mathcal{A}$.
- (b) If ρ has the Fatou property and $|\mu|$ exists properly on a set $A \in \mathcal{A}$, then $\mu_\rho(A) = \rho(|\mu|(A))$.
- (c) If ρ is a seminorm, then for every $A \in \mathcal{A}$, $\bar{\mu}_\rho(A) \leq \mu_\rho(A)$; and if ρ is an M -seminorm, then $\bar{\mu}_\rho(A) = \mu_\rho(A)$.

If $\rho = \|\cdot\|$ is the norm of a Banach lattice E , then we shall write $\|\mu\|_a(\cdot)$ and $\|\mu\|(\cdot)$ for the associated absolute semivariation and semivariation of μ , respectively.

Remark. In [DW, Thm. 1] it has been shown that for the indefinite Bochner integrals $\mu = \int f d\nu$ in Banach lattices one always has $\|\mu\|_a(A) = \|\int_A |f| d\nu\| = \|\mu|(A)\|$.

Proposition 3.2. Let $E = (E, \|\cdot\|)$ be a Banach lattice, $\kappa: E \rightarrow E^{**}$ the canonical embedding, and $\mu: \mathcal{A} \rightarrow E$ a measure. Then $\|\mu\|_a(S) < \infty$ if and only if the measure $\kappa \circ \mu: \mathcal{A} \rightarrow E^{**}$ is order bounded. Moreover, in this case the modulus $|\kappa \circ \mu|$ exists properly and $\|\mu\|_a(A) = \|\kappa \circ \mu|(A)\|$ for every $A \in \mathcal{A}$.

Proof. “If”: Since E^{**} is Dedekind complete and has the Fatou property, $|\kappa \circ \mu|$ exists properly and, by Proposition 3.1 (b), $\|\mu\|_a(A) = \|\kappa \circ \mu\|_a(A) = \|\kappa \circ \mu|(A)\|$ for all $A \in \mathcal{A}$.

“Only if”: Let $A \in \mathcal{A}$. The net $(\sum_{D \in \pi} |\mu(D)| : \pi \in \Pi(A))$ is increasing and norm bounded by $\|\mu\|_a(A)$. As E^{**} has the Levi property, $\sup_\pi \sum_{D \in \pi} |\mu(D)|$ exists in E^{**} . It follows that $|\kappa \circ \mu|$ exists properly. ■

Proposition 3.3. If $\mu: \mathcal{A} \rightarrow E$ is a σ -additive measure and ρ is a continuous Riesz pseudonorm on E , then the submeasure μ_ρ is σ -subadditive.

Proof. Let (A_n) be a disjoint sequence in \mathcal{A} such that also its union A is in \mathcal{A} . Denote $B_N = \bigcup_{n=N+1}^\infty A_n$, and let $\pi \in \Pi(A)$. First observe that for each $D \in \pi$ and $N \in \mathbb{N}$ one has $|\mu(D)| \leq \sum_{n=1}^N |\mu(A_n \cap D)| +$

$|\mu(B_N \cap D)|$. Hence

$$\begin{aligned} \rho\left(\sum_{D \in \pi} |\mu(D)|\right) &\leq \sum_{n=1}^N \rho\left(\sum_{D \in \pi} |\mu(A_n \cap D)|\right) + \rho\left(\sum_{D \in \pi} |\mu(B_N \cap D)|\right) \\ &\leq \sum_{n=1}^{\infty} \mu_{\rho}(A_n) + \rho\left(\sum_{D \in \pi} |\mu(B_N \cap D)|\right). \end{aligned}$$

Since μ is σ -additive and $B_N \downarrow \emptyset$, the last term on the right tends to zero as $N \rightarrow \infty$, and we easily conclude that $\mu_{\rho}(A) \leq \sum_{n=1}^{\infty} \mu_{\rho}(A_n)$. ■

Let us say that a measure $\mu: \mathcal{A} \rightarrow E$ is *absolutely exhaustive* if for every disjoint sequence (C_j) in \mathcal{A} the series $\sum_j |\mu(C_j)|$ is Cauchy. Evidently, if μ admits an exhaustive absolute majorant (in particular, if $|\mu|$ exists and is exhaustive), then μ is absolutely exhaustive. However, a σ -additive measure that is absolutely exhaustive need not even be order bounded. Such are, for instance, the c_0 -valued measures in Examples 1.9 and 2.4.

Proposition 3.4. *Let P be a family of Riesz pseudonorms defining the topology of E . Then for every measure $\mu: \mathcal{A} \rightarrow E$ the following are equivalent:*

- (a) *For every $\rho \in P$, the absolute semivariation μ_{ρ} is exhaustive.*
- (b) *μ is absolutely exhaustive.*

Proof. (a) \implies (b): Let (C_j) be a disjoint sequence in \mathcal{A} . Since μ_{ρ} is exhaustive and $\rho(\sum_{j=m}^n |\mu(C_j)|) \leq \mu_{\rho}(\bigcup_{j=m}^n C_j)$ for all $n \geq m$, the series $\sum_j |\mu(C_j)|$ is Cauchy.

(b) \implies (a): Suppose μ_{ρ} is not exhaustive. Thus there is a disjoint sequence (A_n) in \mathcal{A} such that $\mu_{\rho}(A_n) > \varepsilon$ for all n and some $\varepsilon > 0$. Then, by the definition of μ_{ρ} , for every n there exists $\pi(n) \in \Pi(A_n)$ such that $\rho(\sum_{B \in \pi(n)} |\mu(B)|) > \varepsilon$. Arrange the families of sets $\pi(1), \pi(2), \dots$ in a single sequence C_1, C_2, \dots so that $\pi(n) = \{C_j : j \in J(n)\}$, where $(J(n))$ is a sequence of consecutive intervals in \mathbb{N} . Then $\rho(\sum_{j \in J(n)} |\mu(C_j)|) > \varepsilon$ for every n . This is impossible because the C_j 's are pairwise disjoint and so the series $\sum_j |\mu(C_j)|$ is Cauchy in E , by (b). ■

Proposition 3.5. *Let $\mu: \mathcal{A} \rightarrow E$ be a σ -additive measure and ρ a continuous Riesz pseudonorm on E . If the absolute semivariation μ_{ρ} is exhaustive, then it is also order continuous.*

Proof. This follows from Proposition 3.3 and [D1, Part II, Thm. 5.3]. A direct argument showing this runs as follows: Suppose μ_ρ is not order continuous. Thus there is a sequence (A_n) in Σ with $A_n \downarrow \emptyset$ such that $\mu_\rho(A_n) > \varepsilon$ for all n and some $\varepsilon > 0$. Then for every n there exists $\pi(n) \in \Pi(A_n)$ such that $\rho(\sum_{B \in \pi(n)} |\mu(B)|) > \varepsilon$. Note that $\mu(B \setminus A_n) \rightarrow \mu(B)$ for every $B \in \Sigma$, hence also $\sum_{B \in \pi} |\mu(B \setminus A_n)| \rightarrow \sum_{B \in \pi} |\mu(B)|$ for every finite $\pi \subset \Sigma$.

Using this, we find by an easy induction a sequence $1 = n_0 < n_1 < n_2 < \dots$ such that

$$\rho\left(\sum_{B \in \pi(n_{k-1})} |\mu(B \setminus A_{n_k})|\right) > \varepsilon \quad \text{for } k = 1, 2, \dots$$

Hence $\mu_\rho(A_{n_{k-1}} \setminus A_{n_k}) > \varepsilon$ for each k , contradicting the exhaustivity of μ_ρ . ■

Corollary 3.6. *If a measure $\mu: \mathcal{A} \rightarrow E$ is σ -additive and admits an exhaustive absolute majorant, then the absolute semivariation μ_ρ is exhaustive and order continuous for every continuous Riesz pseudonorm ρ on E .*

As a consequence of Propositions 3.4 and 3.5 we also have the following.

Corollary 3.7. *Let P be a family of Riesz pseudonorms defining the topology of E . Then for a σ -additive measure $\mu: \Sigma \rightarrow E$ the following are equivalent:*

- (a) *For every $\rho \in P$, the absolute semivariation μ_ρ is order continuous.*
- (b) *For every $\rho \in P$, the absolute semivariation μ_ρ is exhaustive.*
- (c) *μ is absolutely exhaustive.*

Proposition 3.8. *If a measure $\mu: \Sigma \rightarrow E$ is σ -additive and absolutely exhaustive, then $\mu_\rho(S) < \infty$ for every continuous Riesz pseudonorm ρ on E .*

Proof. By Corollary 3.7, μ_ρ is order continuous. Consider a maximal family \mathcal{C} of disjoint sets $C \in \Sigma$ with $0 < \mu_\rho(C) < \infty$. Clearly, \mathcal{C} has to be countable, say $\mathcal{C} = \{C_1, C_2, \dots\}$. Let $S_0 = \bigcup \mathcal{C}$. Since $\mu_\rho(\bigcup_{n \geq m} C_n) \rightarrow 0$ as $m \rightarrow \infty$, $\mu_\rho(S_0) < \infty$. To finish the proof, it is enough to show that $\mu_\rho(S \setminus S_0) = 0$. Otherwise $\mu_\rho(S \setminus S_0) = \infty$ and, as easily seen,

$S \setminus S_0$ must contain an atom A for μ_ρ with $\mu_\rho(A) = \infty$. Then for any $D \subset A$ either $\rho(\mu(D)) = 0$ or $\rho(\mu(A \setminus D)) = 0$. It follows that for every $\{D_1, \dots, D_n\} \in \Pi(A)$ we have $\rho(\sum_i |\mu(D_i)|) = \rho(\mu(A))$. In consequence, $\mu_\rho(A) = \rho(\mu(A)) < \infty$; a contradiction. ■

Applying Proposition 3.8 and Corollary 3.7 in the setting of Proposition 3.2, we have the following complement to the latter result.

Corollary 3.9. *Let E be a Banach lattice, and $\kappa: E \rightarrow E^{**}$ the canonical embedding. If a measure $\mu: \Sigma \rightarrow E$ is σ -additive and absolutely exhaustive, then the measure $\kappa \circ \mu: \Sigma \rightarrow E^{**}$ has a σ -additive proper modulus $|\kappa \circ \mu|$.*

4 Topological Riesz spaces with Properties (S) and (CS)

Recall that $E = (E, \tau)$ is a TRS. Consider a series $(*) \sum_{n=1}^{\infty} x_n$ in E .

We shall say that the series $(*)$ is *perfectly order bounded* if the set of all its finite sums, $\{\sum_{n \in M} x_n : M \subset \mathbb{N} \text{ finite}\}$, is order bounded. In view of the fact stated before Proposition 1.2, this is equivalent to the requirement that the series $\sum_n |x_n|$ is order bounded, that is, the sequence $\sum_{i=1}^n |x_i|$ ($n \in \mathbb{N}$) of its partial sums is order bounded.

Now assume that the series $(*)$ is subseries convergent in E . Then we define the associated (σ -additive) measure $\xi: \mathcal{P}(\mathbb{N}) \rightarrow E$ by $\xi(M) = \sum_{n \in M} x_n$. As easily seen, the series $(*)$ is perfectly order bounded iff the measure ξ is order bounded.

The following extends a result from [DW, Cor.], where E was a Banach lattice. It can be easily verified by a direct argument, or deduced from Propositions 6.4 and 8.6 below.

Proposition 4.1. *Let $\sum_n x_n$ be a subseries convergent series in E with the associated measure $\xi: \mathcal{P}(\mathbb{N}) \rightarrow E$. Then the following hold.*

(a) *The modulus $|\xi|$ exists properly iff the series $\sum_n |x_n|$ is subseries (o)-convergent, in which case*

$$|\xi|(M) = (o)\text{-}\sum_{n \in M} |x_n| \quad \text{for all } M \subset \mathbb{N};$$

(b) *The modulus $|\xi|$ exists and is σ -additive iff the series $\sum_n |x_n|$ is*

subseries convergent, in which case

$$|\xi|(M) = \sum_{n \in M} |x_n| \quad \text{for all } M \subset \mathbb{N},$$

and hence $|\xi|$ exists properly.

We now introduce a few definitions.

We shall say that the TRS E (or its topology) has *Property (S)* if for every subseries convergent and perfectly order bounded series $\sum_n x_n$ in E , the series $\sum_n |x_n|$ is convergent. Note that then every subseries of the latter series must be convergent. In other words, E has Property (S) iff for every order bounded σ -additive measure $\mu: \mathcal{P}(\mathbb{N}) \rightarrow E$ its modulus $|\mu|$ exists and is σ -additive. As was already mentioned in the Introduction, for Banach lattices E this property was introduced by Groenewegen [G] as *property (C)*; see also [GvR], where such E are said to be *order bounded AM*. Some characterizations of this property, mostly of measure or operator theoretic type, were given in [G, Ch. III and V] and [GvR], and more recently in [CW, Thm. 2.6].

A Cauchy type version of (S) will also be useful. But first recall that a series $\sum_n x_n$ is called *unconditionally Cauchy* if for every neighborhood U of zero there is a finite set $M \subset \mathbb{N}$ such that $\sum_{n \in N} x_n \in U$ for all finite sets $N \subset \mathbb{N}$ disjoint from M . This is equivalent to requiring that every subseries $\sum_k x_{n_k}$ of the series $\sum_n x_n$ is Cauchy (in the usual sense). Obviously, for positive series “unconditionally Cauchy” and “Cauchy” is the same. Now, we shall say that the TRS E (or its topology) has *Property (CS)* if for every unconditionally Cauchy and perfectly order bounded series $\sum_n x_n$ in E , the series $\sum_n |x_n|$ is (unconditionally) Cauchy. Evidently, if E is sequentially complete, or merely has sequentially complete order intervals (comp. with [AB1, Thms. 13.2 and 17.6]), then Properties (S) and (CS) are equivalent.

We will also need a weaker, “order bounded” variant of the (σ -) monotone completeness property, (σ -) MCP (see [AB1, Def. 7.4 and Thm. 16.1]). We shall say the TRS E has the σ -MCP *for intervals*, if every monotone order bounded Cauchy sequence in E is convergent; equivalently, every Cauchy and order bounded positive series in E is convergent. The general MCP *for intervals* is defined similarly, using nets in place of sequences.

This type of completeness goes back to Veksler's paper [V], and was later studied in [La]. It is known (see [V] and [La, Prop. 5.2]) that a metrizable Riesz space E is intervally complete, i.e., has (topologically) complete order intervals, iff E has the σ -MCP for intervals. In consequence, if a TRS E has the σ -MCP for intervals, then it is *relatively uniformly complete*. That is, for every $0 \leq u \in E$ the ideal E_u generated by u is a Banach space under the norm $\|\cdot\|_u$ whose closed unit ball is equal to the order interval $[-u, u]$.

Proposition 4.2. (a) *If E has Property (CS), then every exhaustive order bounded measure $\mu : \mathcal{A} \rightarrow E$ is absolutely exhaustive.*

(b) *If E has the σ -MCP for intervals, Σ is an infinite σ -algebra, and every σ -additive order bounded measure $\mu : \Sigma \rightarrow E$ is absolutely exhaustive, then E has Property (S).*

Proof. (b): Let $\sum_n x_n$ be a subseries convergent and perfectly order bounded series in E . Since Σ is infinite, we may choose a disjoint sequence (A_n) in $\Sigma \setminus \{\emptyset\}$, and for each n pick a point $s_n \in A_n$. Define $\mu : \Sigma \rightarrow E$ by $\mu(A) = \sum_n 1_A(s_n)x_n$. Then μ is a σ -additive and order bounded measure. By assumption, μ is absolutely exhaustive. Hence, the series $\sum_n |\mu(A_n)| \equiv \sum_n |x_n|$ is Cauchy in E . Since it is also order bounded, and E has the σ -MCP for intervals, it is convergent. ■

Proposition 4.3. *Assume that E has both the σ -MCP for intervals and Property (CS). Then for every unconditionally Cauchy and perfectly order bounded series $\sum_n x_n$ in E , both the series $\sum_n x_n$ and $\sum_n |x_n|$ are subseries convergent. In particular, E has Property (S).*

Proof. In view of (CS), every subseries of the series $\sum_n |x_n|$ is Cauchy and order bounded, hence converges in E , by the σ -MCP for intervals. Since $\sum_{i=m}^n x_i^\pm \leq \sum_{i=m}^n |x_i|$ whenever $m \leq n$, the positive series $\sum_n x_n^+$ and $\sum_n x_n^-$ are Cauchy and order bounded. Applying again the σ -MCP for intervals, it follows that the series $\sum_n x_n$ is subseries convergent. ■

Corollary 4.4. *Assume that E has the σ -MCP for intervals and Property (CS), and let $\mu : \mathcal{A} \rightarrow E$ be an exhaustive order bounded measure. Then for every disjoint sequence (D_n) in \mathcal{A} both the series $\sum_n \mu(D_n)$ and $\sum_n |\mu(D_n)|$ are convergent, and for every monotone sequence (A_n) in \mathcal{A} the limit $\lim_n \mu(A_n)$ exists.*

As will be seen from Propositions 4.5, 4.7 and 4.8, the classes of

TRS's with Property (S) or (CS) are fairly wide. In particular, every AM-space has Property (S), because:

Proposition 4.5. *For a Banach lattice $E = (E, \|\cdot\|)$ the following are equivalent:*

- (a) *E is Riesz homeomorphic to an AM-space.*
- (b) *For every unconditionally convergent series $\sum_n x_n$ in E , the series $\sum_n |x_n|$ is convergent.*
- (c) *Every unconditionally convergent series in E is order bounded.*
- (d) *For every unconditionally convergent series $\sum_n x_n$ in E , the series $\sum_n |x_n|$ is norm bounded (i.e., the sequence of its partial sums is norm bounded).*

Proof. (a) \implies (b): We may assume that E is an AM-space. Let $\sum_n x_n$ be an unconditionally convergent series in E . Since the norm in E is an M-norm, using the fact stated before Proposition 1.2 (or, alternatively, Proposition 3.1 (c)) we see that

$$\left\| \sum_{n \in J} |x_n| \right\| \leq 2 \sup_{K \subset J} \left\| \sum_{n \in K} x_n \right\| \quad \text{for every finite subset } J \text{ of } \mathbb{N}.$$

It follows that the series $\sum_n |x_n|$ converges.

(b) \implies (c) and (c) \implies (d) are obvious.

(d) \implies (a): By Theorem 3.2 in [APY], E is isomorphic to an AM-space (or admits an equivalent M-norm) if and only if the following holds:

(*) There exists a constant $d > 0$ such that

$$\left\| \sum_{i=1}^n |x_i| \right\| \leq d \sup \left\{ \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| : \varepsilon_i = \pm 1 \right\} \quad \text{for all } x_1, \dots, x_n \text{ in } E.$$

Suppose (*) is false. Then we can find an infinite sequence (x_n) in E such that for a certain partition (J_k) of \mathbb{N} into consecutive intervals,

$$\sup \left\{ \left\| \sum_{i \in J_k} \varepsilon_i x_i \right\| : \varepsilon_i = \pm 1 \right\} \leq 2^{-k} \quad \text{and} \quad \left\| \sum_{i \in J_k} |x_i| \right\| = k, \quad k = 1, 2, \dots$$

Clearly, the series $\sum_n x_n$ is unconditionally convergent, while the series $\sum_n |x_n|$ is not norm bounded. A contradiction. ■

Remark. The implication (a) \implies (b) is a result of Jameson [Ja2, Prop. 5], while (b) \implies (a) was shown in [APY, Thm. 3.2]. See also [CL, Cor. 2], [Le, Cor. 2.2], and [hS, IV, ex. 16]. The equivalence of (a) and (c) was established in [G, Thm. III.4.8]. It is worth pointing out that Jameson's result is essentially an abstract version of an old result due to W. Sierpiński (1910 for real functions; [S] for complex functions): *A series of functions $\sum_n f_n$ is unconditionally uniformly convergent iff the series $\sum_n |f_n|$ is uniformly convergent.* Similar results for the subseries a.e. uniform convergence of series of bounded measurable or continuous functions were obtained by Orlicz [O; Thms. 5, 6, and 7].

We note the following consequence of Proposition 4.5.

Corollary 4.6. *For a Banach lattice E the following are equivalent:*

- (a) *E is Riesz homeomorphic to an AM-space.*
- (b) *Every σ -additive measure $\mu: \mathcal{P}(\mathbb{N}) \rightarrow E$ is absolutely exhaustive.*
- (c) *Every σ -additive E -valued measure defined on a σ -algebra is absolutely exhaustive.*

By an *M-topological Riesz space* (M-TRS) we shall mean a TRS E whose topology can be defined by a family of Riesz M-seminorms or, equivalently, a TRS E which is Riesz homeomorphic to a Riesz subspace of the product of a family of AM-spaces. (Cf. [Ja1] and [C, Cor. 5.8.24].)

We shall say that a Riesz subspace F of a TRS E is *sequentially obm-closed* in E if, whenever a monotone (or just increasing) sequence in F is order bounded in F and converges in E , then its limit is actually in F . Equivalently, whenever a positive series in F is order bounded in F and converges in E , its sum is in F . Evidently, if F is an ideal in E , or F is sequentially closed in E , or F is sequentially order closed in E , then F is sequentially obm-closed in E . Note that, in the space of bounded real functions on $[0, 1]$ considered with the topology of pointwise convergence, the subspace consisting of Borel measurable functions is both sequentially closed and sequentially order closed, but not an ideal. Furthermore, the linear span of the unit vectors in ℓ_∞ is a Riesz subspace that is neither topologically nor order sequentially closed. However, it is

sequentially obm-closed; in fact, its order intervals are even norm complete. This example provides also a simple illustration to part (a) of the proposition below.

Proposition 4.7. *The following statements hold.*

- (a) *If a TRS has Property (S), so do all its sequentially obm-closed Riesz subspaces.*
- (b) *The product of any family of TRS with Property (S) has Property (S).*
- (c) *Every M-TRS with the σ -MCP for intervals has Property (S).*
- (d) *If a TRS is σ -Dedekind complete and σ -Lebesgue, then it has the σ -MCP for intervals and Property (S).*

Exact analogues of statements (a), (b) and (c) are also valid for Property (CS), with “sequentially obm-closed” in (a), and “the σ -MCP for intervals” in (c) omitted. In addition,

- (e) *If a TRS is pre-Lebesgue (in particular, Lebesgue), then it has Property (CS).*

Proof. Statements (a) and (b) are obvious, and (c) follows easily from Proposition 4.5. As for (d), note that a TRS E is σ -Dedekind complete and σ -Lebesgue iff every order bounded positive series in E is convergent. Finally, if E is Lebesgue, then it is pre-Lebesgue (see [AB1, Thm. 10.2]), and the latter property implies (CS) in a trivial way. ■

The following two easy facts are also worth mentioning.

Proposition 4.8.

- (a) *If $\{\tau_\alpha : \alpha \in A\}$ is a family of locally solid topologies on a Riesz space E , each having Property (CS), then also the supremum topology, $\sup_\alpha \tau_\alpha$, has Property (CS).*
- (b) *Let (E_α, τ_α) , $\alpha \in A$, be a family of TRS’s having Property (S) and such that each (E_α, τ_α) is continuously included as a Riesz subspace in a Riesz space F equipped with a Hausdorff topology τ_F . Then the TRS (E, τ) , where $E = \bigcap_\alpha E_\alpha$ and $\tau = \sup_\alpha (\tau_\alpha|_E)$, has Property (S).*

5 The σ -MCP for intervals and positive copies of c or ℓ_∞

We first prove a result that is complementary to analogous characterizations obtained in [DL, Sec. 2].

Proposition 5.1. *For a TRS E the following are equivalent.*

- (a) E is σ -Dedekind complete and σ -Lebesgue.
- (b) E has the σ -MCP for intervals and contains no positive copy of c ; that is, there is no positive linear isomorphic embedding $T: c \rightarrow E$.

Proof. (b) \implies (a): Suppose (a) is false. Then (see the proof of Proposition 4.7), there is a nonconvergent positive series $\sum_n x_n$ in E such that $x_1 + \dots + x_n \leq x$ for some $x \in E$ and all n . Then the series is not Cauchy, because E has the σ -MCP for intervals. Hence we may assume without loss of generality that $x_n \not\rightarrow 0$. Let $a = (\alpha_n) \in c_0$. Then

$$\sum_{i=m}^n |\alpha_i x_i| \leq \sup_{m \leq i \leq n} |\alpha_i| \sum_{i=m}^n |x_i| \leq \sup_{m \leq i \leq n} |\alpha_i| \cdot x \quad \text{whenever } m \leq n,$$

hence the positive series $\sum_n |\alpha_n x_n|$ is Cauchy. As in the proof of Proposition 4.3, it follows that the series $\sum_n \alpha_n x_n$ converges in E . Clearly, the formula $T_0(a) = \sum_n \alpha_n x_n$ defines a positive linear operator from c_0 to E mapping the unit ball of c_0 into the interval $[-x, x]$. Hence T_0 is also continuous, and $T_0(e_n) = x_n \not\rightarrow 0$. By [D3], there exists an infinite subset M of \mathbb{N} such that T_0 is an isomorphic (and positive) embedding of $c_0(M) \cong c_0$ into E .

We may assume of course that $M = \mathbb{N}$ and that $x \notin T_0(c_0)$. Define a linear operator $T: c \rightarrow E$ as follows: Given $a = (\alpha_n) \in c$, denote $\alpha_\infty = \lim_n \alpha_n$ and $a_0 = (\alpha_n - \alpha_\infty) \in c_0$, and then set

$$T(a) = \alpha_\infty x + T_0(a_0) = \alpha_\infty x + \sum_{n=1}^{\infty} (\alpha_n - \alpha_\infty) x_n. \quad (*)$$

If $a \geq 0$, then for every n ,

$$\alpha_\infty x + \sum_{i=1}^n (\alpha_i - \alpha_\infty) x_i = \alpha_\infty \left(x - \sum_{i=1}^n x_i \right) + \sum_{i=1}^n \alpha_i x_i \geq 0,$$

and it follows that $T(a) \geq 0$. Thus T is positive. Let $e_\infty = (1, 1, 1, \dots) \in c$. Then T maps isomorphically each summand in $c = \mathbb{R} \cdot e_\infty \oplus c_0$ onto the corresponding summand in $T(c) = \mathbb{R} \cdot x \oplus T_0(c_0)$, hence T is an isomorphic embedding.

(a) \implies (b): That E has to enjoy the σ -MCP for intervals is obvious. Suppose $T: c \rightarrow E$ is a positive isomorphic embedding, and denote $x_n = Te_n$. Then $x_1 + \dots + x_n \leq T(e_\infty)$ for each n so that the positive series $\sum_n x_n$ is order bounded. However, it is not convergent because $x_n \not\rightarrow 0$, which contradicts (a). ■

Remark 5.2. For E a Banach lattice the above result states that E is Lebesgue (or has order continuous norm) iff E contains no positive copy of c (comp. [AB2, Thms. 14.9, 14.21]). It is worth noting that “ E contains no positive copy of c ” cannot be replaced here with “ E contains no Riesz homeomorphic copy of c ”. Indeed, the Banach lattice $C([0, 1])$ is not Lebesgue, and though it obviously contains a positive copy of c , it contains no Riesz homeomorphic copy of c .

For, suppose it does. Then there are a disjoint sequence (f_n) in $C[0, 1]$ and an $f \in C[0, 1]$ such that $\|f_n\| > 1$, $0 \leq f_n \leq f$, and $(f - f_n) \wedge f_n = 0$ for all n . For each n choose a point $t_n \in [0, 1]$ so that $f_n(t_n) = \|f_n\|$, and let $[a_n, b_n]$ be a maximal closed interval containing t_n and such that $f(t) > 0$ for $t \in (a_n, b_n)$. Observe that $f(s_n) = 0$ for some $s_n \in \{a_n, b_n\}$ and that $b_n - a_n \rightarrow 0$ as $n \rightarrow \infty$. In consequence, also $t_n - s_n \rightarrow 0$. Now, $f(t_n) - f(s_n) = f_n(t_n) - f_n(s_n) = f_n(t_n) > 1$ for every n , contradicting the uniform continuity of f . ■

The next result is taken from [DL, Prop. 2.8]; it will be of decisive importance in the proof of Theorem 6.7 in the next section.

Proposition 5.3. *Let a TRS E be relatively uniformly complete (in particular, let E have the σ -MCP for intervals). Then E contains no positive copy of ℓ_∞ iff every measure $\nu: \mathcal{P}(\mathbb{N}) \rightarrow E_+$ is exhaustive.*

6 Exhaustivity and σ -additivity of the modulus of a measure

We first tackle the problem of exhaustivity and σ -additivity of the modulus by applying the results of the preceding section, thus implicitly using the associated absolute semivariations μ_ρ . Theorem 6.2 (i) (b) and

Theorem 6.2 (ii) (b) below are generalizations of the results first proved by Schaefer and Zhang in [hSZ, Thm. 5] in the case of Banach lattices, with the latter result for the particular case of Dedekind complete AM-spaces with units. For the same particular case, Theorem 6.2 (ii) (a) was obtained earlier by Schmidt [kS2, Thm. 1.3].

Directly from Propositions 3.1 (b) and 3.4, and Corollary 3.7 we have the following.

Proposition 6.1. *Let E have the Fatou property.*

- (a) *If a measure $\mu: \mathcal{A} \rightarrow E$ has a proper modulus $|\mu|$, then $|\mu|$ is exhaustive iff μ is absolutely exhaustive.*
- (b) *If a σ -additive measure $\mu: \Sigma \rightarrow E$ has a proper modulus $|\mu|$, then $|\mu|$ is σ -additive iff μ is absolutely exhaustive.*

Combining the above with Propositions 4.2 and 4.7 we obtain the following result. (Recall that if a TRS is Lebesgue, then it is also Fatou, see [AB1, Thm. 11.6].)

Theorem 6.2. *If E has both the Fatou property and Property (CS), in particular,*

- (i) *E is Lebesgue, or*
- (ii) *E is an M -TRS with the Fatou property,*

then the following statements hold.

- (a) *If an exhaustive measure $\mu: \mathcal{A} \rightarrow E$ has a proper modulus $|\mu|$, then $|\mu|$ is exhaustive.*
- (b) *If a σ -additive measure $\mu: \Sigma \rightarrow E$ has a proper modulus $|\mu|$, then $|\mu|$ is σ -additive.*

Remark. In the case (i) statement (a) could be formulated in a stronger form: *If a measure $\mu: \mathcal{A} \rightarrow E$ has a proper modulus $|\mu|$, then $|\mu|$ is exhaustive.* In fact, by the Lebesgue property, every positive order bounded measure with values in E is automatically exhaustive.

In the remaining part of this section we deal with the above problem using more direct methods. Our first result, Theorem 6.3, is a variant of Theorem 6.2 (ii). In proving the proper existence of $|\mu|$, we follow closely the ideas already used in [kS1, Thm. 5.3] (and later in [DW, Ex. 1, (b)]) for E an AM-space. Theorem 6.3, (b) for E an AM-space, with μ assumed additionally to be order bounded, was obtained in [hSZ, Cor. on p. 157] (cf. also [kS3, Thm. 5.2.7]).

Theorem 6.3. *Let E be a quasi-complete M-TRS.*

- (a) *If a measure $\mu: \mathcal{A} \rightarrow E$ has relatively compact range, then $|\mu|$ exists properly and is exhaustive.*
- (b) *If a measure $\mu: \Sigma \rightarrow E$ has relatively compact range and is σ -additive, then $|\mu|$ exists properly and is σ -additive.*

Proof. (b): For $A \in \Sigma$ denote by \mathfrak{F}_A the class of all finite families Δ of sets from Σ_A directed upward by inclusion. Furthermore, let $M_A = \{\sup_{B,C \in \Delta} |\mu(B) - \mu(C)| : \Delta \in \mathfrak{F}_A\}$.

Now, by assumption, E is a quasi-complete M-TRS and the range of μ is relatively compact. Therefore, by [AB2, Thm. 12.29], which can be generalized easily to quasi-complete M-TRS's, also the set M_A is relatively compact, hence its sup exists and belongs to the closure of M_A . Evidently, $\sup M_A = \sup \{|\mu(B) - \mu(C)| : B, C \subset A\}$. It follows that $|\mu|$ exists properly and that $|\mu|(A) = \sup M_A$ for all $A \in \Sigma$. Moreover, by compactness, $|\mu|(A) = (\tau)\text{-}\lim_{\Delta} \sup_{B,C \in \Delta} |\mu(B) - \mu(C)|$. In consequence, for every continuous M-seminorm ρ on E ,

$$\rho(|\mu|(A)) \leq \sup_{\Delta} \sup_{B,C \in \Delta} \rho(\mu(B) - \mu(C)) \leq 2 \sup_{B \subset A} \rho(\mu(B)).$$

From this the σ -additivity of $|\mu|$ follows immediately.

The proof of (a) is similar noting that the relative compactness of $\mu(\mathcal{A})$ implies μ is exhaustive (by a result in [R]). ■

Remark. In the theorem above E is assumed quasi-complete. However, it is not hard to see that, in general, for a TRS the notions of completeness and quasi-completeness coincide (see [We, Prop. 2.8], [W2, Prop. 1.4]).

Proposition 6.4. *Let $\mu: \mathcal{A} \rightarrow E$ be a σ -additive measure such that the modulus $|\mu|$ exists. Assume that whenever $A_n \uparrow A$ in \mathcal{A} , the limit $\lim_n |\mu|(A_n)$ exists in E . Then $|\mu|$ is σ -additive.*

Proof. Let $A_n \uparrow A$ in \mathcal{A} . Define a measure $\nu: \mathcal{A}_A \rightarrow E$ by $\nu(B) = \lim_n |\mu|(A_n \cap B)$. Obviously, $\nu \leq |\mu| \upharpoonright \mathcal{A}_A$. Since $|\mu(B)| = \lim_n |\mu|(A_n \cap B)| \leq \nu(B)$ for each $B \in \mathcal{A}_A$, the measure ν is an absolute majorant for $\mu \upharpoonright \mathcal{A}_A$. From this and Proposition 1.1 it follows that $|\mu| \upharpoonright \mathcal{A}_A \leq \nu$ and, consequently, $\nu = |\mu| \upharpoonright \mathcal{A}_A$. In particular, $\nu(A) = |\mu|(A)$. ■

Note that the condition imposed on $|\mu|$ above is satisfied when $|\mu|$ is exhaustive and E has the σ -MCP for intervals. It is so in particular when E is σ -Dedekind complete and σ -Lebesgue. In fact, in the latter case, it follows from Corollary 1.3 that every order bounded measure $\mu: \mathcal{A} \rightarrow E$ is exhaustive. Hence Proposition 6.4 has the following two consequences of which the second is a remarkable improvement of Theorem 6.2 (i).

Corollary 6.5. *Let E have the σ -MCP for intervals. If $\mu: \mathcal{A} \rightarrow E$ a σ -additive measure such that the modulus $|\mu|$ exists and is exhaustive, then it is also σ -additive.*

Corollary 6.6. *Let E be σ -Lebesgue and σ -Dedekind complete. If $\mu: \mathcal{A} \rightarrow E$ is a σ -additive measure such that the modulus $|\mu|$ exists, then $|\mu|$ is σ -additive.*

Our next result is also an improvement of Theorem 6.2 (i) and, at the same time, a generalization of a very special case of a result due to Kronsbein and Meyer-Nieberg [KM-N, Thm. 11] for E a Banach lattice without isomorphic copies of ℓ_∞ . (In that case a theorem of Diestel and Faires [DU, Thm. I.4.2] can be used in place of our Proposition 5.3.)

Theorem 6.7. *Let E have the σ -MCP for intervals and contain no positive copy of ℓ_∞ . If $\mu: \Sigma \rightarrow E$ is a σ -additive measure such that the modulus $|\mu|$ exists, then $|\mu|$ is σ -additive.*

Proof. By Proposition 5.3, $|\mu|$ is exhaustive. To finish, apply Corollary 6.5. ■

Remark. The noncontainment of a positive copy of ℓ_∞ is by no means necessary for the assertion of Theorem 6.7 to hold. This should be clear from Theorem 6.2 (ii) (or [hSZ, Thm. 5(1)]). In fact, it is rather straightforward to see that for every σ -additive measure $\mu = (\mu_n): \Sigma \rightarrow \ell_\infty$ its modulus $|\mu| = (|\mu_n|)$ is also σ -additive. (Uniform σ -additivity or exhaustivity of the μ_n 's implies the same property for the $|\mu_n|$'s.)

We conclude with a result that can be regarded as a complement to Corollary 6.5.

Proposition 6.8. *Let E have the MCP for intervals. If a σ -additive measure $\mu: \mathcal{A} \rightarrow E$ admits an exhaustive absolute majorant γ , then it also admits a smaller σ -additive absolute majorant ν .*

Proof. Let (D_n) be a disjoint sequence in \mathcal{A} with union $A \in \mathcal{A}$. Then $|\mu(D_1 \cup \dots \cup D_k)| \leq \gamma(D_1) + \dots + \gamma(D_k) \leq \gamma(A)$. Hence, since μ is σ -additive,

$$|\mu(A)| \leq \sum_{n=1}^{\infty} \gamma(D_n) \leq \gamma(A), \quad (*)$$

where the series is τ -convergent.

Let ν be the σ -additive summand in the Hewitt-Yosida decomposition of γ . Recall that, for every $A \in \mathcal{A}$,

$$\nu(A) = \tau\text{-}\lim_{\pi} \sum_{D \in \pi} \gamma(D),$$

where the limit is taken over the net of all *countable* \mathcal{A} -partitions π of A . (See [D2, Sec. 3]; in the present setting, the MCP for intervals in E ensures the existence of the limit above.) From (*) it follows that $|\mu(A)| \leq \nu(A) \leq \gamma(A)$, which concludes the proof. ■

7 σ -additive measures with a non- σ -additive modulus

At the end of their paper [hSZ], Schaefer and Zhang raised the problem of whether the modulus of a σ -additive order bounded measure μ with values in a Dedekind complete Banach lattice is always σ -additive. In [DW, Ex. 3], we showed that it is not so for the Banach lattice $\ell_{\infty}(L_2) = (L_2 \oplus L_2 \oplus \dots)_{\ell_{\infty}}$, where $L_2 = L_2[0, 1]$, even if the domain of μ is the σ -algebra $\mathcal{P}(\mathbb{N})$. That is, we proved that the Dedekind complete Banach lattice $\ell_{\infty}(L_2)$ lacks Property (S) by exhibiting an unconditionally convergent and perfectly order bounded series $\sum_n f_n$ in $\ell_{\infty}(L_2)$ such that the series $\sum_n |f_n|$ is not convergent. In our construction, the terms of the “bad” series $\sum_n f_n$ come up in consecutive blocks ($f_n: n \in J_k$), where the k th block is formed by the first n_k Rademacher functions in the k th copy of L_2 , each multiplied by n_k^{-1} , and $n_k \rightarrow \infty$. Since, for each k , we have $\|\sum_{n \in J} f_n\|_2 \leq n_k^{-1/2}$ for $J \subset J_k$, while $\sum_{n \in J_k} |f_n| = 1 = 1_{[0,1]}$, the series $\sum_n f_n$ is as required.

We take this opportunity to acknowledge the fact that the same example $\ell_\infty(L_2)$, with essentially the same series $\sum_n f_n$, was given earlier in [GvR, (4.3)]. Expanding a remark made at the end of [GvR], we arrive at the following.

Example 7.1. *There exists a σ -additive order bounded measure $\varphi: \mathcal{P}(\mathbb{N}) \rightarrow c(L_2)$ that has no absolute majorant. Here $c(L_2)$ denotes the closed separable sublattice of $\ell_\infty(L_2)$ consisting of convergent sequences in L_2 .*

Consider the series $\sum_n f_n$ described above. Note that the sums of all its subseries, and the order sum $(o)\text{-}\sum_n |f_n| = (1, 1, \dots)$, actually “sit” in $c(L_2)$. Thus $c(L_2)$ does not have Property (S). Let $\varphi: \mathcal{P}(\mathbb{N}) \rightarrow c(L_2)$ be the measure associated with the series $\sum_n f_n$. Evidently, it is σ -additive and order bounded. However, it does not have an absolute majorant $\nu: \mathcal{P}(\mathbb{N}) \rightarrow c(L_2)$. For, if such ν existed, it would be exhaustive, by Proposition 5.3 or [DU, Thm. I.4.2]. From this, as $\sum_{n \in J} |f_n| \leq \nu(J)$ for finite sets $J \subset \mathbb{N}$, it would follow that the series $\sum_n |f_n|$ is Cauchy. A contradiction.

As we will see in Theorem 7.7, the absence of Property (S) is quite a typical phenomenon in some non-order continuous Banach lattices. Moreover, the basic ideas employed in the construction of the series $\sum_n f_n$ mentioned above, suitably modified, can also be used when dealing with those more general cases. After “discovering” [GvR] we realized that these ideas (employed also in Example 7.3) are in perfect accordance with the characterization of Banach lattices that are not order bounded AM , established in [GvR, Thm. 4.2].

But first, let us make a trivial observation that in our construction the “disjoint” blocks $(f_n : n \in J_k)$ consist of *simple* functions, and having realized that it is easy to give a sequential version of the construction — a solid Banach sequence space without Property (S).

Example 7.2. Let (M_k) be a partition of \mathbb{N} into consecutive intervals with $|M_k| = m_k = 2^{n_k}$, where $n_k \rightarrow \infty$. Define E to be the space of all real sequences $x = (t_n)$ such that

$$\|x\| := \sup_k \left(\frac{1}{m_k} \sum_{n \in M_k} |t_n|^2 \right)^{1/2} < \infty,$$

equipped with the norm given by this equality. Then E is a Dedekind complete Banach lattice without Property (S). A required “bad” series

can be constructed as described above using Rademacher-type functions on the intervals M_k .

Clearly, E is Riesz isometric to the Banach lattice $F = (\sum_k \ell_2^{m_k})_{\ell_\infty}$. Thus also F fails to have Property (S). In fact, it is so for any F defined in this way as soon as $\sup_k m_k = \infty$.

Next, we give a more direct extension of our original example $\ell_\infty(L_2)$ based on ideas quite different from those employed in [DW, Ex. 3] or [GvR, (4.3)].

Example 7.3. If E is a Banach lattice which is not Riesz isomorphic to an AM-space, then the Banach lattice $\ell_\infty(E)$ does not have Property (S).

As in the proof of Proposition 4.5, we can find an infinite sequence (x_n) in E such that for a certain partition (M_k) of \mathbb{N} into consecutive intervals,

$$\sup \left\{ \left\| \sum_{n \in M_k} \varepsilon_n x_n \right\|_E : \varepsilon_n = \pm 1 \right\} \leq 2^{-k} \quad \text{and}$$

$$\left\| \sum_{n \in M_k} |x_n| \right\|_E = 1, \quad k = 1, 2, \dots$$

Define a sequence (\mathbf{x}_n) in $\ell_\infty(E)$ by setting $\mathbf{x}_n = (0, \dots, 0, x_n, 0, \dots)$ for $n \in M_k$, where x_n occupies the k -th position, $k = 1, 2, \dots$. Since

$$\left\| \sum_{n \in M_k} \varepsilon_n \mathbf{x}_n \right\|_{\ell_\infty(E)} = \left\| \sum_{n \in M_k} \varepsilon_n x_n \right\|_E \leq 2^{-k} \quad \text{for } \varepsilon_n = \pm 1 \text{ and every } k,$$

the series $\sum_n \mathbf{x}_n$ is unconditionally convergent in $\ell_\infty(E)$. Moreover, denoting $u_k = \sum_{n \in M_k} |x_n|$, we have $\|u_k\|_E = 1$ so that $\mathbf{u} := (u_k) \in \ell_\infty(E)$, and $\sum_{n=1}^N |\mathbf{x}_n| \leq \mathbf{u}$ for every N . Thus the series $\sum_n \mathbf{x}_n$ is order bounded in $\ell_\infty(E)$. However,

$$\left\| \sum_{n \in M_k} |\mathbf{x}_n| \right\|_{\ell_\infty(E)} = \left\| \sum_{n \in M_k} |x_n| \right\|_E = 1, \quad \text{for every } k,$$

hence the series $\sum_n \mathbf{x}_n$ is not convergent in $\ell_\infty(E)$. ■

Remark 7.4. More generally, consider a sequence (E_n) of Banach lattices that are not uniformly Riesz homeomorphic to AM-spaces. That

is (see [APY, Thm. 3.1]), there is no finite constant d such that condition (*) stated in the proof of Proposition 4.5 holds for each of the spaces E_n . Then, by the same construction as above, the Banach lattice $(\sum_n E_n)_{\ell_\infty}$ does not have Property (S). In particular, in view of [APY, Thm. 2.1], this includes the Banach lattices constructed in Example 7.2 and, moreover, allows to replace the finite dimensional ℓ_2 -spaces used there with, for instance, finite dimensional ℓ_p -spaces, $1 \leq p < \infty$. Of just this type is the earliest known example of a Banach lattice without Property (S), which was given in [G, Ex. 4.5, p. 95]: the ℓ_∞ -sum of a sequence of finite dimensional ℓ_1 -spaces.

For every Banach lattice E , we define

$$E_A = \{x \in E : |x| \geq x_\alpha \downarrow 0 \implies \|x_\alpha\| \rightarrow 0\}.$$

Thus E_A is the largest ideal in E on which the norm is order continuous, and E_A is closed in E . For more information, see [W1].

We now proceed to the promised general result.

Sublemma 7.5. *Let F be a Banach lattice of measurable functions over a finite measure space $(X, \mathfrak{X}, \lambda)$. Let $g \in F_A$. Then $\|f_n\| \rightarrow 0$ for every sequence (f_n) in F such that $|f_n| \leq |g|$ and $\|f_n\|_2 \rightarrow 0$. In other words, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $f \in F$, $|f| \leq |g|$ and $\|f\|_2 \leq \delta$, then $\|f\| \leq \varepsilon$.*

Proof. It is enough to note that $|f_n| \leq |g|$ implies that the f_n 's have equi-absolutely continuous norms, and $\|f_n\|_2 \rightarrow 0$ implies that $f_n \rightarrow 0$ in measure. As is well known, these two things assure that $\|f_n\| \rightarrow 0$. ■

Lemma 7.6. *Let E be an atomless Banach lattice. If $0 < z \in E_A$, then for every $\varepsilon > 0$ there exists a finite family $\{x_i : i \in I\}$ of elements in E such that*

$$\left\| \sum_{j \in J} x_j \right\| \leq \varepsilon, \quad \forall J \subset I \quad \text{and} \quad \sum_{i \in I} |x_i| = z.$$

Proof. Consider the closed ideal $E_z \subset E_A$ generated by z . By [LT, Thm. 1.b.14], E_z is order isometric to a solid order continuous Banach lattice F of measurable functions over some nonatomic probability measure space $(X, \mathfrak{X}, \lambda)$, with z corresponding to the constant function 1 on

X . Thus it is enough to show the existence of a finite family $\{g_i : i \in I\}$ in F such that

$$\left\| \sum_{j \in J} g_j \right\| \leq \varepsilon, \quad \forall J \subset I \quad \text{and} \quad \sum_{i \in I} |g_i| = 1.$$

Fix a $\delta > 0$ provided by Sublemma 7.5 for the given number ε and the function $g = 1$. Choose an integer $N > \delta^{-2}$, denote $I = \{1, \dots, N\}$, and let ρ_i for $i \in I$ be the first N Rademacher-type functions on $(X, \mathfrak{X}, \lambda)$. Set $g_i = N^{-1}\rho_i$ for $i \in I$. Then, for any subset J of I ,

$$\left| \sum_{j \in J} g_j \right| \leq 1 \quad \text{and} \quad \int_X \left| \sum_{j \in J} g_j \right|^2 d\lambda = |J| \cdot N^{-2} \leq N^{-1} \leq \delta^2,$$

and, in view of the sublemma, it follows easily that the g_i 's are as required. ■

Theorem 7.7. *If E is an atomless Banach lattice in which E_A is not a band, then E lacks Property (S). In fact, there exists an unconditionally convergent series $\sum_n x_n$ in E_A that is order bounded in E and for which the series $\sum_n |x_n|$ does not converge.*

Proof. Without loss of generality we may assume that E_A is order dense in E (replacing, if necessary, E by E^{dd} ; cf. [AB1, Thm. 1.11], [AB2, Thm. 3.4]). Fix $z \in E_+ \setminus E_A$. By [W1, Lemma 3], there exists a disjoint sequence (z_k) in E_A such that $0 \leq z_k \leq z$ and $\inf_k \|z_k\| > 0$. Applying Lemma 7.6, we find a partition (I_k) of \mathbb{N} into consecutive intervals and, for each k , a family $\{x_n : n \in I_k\}$ of elements in E such that

$$\left\| \sum_{n \in J} x_n \right\| \leq 2^{-k}, \quad \forall J \subset I_k \quad \text{and} \quad \sum_{n \in I_k} |x_n| = z_k.$$

Evidently, the series $\sum_n x_n$ is unconditionally convergent and order bounded, but the series $\sum_n |x_n|$ is not convergent. ■

Remarks 7.8. (a) Theorem 7.7 does not extend to atomic (or discrete) Banach lattices. Indeed, $(\ell_\infty)_A = c_0$ is not a band in ℓ_∞ , and yet ℓ_∞ has Property (S). On the other hand, as we saw in Example 7.2 (see also

the remark after Example 7.3), all Banach lattices $F = (\sum_k \ell_2^{m_k})_{\ell_\infty}$, where $m_k \rightarrow \infty$, fail to have Property (S), and for each of them $F_A = (\sum_k \ell_2^{m_k})_{c_0}$ is not a band. (In fact, for a solid Banach lattice $F \subset \omega = \mathbb{R}^{\mathbb{N}}$, F_A is not a band in F unless $F_A = F$.) We do not know, however, whether all Orlicz sequence spaces ℓ_φ have Property (S).

(b) If E is a Banach lattice and E_A is a band (it is so, in particular, when $E_A = \{0\}$), then E may or may not have Property (S). We illustrate this statement by the following examples.

Let $F := L_2(0, \infty) \cap L_\infty(0, \infty)$ with the norm $\|f\| = \max(\|f\|_2, \|f\|_\infty)$. Then F is Dedekind complete, has the Fatou property, $F_A = \{0\}$, and F is not isomorphic to an AM-space. Nevertheless, as both $L_2(0, \infty)$ and $L_\infty(0, \infty)$ have Property (S), so does F . However, if $E := \ell_\infty(F)$, then $E_A = \{0\}$ and, by Example 7.3, E lacks Property (S).

Likewise, if $G := L_\infty[0, 1]$ and $H := \ell_2(G)$, then $H_A = \{0\}$ and, as easily seen, H has Property (S). However, H is not isomorphic to an AM-space so that, by Example 7.3, $\ell_\infty(H)$ lacks Property (S) while $\ell_\infty(H)_A = \{0\}$.

Proposition 7.9. *Let E be a σ -Dedekind complete Banach lattice without Property (S). If the σ -algebra Σ admits an atomless probability measure m , then there exists an atomless order bounded σ -additive measure $\mu: \Sigma \rightarrow E$ with a non- σ -additive proper modulus $|\mu|$.*

Proof. Take an unconditionally convergent and perfectly order bounded series $\sum_n x_n$ in E such that the series $\sum_n |x_n|$ is not convergent. Next, fix an infinite disjoint sequence (A_n) in Σ with all $a_n := m(A_n) > 0$, and define (mutually singular) atomless probability measures m_n on Σ by $m_n(A) = a_n^{-1} m(A \cap A_n)$. Finally, define a measure $\mu: \Sigma \rightarrow E$ by $\mu(A) = \sum_{n=1}^\infty m_n(A) x_n$. Obviously, μ is σ -additive, order bounded, and atomless. Since E is σ -Dedekind complete, the modulus $|\mu|$ exists. In fact, as easily seen, $|\mu|(A) = (o)\text{-}\sum_n a_n^{-1} m(A \cap A_n) |x_n|$ for every $A \in \Sigma$. However, $|\mu|$ is not σ -additive because $|\mu|(A_n) = |x_n|$ for every n . ■

We conclude this section with yet another example of a σ -additive measure with a proper non- σ -additive modulus. It also shows that the assumption that E is Fatou can be removed from neither Proposition 6.1 nor Theorem 6.2, and that Property (S) alone is not enough for the transition of σ -additivity from a measure to its modulus. The example below is, essentially, only a slightly modified version of Example 5.14 on

p. 106 (see also an observation concerning it on p. 112) in [G] (cf. also [GvR, (3.4)]). Since, however, [G] is not easily available, we give it *in extenso*.

Example 7.10. A σ -additive order bounded measure μ with values in a Dedekind complete AM-space whose modulus $|\mu|$ is not exhaustive and, a fortiori, not σ -additive.

For each $n \in \mathbb{N}$, let $E_n = (\ell_\infty, \|\cdot\|_n)$, where $\|x\|_n = \max\{\|x\|_\infty, n \limsup_k |t_k|\}$ for $x = (t_k) \in \ell_\infty$. Clearly, E_n is an AM-space, $\|x\|_n = \|x\|_\infty$ for $x \in c_0$, and $0 \leq x_j \uparrow x$ in E_n implies $\|x\|_n \leq n \lim_j \|x_j\|_n$, where the constant factor n is best possible. Let $E = (E_1 \oplus E_2 \oplus \dots)_{\ell_\infty}$. Then E is a Dedekind complete AM-space without the Fatou property, and it can be viewed as an ideal in $\omega = \mathbb{R}^\mathbb{N}$.

Recall that we denote by \mathcal{B} the Borel σ -algebra on the interval $[0, 1]$. Consider a measure $\mu = (\mu_n) : \mathcal{B} \rightarrow E$, where each $\mu_n : \mathcal{B} \rightarrow E_n$ is defined by

$$\mu_n(A) = \frac{2^n}{n} \left(\int_{A \cap I_n} r_k d\lambda \right)_{k \in \mathbb{N}}$$

and $I_n := (2^{-n}, 2^{-n+1})$. Note that μ_n is c_0 -valued so that

$$\|\mu_n(A)\|_n = \|\mu_n(A)\|_\infty = \frac{2^n}{n} \sup_k \left| \int_{A \cap I_n} r_k d\lambda \right| \leq \frac{2^n}{n} \lambda(A \cap I_n) \leq \frac{1}{n}.$$

From this it follows that each μ_n is σ -additive and that $\mu(A) \in (E_1 \oplus E_2 \oplus \dots)_{c_0}$ for all $A \in \mathcal{B}$. Moreover, for any $m \in \mathbb{N}$,

$$\begin{aligned} \|\mu(A)\| &= \sup_n \|\mu_n(A)\|_n \leq \sup_n \frac{2^n}{n} \lambda(A \cap I_n) \leq \\ &\max \left\{ \sup_{1 \leq n \leq m} \frac{2^n}{n} \lambda(A \cap I_n), \frac{1}{m} \right\}. \end{aligned}$$

Hence $\|\mu(A_j)\| \rightarrow 0$ whenever $A_j \downarrow \emptyset$ in \mathcal{B} . Thus μ assumes its values in E and is σ -additive. Next, for every $A \in \mathcal{B}$ and $n \in \mathbb{N}$,

$$|\mu_n|(A) = \frac{2^n}{n} (\lambda(A \cap I_n), \lambda(A \cap I_n), \dots) \quad \text{and} \quad \|\mu_n|(A)\|_n = 2^n \lambda(A \cap I_n).$$

By Corollary 1.8, the modulus $|\mu|$ exists properly and $|\mu| = (|\mu_n|)$. Since $\|\mu|(I_j)\| = 2^j \lambda(I_j) = 1$ for all j , we see that $|\mu|$ is not exhaustive.

8 Remarks on the proper existence of the modulus of a measure

In this section, unless stated otherwise, E is assumed to be an arbitrary Riesz space.

We first show that if a measure admits a Hahn decomposition, then it has a proper modulus. In the proposition below, measures $\mu_1, \mu_2 : \mathcal{A} \rightarrow E$ are called *mutually singular* if there is $A_0 \in \mathcal{A}$ such that $\mu_1(\mathcal{A} \cap A_0) = \mu_2(\mathcal{A} \cap (S \setminus A_0)) = \{0\}$.

Proposition 8.1. *For a measure $\mu : \mathcal{A} \rightarrow E$, the following are equivalent:*

- (a) $\mu(\mathcal{A})$ has a largest element.
- (b) μ admits a Hahn decomposition: There exist mutually singular positive measures $\mu_1, \mu_2 : \mathcal{A} \rightarrow E$ such that $\mu = \mu_1 - \mu_2$.

Moreover, if (b) holds, then $|\mu|$ exists properly and $|\mu| = \mu_1 + \mu_2$.

Proof. Let $A_0 \in \mathcal{A}$ be such that $\mu(A) \leq \mu(A_0)$ for all $A \in \mathcal{A}$. If $B \subset A_0$ and $C \subset S \setminus A_0$, then $\mu(A_0) - \mu(B) \leq \mu(A_0)$ and $\mu(C) + \mu(A_0) \leq \mu(A_0)$, hence $\mu(B) \geq 0$ and $\mu(C) \leq 0$. It follows that the measures μ_1 and μ_2 defined by $\mu_1(A) = \mu(A \cap A_0)$ and $\mu_2(A) = -\mu(A \setminus A_0)$ are as required in (b).

Conversely, let μ_1, μ_2 satisfy (b) and let A_0 be as required in the definition of their mutual singularity. Then, for every $A \in \mathcal{A}$, $\mu(A) \leq \mu_1(A) \leq \mu_1(A_0) = \mu(A_0)$ so that $\mu(A_0)$ is the largest element in $\mu(\mathcal{A})$. Moreover, if $\pi \in \Pi(A)$ and π is finer than the partition $\{A \cap A_0, A \setminus A_0\}$ of A , then one easily verifies that $\sum_{D \in \pi} |\mu(D)| = \mu_1(A) + \mu_2(A)$. It follows that $|\mu|(A) = \mu_1(A) + \mu_2(A)$ and that $|\mu|$ exists properly. ■

We next present some results concerning the *proper* existence of the modulus $|\mu|$ of an order bounded measure μ when it is a priori known that $|\mu|$ exists.

Proposition 8.2. *If the modulus $|\mu|$ of a measure $\mu : \mathcal{A} \rightarrow E$ exists, then the class of all sets $A \in \mathcal{A}$ on which it exists properly is an ideal in \mathcal{A} .*

Proof. Denote the class of such sets A by \mathcal{A}' . We first show that if $B \in \mathcal{A}$ and $B \subset A \in \mathcal{A}'$, then $B \in \mathcal{A}'$. Suppose $u \in E$ is such that

$\sum_{D \in \pi} |\mu(D)| \leq u$ for all $\pi \in \Pi(B)$. Then for every $\pi \in \Pi(A)$,

$$\sum_{D \in \pi} |\mu(D)| \leq \sum_{D \in \pi} |\mu(D \cap B)| + \sum_{D \in \pi} |\mu(D \cap (A \setminus B))| \leq u + |\mu|(A \setminus B).$$

It follows that $|\mu|(B) + |\mu|(A \setminus B) \leq u + |\mu|(A \setminus B)$. In consequence, $|\mu|(B) \leq u$ which means that $|\mu|(B) = \sup \{ \sum_{D \in \pi} |\mu(D)| : \pi \in \Pi(B) \}$ so that $B \in \mathcal{A}'$.

It remains to show if $A, B \in \mathcal{A}'$, then $A \cup B \in \mathcal{A}'$. In view of what we have proved above it can be assumed that $A \cap B = \emptyset$, and then a verification that $|\mu|$ exists properly on $A \cup B$ is straightforward. ■

Corollary 8.3. *If the modulus $|\mu|$ of a measure $\mu : \mathcal{A} \rightarrow E$ exists, and if it exists properly on S , then $|\mu|$ exists properly.*

A measure $\mu : \mathcal{A} \rightarrow E_+$ is said to be *order-countably additive*, or *(o)- σ -additive*, if $\mu(A) = (o)\text{-}\sum_n \mu(A_n) := (o)\text{-}\lim_n \sum_{i=1}^n \mu(A_i)$ whenever $A \in \mathcal{A}$ and (A_n) is a disjoint sequence in \mathcal{A} with union A ; equivalently, $\mu(A) = (o)\text{-}\lim_n \mu(A_n)$ whenever $A_n \uparrow A$ in \mathcal{A} .

Proposition 8.4. *If the modulus $|\mu|$ of a measure $\mu : \Sigma \rightarrow E$ exists and is (o)- σ -additive, then the class of all sets $A \in \Sigma$ on which $|\mu|$ exists properly is a σ -ideal in Σ .*

Proof. Denote the class of such sets A by Σ' . In view of Proposition 8.2 we only have to verify that for every increasing sequence (A_n) in Σ' its union A is also in Σ' .

Take any $x \in E_+$ such that $\sum_{D \in \pi} |\mu(D)| \leq x$ for each $\pi \in \Pi(A)$. From this, as the A_n 's are in Σ' , it follows that $|\mu|(A_n) \leq x$ for all n . Since $|\mu|$ is (o)- σ -additive, $|\mu|(A) = (o)\text{-}\lim_n |\mu|(A_n) \leq x$. Therefore, $|\mu|(A) = \sup \{ \sum_{D \in \pi} |\mu(D)| : \pi \in \Pi(A) \}$. ■

Remark. According to Proposition 8.6 below, if E is a σ -Dedekind complete TRS and $\mu : \Sigma \rightarrow E$ is a σ -additive measure whose modulus $|\mu|$ exists, then $|\mu|$ is (o)- σ -additive.

Corollary 8.5. *Let $\mu : \Sigma \rightarrow E$ be a measure such that the modulus $|\mu|$ exists and is (o)- σ -additive. Also, assume that every disjoint family of non- μ -null sets in Σ is countable. Then there exists $S_0 \in \Sigma$ such that $|\mu|$ exists properly on every set $A \in \Sigma$ contained in S_0 , and does not exist properly on any non- μ -null set $A \in \Sigma$ that is disjoint from S_0 .*

Proof. Take a maximal disjoint family \mathcal{D} of non- μ -null sets on which $|\mu|$ exists properly. By assumption, it is countable, and its union, S_0 , is as required, by Proposition 8.4 and maximality of \mathcal{D} . ■

The following result has the same proof as Proposition 6.4, replacing \lim in the definition of ν with (o)- \lim .

Proposition 8.6. *Let E be a TRS and $\mu: \mathcal{A} \rightarrow E$ a σ -additive measure such that the modulus $|\mu|$ exists. If for every increasing sequence (A_n) in \mathcal{A} the order limit $(o)\text{-}\lim_n |\mu|(A_n)$ exists in E (it is so e.g. when E is σ -Dedekind complete), then $|\mu|$ is (o)- σ -additive.*

Note that if $|\mu|$ is σ -additive, then it is also (o)- σ -additive.

We are now going to show that there is an intimate relation between the problem of the proper existence of the modulus $|\mu|$ of a measure μ , and a similar problem concerning the induced integral operator I_μ . The latter problem, in its general form, can be formulated as follows (see [AB2, Chap 1, Sec. 1] and [vR, Problem 1.2] for more information):

Let E and F be Riesz spaces, and let $T: E \rightarrow F$ be a regular operator; that is, T admits an absolute majorant, a positive operator $P: E \rightarrow F$ with $|T(x)| \leq P(x)$ for all $x \in E_+$. The smallest of such majorants for T , if exists, is called the modulus of T and denoted $|T|$. If $\sup T([-x, x])$ exists in F for each $x \in E_+$, then the modulus $|T|$ exists and $|T|(x) = \sup T([-x, x])$ for all $x \in E_+$; that is, $|T|$ exists properly. However, it is unknown whether the mere existence of the modulus $|T|$ always implies its proper existence.

In Proposition 8.8 below we show that, in particular, if E is a Banach lattice, then for an order bounded measure $\mu: \mathcal{A} \rightarrow E$ and the induced integral operator $I_\mu: B(\mathcal{A}) \rightarrow E$, the questions of proper existence of $|\mu|$ and $|I_\mu|$ are equivalent. Here, following [DU, p. 6], we denote by $B(\mathcal{A}) = B(S, \mathcal{A})$ the Banach lattice consisting of functions $f: S \rightarrow \mathbb{R}$ that are uniform limits of (\mathcal{A} -) simple functions, equipped with the sup norm.

Actually, we will work in the more general setting where E is an arbitrary Riesz space that is relatively uniformly complete. Recall that the latter means that for every $0 \leq u \in E$ the ideal E_u generated by u is a Banach space under the norm $\|\cdot\|_u$ whose closed unit ball = $[-u, u]$. We denote by $oba(\mathcal{A}, E)$ and $L_b(B(\mathcal{A}), E)$ the ordered vector spaces of all order bounded measures $\mu: \mathcal{A} \rightarrow E$ and all order bounded operators

$T: B(\mathcal{A}) \rightarrow E$, respectively. For every $\mu \in \text{oba}(\mathcal{A}, E)$ and $f \in B(\mathcal{A})$, let $I_\mu(f) = \int_S f d\mu$. This makes sense because choosing $0 \leq u \in E$ so that $\mu(\mathcal{A}) \subset [-u, u]$, we may view μ as a bounded measure with values in the Banach space $(E_u, \|\cdot\|_u)$, and then define the integral $\int_S f d\mu$ in the usual way (see e.g. [DU, p. 6]). Note that the operator $I_\mu: B(\mathcal{A}) \rightarrow (E_u, \|\cdot\|_u)$ is continuous.

In the final stage of the proof of Proposition 8.8 we will need the following (certainly well known)

Lemma 8.7. *For every $0 \leq f \in B(\mathcal{A})$ there is a sequence (f_n) of simple functions such that $0 \leq f_n(s) \uparrow f(s)$ for all $s \in S$ and $\|f - f_n\| \rightarrow 0$.*

Proof. Given any $\varepsilon > 0$, there exists a simple function $g = \sum_{i=1}^n a_i 1_{A_i}$ with disjoint $A_i \in \mathcal{A}$, such that $\|f - g\| \leq \varepsilon$. Thus $a_i - \varepsilon \leq f(s) \leq a_i + \varepsilon$ for $s \in A_i$, $i = 1, \dots, n$. Let $b_i = \max\{a_i - \varepsilon, 0\}$. Then $h = \sum_{i=1}^n b_i 1_{A_i}$ is a simple function and $0 \leq h \leq f$. Moreover, if $s \in A_i$, then $0 \leq f(s) - h(s) = f(s) - b_i \leq (a_i + \varepsilon) - b_i$. Observe that $(a_i + \varepsilon) - b_i = (a_i + \varepsilon) - (a_i - \varepsilon) = 2\varepsilon$ when $b_i = a_i - \varepsilon \geq 0$; and $(a_i + \varepsilon) - b_i = a_i + \varepsilon < 2\varepsilon$ when $b_i = 0$, because then $a_i - \varepsilon < 0$, or $a_i < \varepsilon$. It follows that $0 \leq f - h \leq 2\varepsilon$ on S .

Applying the above with $\varepsilon = (2k)^{-1}$ gives a sequence (h_k) of simple functions such that $0 \leq f - h_k \leq 1/k$. Then, setting $f_n = h_1 \vee \dots \vee h_n$, we get an increasing sequence (f_n) of simple functions such that $0 \leq f_n \leq f$ and $f_n \rightarrow f$ uniformly. ■

Proposition 8.8. *Let E be a relatively uniformly complete Riesz space. Then the map $\mu \rightarrow I_\mu$ is a linear and order isomorphism between the ordered vector spaces $\text{oba}(\mathcal{A}, E)$ and $L_b(B(\mathcal{A}), E)$. Moreover, if $\mu \in \text{oba}(\mathcal{A}, E)$, then*

- (a) μ admits an absolute majorant iff I_μ admits an absolute majorant;
- (b) $|\mu|$ exists iff $|I_\mu|$ exists, and if this is the case, then $|I_\mu| = I_{|\mu|}$;
- (c) the modulus $|\mu|$ exists properly iff the modulus $|I_\mu|$ exists properly.

Proof. Only part (c) requires a proof. Note: Below, we will use continuity of I_μ and $I_{|\mu|}$ as operators from $B(\mathcal{A})$ to $(E_u, \|\cdot\|_u)$ for suitably chosen u 's (not indicated explicitly).

For each $A \in \mathcal{A}$ denote $M_A = \{\sum_{D \in \pi} |\mu(D)| : \pi \in \Pi(A)\}$.

Suppose $|I_\mu| = I_{|\mu|}$ exists properly. Thus

$$|I_\mu|(f) = \sup\{|I_\mu(g)| : |g| \leq f\} = \sup\{I_\mu(g) : |g| \leq f\} \tag{*}$$

for all $0 \leq f \in B(\mathcal{A})$.

Fix $A \in \mathcal{A}$ and let x be an upper bound of M_A . If $h \in B(\mathcal{A})$ is simple and $|h| \leq 1_A$, then clearly $|I_\mu(h)| \leq x$. If $g \in B(\mathcal{A})$ is arbitrary and $|g| \leq 1_A$, then choose sequences $(u_n), (v_n)$ of simple functions so that $0 \leq u_n, v_n \leq 1_A$ and $u_n \uparrow g^+, v_n \uparrow g^-$ uniformly. Then $|u_n - v_n| \leq 1_A$, hence $|I_\mu(u_n - v_n)| \leq x$, and the continuity of I_μ gives $|I_\mu(g)| \leq x$. Therefore $|\mu|(A) = I_{|\mu|}(1_A) = |I_\mu|(1_A) \leq x$. In consequence, $|\mu|(A) = \sup M_A$; that is, $|\mu|(A)$ exists properly.

Suppose now that $|\mu|$ exists properly. Thus $|\mu|(A) = \sup M_A$ for every $A \in \mathcal{A}$. By (b), $|I_\mu|$ exists and $|I_\mu| = I_{|\mu|}$. Hence, for every $A \in \mathcal{A}$, $|\mu|(A) = |I_\mu|(1_A)$ is an upper bound of the set $\{|I_\mu(g)| : |g| \leq 1_A\}$. In fact, it is its smallest upper bound because $\sum_i |\mu(D_i)| = \sup\{|I_\mu(\sum_i \varepsilon_i 1_{D_i})| : \varepsilon_i = \pm 1\}$ for all $\{D_1, \dots, D_n\} \in \Pi(A)$. It follows that (*) holds for $f = 1_A$, hence also for $f = a1_A$, where $a \in \mathbb{R}_+$.

Now, take any simple function $f = \sum_{i=1}^n a_i 1_{A_i}$, where $a_i \geq 0$ and $A_i \in \mathcal{A}$ are disjoint. Note that $\{g : |g| \leq f\} = \sum_{i=1}^n \{g_i : |g_i| \leq a_i 1_{A_i}\}$ and that, by the preceding part, $\sup\{I_\mu(g_i) : |g_i| \leq a_i 1_{A_i}\}$ exists and equals $|I_\mu|(a_i 1_{A_i})$ for $i = 1, \dots, n$. Hence $\sup\{I_\mu(g) : |g| \leq f\}$ exists and equals $\sum_{i=1}^n |I_\mu|(a_i 1_{A_i}) = |I_\mu|(f)$. Thus (*) holds for nonnegative simple functions $f \in B(\mathcal{A})$.

Finally, let $0 \leq f \in B(\mathcal{A})$ be arbitrary. By the definition of $|I_\mu|$, $|I_\mu|(f)$ is an upper bound of the set $\{I_\mu(g) : |g| \leq f\}$. Let x be any of its upper bounds. Applying Lemma 8.7, choose a sequence (f_n) of positive simple functions uniformly increasing to f . Clearly, x is an upper bound of each set $\{I_\mu(g) : |g| \leq f_n\}$ and so $|I_\mu|(f_n) \leq x$. By the continuity of $|I_\mu|$, $|I_\mu|(f) \leq x$. Therefore, $|I_\mu|(f) = \sup\{I_\mu(g) : |g| \leq f\}$. This completes the proof. ■

9 Absolutely exhaustive operators

We shall say that a continuous linear operator T from $B(\mathcal{A})$ to a TVS E is *exhaustive* (see [K]) if, for every bounded sequence (f_n) in $B(\mathcal{A})$ consisting of disjoint (= disjointly supported) functions, $T(f_n) \rightarrow 0$ or, equivalently, the series $\sum_n T(f_n)$ is (unconditionally) Cauchy. (For

E a Banach space this is equivalent to T being weakly compact.) If E is a TRS and T satisfies a stronger requirement that, for all (f_n) as above, the series $\sum_n |T(f_n)|$ is Cauchy, then we shall say that T is *absolutely exhaustive*. Evidently, if T has an exhaustive absolute majorant $P: B(\mathcal{A}) \rightarrow E$, then T is absolutely exhaustive.

Proposition 9.1. *Let E be a TRS. A continuous linear operator $T: B(\mathcal{A}) \rightarrow E$ is absolutely exhaustive iff the associated (bounded) measure $\mu_T: \mathcal{A} \rightarrow E$, defined by $\mu_T(A) = T(1_A)$, is absolutely exhaustive.*

Proof. Denote by W the solid hull of the image by T of the closed unit ball in $B(\mathcal{A})$. Note that W is a bounded subset of E .

Let μ_T be absolutely exhaustive. Take a bounded disjoint sequence $(g_n) \subset B(\mathcal{A})$ consisting of simple functions. We may assume that $\|g_n\|_\infty \leq 1$ for each n . Then, denoting by A_n the support of g_n , we have $g_n = \sum_{j \in J_n} c_j 1_{C_j}$, where (J_n) is a partition of \mathbb{N} into a sequence of consecutive intervals, $\{C_j: j \in J_n\}$ is an \mathcal{A} -partition of A_n , and $|c_j| \leq 1$. Clearly, $|T(g_n)| \leq \sum_{j \in J_n} |\mu_T(C_j)|$. Since, by assumption, the series $\sum_j |\mu_T(C_j)|$ is Cauchy, it follows that so is the series $\sum_n |T(g_n)|$.

Now, let (f_n) be an arbitrary bounded and disjoint sequence in $B(\mathcal{A})$. Fix any solid zero-neighborhood $V = V_0$ in E , and next choose a sequence (V_n) of solid zero-neighborhoods in E so that $V_n + V_n \subset V_{n-1}$ for all $n \geq 1$. Furthermore, for each $n \geq 1$ choose $\alpha_n > 0$ such that $\alpha_n W \subset V_n$. By Lemma 8.7 and the continuity of T , we find a sequence (g_n) of simple functions such that $|g_n| \leq |f_n|$ and $|T(f_n - g_n)| \in \alpha_n W$ for every n . Of course, the sequence (g_n) is bounded and disjoint. Note that $||T(f_n)| - |T(g_n)|| \leq |T(f_n - g_n)| \in \alpha_n W$, hence $|T(f_n)| \leq |T(g_n)| + \alpha_n w_n$ for some $w_n \in W$. Since, by the first part of the proof, the series $\sum_n |T(g_n)|$ is Cauchy, there is k such that $\sum_{n=k}^m |T(g_n)| \in V_1$ for $m \geq k$. Now, if $m \geq k$, then

$$\sum_{n=k}^m |T(f_n)| \leq \sum_{n=k}^m |T(g_n)| + \sum_{n=k}^m \alpha_n w_n \in V_1 + \sum_{n=k}^m V_n \subset V_1 + V_{k-1} \subset V_0 = V,$$

and it follows that the series $\sum_n |T(f_n)|$ is Cauchy.

The other direction is obvious. ■

Corollary 9.2. *Let E be a Banach lattice with the Fatou property, and $T: B(\mathcal{A}) \rightarrow E$ an order bounded linear operator such that the modu-*

lus $|T|$ exists properly. Then $|T|$ is weakly compact iff T is absolutely exhaustive.

Proof. By Proposition 8.8, $T = I_\mu$ for some $\mu \in \text{oba}(\mathcal{A}, E)$, and the modulus $|\mu|$ exists properly. Now, by Propositions 9.1 and 6.1 (a), and in view of [DU, Thm. VI.1.1], we have the following chain of equivalences: T is absolutely exhaustive iff μ is absolutely exhaustive iff $|\mu|$ is exhaustive iff $|T| = I_{|\mu|}$ is weakly compact. ■

Similarly, using Theorem 6.2 one obtains the following.

Corollary 9.3. *Let E be a Banach lattice with the Fatou property and Property (S), and $T: B(\mathcal{A}) \rightarrow E$ an order bounded weakly compact linear operator. If the modulus $|T|$ exists properly, then it is weakly compact.*

Remark. Results analogous to those stated above also hold when $B(\mathcal{A})$ is replaced with a space $L_\infty(\lambda)$ of bounded measurable functions. For other results dealing with the question of weak compactness of $|T|$, see [G, Thm. 6.4, p. 112], [GvR], [CW].

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