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HILLE-YOSIDA THEORY IN CONVENIENT ANALYSIS

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Abstract

A Hille-Yosida Theorem is proved on convenient vector spaces, a class, which contains all sequentially complete locally convex spaces. The approach is governed by convenient analysis and the credo that many reasonable questions concerning strongly continuous semigroups can be proved on the subspace of smooth vectors. Examples from literature are reconsidered by these simpler methods and some applications to the theory of infinite dimensional heat equations are given.

1 Introduction

Semigroups of linear operators on locally convex spaces provide the successful setting for the analysis of initial value problems. On Banach spaces a subtle theory for strongly continuous semigroups, namely Hille-Yosida-Theory had been developed (see for example [EN00]). In this article on the one hand generalizations of this theory on locally convex vector spaces are considered. Furthermore we want to point out the credo that several problems occuring in semigroup theory can analysed more effectively on locally convex spaces as on Banach spaces. One example is provided in the last section.

On sequentially complete vector spaces it is sufficient for the analysis of strongly continuous, locally equicontinuous semigroups of continuous operators to analyze smooth semigroups of continuous linear operators.

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Convenient analysis was invented to get a concise formulation for the questions of infinite dimensional differential geometry as diffeomorphism groups or subgroups of them. A general lemma is presented, when it is possible to pass from strongly continuous semigroups on convenient locally convex spaces to smooth semigroups. As approximation procedure a classical product integral procedure is proposed, since for this type of approximation a general convergence theorem can be proved without assuming the existence of the smooth semigroup. Furthermore the quality of the convergence is very good. This theorem was originally proved to obtain some results in infinite dimensional differential geometry, where Banach space theory is not sufficient.

In the third section convenient Hille-Yosida-Theory is presented following the footsteps of [Ouc73]. We prove that his procedure leads to better convergence results as were originally obtained (even without the assumption of local equicontinuity). We prove a theorem on asymptotic resolvents and a reproduction formula. In the fourth section we provide several concrete examples and investigate the connections between Abstract Cauchy Problems and smooth semigroups.

The fifth section finally is devoted to examples and some application to infinite dimensional heat equations demonstrating that this smooth perspective is very useful. We conclude by several statements on the treatment of strongly continuous semigroups on locally convex spaces in the last twenty five years to demonstrate that most of the results can be obtained by the smooth theory in an easier way.

2 Convenient Calculus

Convenient analysis provides the broadest field for analysis on locally convex spaces (see [KM97]). The concept of a smooth curve into a locally convex space is obvious. The class of locally convex spaces where weakly smooth curves are exactly smooth curves is given by convenient vector spaces. A locally convex space is called convenient if it is Mackey-complete. We call a sequence $\{x_n\}_{n\in\mathbb{N}}$ Mackey-converging to x with quality $\{\mu_n\}_{n\in\mathbb{N}}$, where the μ_n are non-negative real numbers converging to 0, if there is a bounded set B such that $x_n - x \in \mu_n B$. Analogously we call a sequence $\{x_n\}_{n\in\mathbb{N}}$ a Mackey-Cauchy-sequence if there is a sequence $\{t_{nm}\}_{n,m\in\mathbb{N}}$ with t_{nm} positive real numbers and $t_{nm} \to 0$ for $n, m \to \infty$, such that $x_n - x_m \in t_{nm}B$, where B is bounded. If every Mackey-Cauchy-sequence converges in a locally convex vector space E we speak of a Mackey-complete vector space. On a Mackey-complete vector space there is in general no natural locally convex topology reproducing only this concept of convergence (see [KM97], ch.1), but there is a finest topology in the set of locally convex topologies compatible with the system of bounded sets, called the bornological topology E_{born}. When we talk of a closed set in a convenient vector space, we mean that the set is closed with respect to the topology of E_{born}. Given a bounded, absolutely convex set $B \subset E$ we can look at the localization $E_B := span(B)$ with the Minkowsky norm p_B as norm (see [Jar81], ch.6). Remark that convenient vector spaces are exactly those locally convex spaces, which are locally complete, i.e. for every closed, bounded and absolutely convex set B the normed space E_B is a Banach space. Mackey-completeness is consequently an apparently weak concept of completeness (see [KM97], ch.1). On convenient vector spaces E a uniform boundedness principle of the following form is valid: A set of linear bounded maps from E to the real numbers is uniformly bounded, i.e. bounded on bounded sets if and only if it is pointwise bounded (see [KM97], ch.1).

The final topology with respect to all smooth curves is called the c^∞-topology. If E is a convenient vector space, $c^\infty E$ need not be a topological vector space, since addition might be discontinuous, however, on Fréchet spaces $E = c^\infty E$. A mapping $f : U \to F$, where U is c^∞-open and F is a convenient vector space is called smooth if smooth curves are mapped to smooth curves, which is even on $\mathbb{R}^2$ not obvious. The differential of a smooth mapping is given by derivatives along affine lines.

We assume the main results of convenient calculus (also on non-open domains such as $\mathbb{R}_{\geq 0}$) as provided in [KM97] and apply them to semigroups of linear operators on convenient vector spaces.

Definition 2.1. *Let E be a convenient locally convex vector space, T a semigroup of continuous linear operators. T is called C_0-semigroup if the trajectories $t \mapsto T_t x$ are bounded on compacts and if $\lim_{t\downarrow 0} T_t x = x$ for $x \in E$.*

The theory of C_0-semigroups on convenient vector spaces can be developed analogously to the theory of smooth semigroups, if enough smooth vectors exist. Remark that this is the case if the space is sequentially complete by integration of bounded, right-continuous curves. Anyway we do not need local equicontinuity (see [Ouc73])!

Proposition 2.2. *Let E be a convenient locally convex vector space, T a C_0-semigroup of continuous linear operators on E, such that the "smooth vectors" $S(\phi,t)x = \int_0^t \phi(s)T_s x ds$ exist in E for $t \geq 0$ and $\phi \in C_c^\infty(\mathbb{R}_{>0})$, then the linear subspace*

$$E^\infty := \{x \in E \mid t \mapsto T_t x \text{ is smooth } \}$$

of smooth vectors is dense in E. Let a denote the infinitesimal generator of T. The initial locally convex topology $(a|_{E^\infty})^n : E^\infty \to E$ for $n \in \mathbb{N}$ is convenient and the restriction $T|_{E^\infty}$ is a smooth semigroup with infinitesimal generator $a|_{E^\infty}$.

Proof. The proof is done by standard semigroup theory:

$$\mathcal{D}(a) := \{x \in E \mid \lim_{t \downarrow 0} \frac{1}{t}(T_t x - x) \text{ exists } \}$$

$$ax := \lim_{t \downarrow 0} \frac{1}{t}(T_t x - x) \text{ for } x \in \mathcal{D}(a)$$

By right continuity we obtain $aT_t x = T_t a x$ for $x \in \mathcal{D}(a)$, which equals the right derivative of T_t, too. Up to now we are only talking about right derivatives and we cannot do better on the whole of E. By the uniform boundedness principle (see [KM97], Theorem 5.18) the set of linear operators $\{T_s\}_{0 \leq s \leq t}$ is uniformly bounded for each $t \geq 0$, since the trajectories are bounded.

We shall apply the indefinite Lebesgue integral for right continuous, bounded curves, which generically takes values in the completion of E.

A standard calculation gives

$$a \int_0^t \phi(s) T_s x ds = -\int_0^t \phi'(s) T_s x ds + \phi(t) T_t x - \phi(0) x$$

for ϕ smooth with support in $\mathbb{R}_{\geq 0}$, if the integrals exists in E:

$$\frac{T_h - id}{h} \int_0^t \phi(s) T_s x ds =$$

$$= \int_0^t \frac{\phi(s) - \phi(s+h)}{h} T_{s+h} x ds + \frac{1}{h}\left(\int_t^{t+h} \phi(s) T_s x ds - \int_0^h \phi(s) T_s x ds\right)$$

for $h > 0$ and therefore for any continuous seminorm p

$$\begin{aligned}p(\int_0^t \frac{\phi(s)-\phi(s+h)}{h} T_{s+h}xds + \int_0^t \phi'(s)T_sxds) \\ \leq \int_0^t |\frac{\phi(s)-\phi(s+h)}{h} + \phi'(s)| p(T_{s+h}x)ds \\ + \int_0^t |\phi'(s)| p(T_{s+h}x - T_sx)ds.\end{aligned}$$

Since T_t is uniformly bounded on compact t-sets, we can conclude by dominated convergence that the above limit is zero as $h \to 0$ and that the given formula holds.

Consequently the notion "smooth vector" is justified as the image under a again lies in $\mathcal{D}(a)$. However these vectors lie dense in E as we can choose a Dirac sequence supported right from zero. Furthermore we have the formula

$$\int_0^t T_saxds = T_tx - x$$

for $t \geq 0$ and $x \in \mathcal{D}(a)$ by the fundamental theorem for right continuous curves. Hence for $x \in \mathcal{D}(a)$ the trajectory $t \mapsto T_tx$ is continuous.

Therefore $E^\infty = \cap_{n\geq 0}\mathcal{D}(a^n)$: the inclusion $\subset$ is clear, then other inclusion $\supset$ follows from the consideration, that for $x \in \cap_{n\geq 0}\mathcal{D}(a^n)$ the curve $t \mapsto T_tx$ is not only right continuous, but continuous and the derivative is not only a right derivative, but a derivative – again by the last formula. By induction we obtain smoothness.

Given a Mackey-converging sequence $x_n \to x$ such that ax_n is defined and converges Mackey to y. Since pointwise bounded sets in $L(E)$ are uniformly bounded, $\{T_s\}_{0\leq s\leq t}$ is uniformly bounded by right continuity at 0, consequently

$$\int_0^t T_syds = T_tx - x$$

which yields $x \in \mathcal{D}(a)$. So $\mathcal{D}(a)$ is a convenient locally convex space $E^{(1)}$ with the operator seminorms. The semigroup $T^{(1)}$ arising from T via restriction to this space is a strongly continuous semigroups of continuous linear operators and smooth vectors exist in $\mathcal{D}(a)$, too.

E^∞ is given as the intersection of all these spaces and equivalently as the domain of definition of all a^k for $k \in \mathbb{N}$. The above described topology is a convenient locally convex topology as it lies in the domain of definition of all a^k, $k \in \mathbb{N}$. Again by the smooth vectors we conclude that E^∞ is dense. ∎

Remark that this proof can be generalized to strongly continuous group homomorphisms of finite dimensional Lie groups to continuous linear operators on locally convex spaces (see [KM97], compare to 49.4.) assuming the existence of smooth vectors.

3 Convenient Hille-Yosida-Theory

We are working out convenient Hille-Yosida-Theory on convenient algebras, which appear to us as most general playground for smooth semigroups. A convenient algebra is a convenient vector space with smooth associative multiplication. Furthermore we always assume it to be unital. Smooth Semigroups are smooth homomorphisms from $\mathbb{R}_{\geq 0}$ to a convenient algebra A. In this chapter we develop a theory of asymptotic resolvents by which one can provide a necessary and sufficient criterion whether a smooth semigroup exists given the infinitesimal generator. The technical ideas from this section stem from the excellent work of [Ouc73].

For the purpose of estimates we need some Landau-like terminology in convenient vector spaces. We shall only apply the symbol O:

Definition 3.1. *Let E be a convenient vector space, $c : D \to E$ for $D \subset \mathbb{R}^n$ some non-empty subset, an arbitrary mapping. Let $d : D \to \mathbb{R}$ be some non-negative function, then we say that c has growth d on D if there is a closed absolutely convex and bounded subset $B \subset E$ so that*

$$c(x) \in d(x)B \text{ for all } x \in D \quad .$$

We write $c = O(d)$ on D or $c(x) = O(d(x))$, applying Landau's symbol O.

Definition 3.2. *Let A be a convenient algebra and $T : \mathbb{R}_{\geq 0} \to A$ a smooth semigroup homomorphism referred to as smooth semigroup, then*

$$a := \lim_{h \downarrow 0} \frac{T_h - e}{h}$$

is called the infinitesimal generator of the smooth semigroup T. Given $b > 0$ the family $\{R(\lambda)\}_{\lambda>0}$ with

$$R(\lambda) := \int_0^b \exp(-\lambda t) T_t dt$$

is a called a standard asymptotic resolvent family of a.

Approximations can be performed by Trotter formulas, which are proved to be a type of existence theorem within the theory, since one does not need the existence of the smooth semigroup to make the Trotter approximation converge (see [Tei99] and [Teib01] for details and its most general formulation for non-autonomuous linear differential equations). This theorem has some surprising application to the fundaments of the theory of infinite dimensional Lie groups (see [Tei99] and [Teib01]):

Theorem 3.3. *Let $c : \mathbb{R}^2 \to A$ a smooth curve into a convenient algebra A with $c(s, 0) = e$ and for any compact s-interval there is $r > 0$ such that*

$$\{c(s, \frac{t}{n})^n \,|\, 0 \leq t \leq r, \, n \in \mathbb{N}\} \text{ is bounded in } A,$$

then the limit $\lim_{n\to\infty} c(s, \frac{t}{n})^n$ exists uniformly on compact intervals in $\mathbb{R}^2$ in all derivatives. Furthermore the resulting family $T_t(s)$ is a smooth family of smooth semigroups with infinitesimal generator $\frac{\partial}{\partial t} c(s, 0)$ for $s \in \mathbb{R}$.

Proposition 3.4. *Let A be a convenient algebra, T a smooth semigroup, then the following formulas are valid:*

1. *Let a be the infinitesimal generator of T, then $\frac{d}{dt} T_t = aT_t = T_t a$ for all $t \in \mathbb{R}_{\geq 0}$.*
2. *The semigroup is uniquely determined by a.*
3. *For all $b \in \mathbb{R}_{\geq 0}$ the following integral exists in A:*

$$R(\lambda) = \int_0^b \exp(-\lambda t) T_t dt \text{ for all } \lambda \in \mathbb{R}$$

4. *For all $\lambda, \mu \in \mathbb{R}$, $b \in \mathbb{R}_{\geq 0}$ we obtain:*

$$\begin{aligned} (\lambda - a)R(\lambda) &= id - \exp(-\lambda b) T_b \\ R(\lambda)R(\mu) &= R(\mu)R(\lambda) \\ R(\lambda)a &= aR(\lambda) \end{aligned}$$

(5) $R : \mathbb{R}_{\geq 0} \to A$ *is real analytic and the set*

$$\{\frac{\lambda^{n+1}}{n!} R^{(n)}(\lambda) \mid \lambda > 0 \text{ and } n \in \mathbb{N}\}$$

is bounded in A.

Proof. The first assertion follows from boundedness of the multiplication:

$$\begin{aligned} T_t a = T_t \lim_{h \downarrow 0} \frac{T_h - id}{h} &= \lim_{h \downarrow 0} \frac{T_{t+h} - T_h}{h} = \frac{d}{dt} T_t \\ &= \lim_{h \downarrow 0} \frac{T_h - id}{h} T_t = T_t a. \end{aligned}$$

Suppose that there is another semigroup associated to a, more precisely, let S, T be smooth semigroups in A with

$$a = \lim_{h \downarrow 0} \frac{T_h - id}{h} = \lim_{h \downarrow 0} \frac{S_h - id}{h},$$

then the curve $c(r) = T(t - r)S(r)$ on $[0, t]$ for $t > 0$ arbitrary is smooth and $c'(r) = -ac(r) + c(r)a = 0$, consequently $T_t = c(0) = c(t) = S_t$.

The existence of the integral and the commutation relations are clear, the only assertion to prove is the asymptotic condition:

$$\begin{aligned} (\lambda - a)R(\lambda) &= \int_0^b \exp(-\lambda t)(\lambda - a) T_t dt = \\ &= \int_0^b -\frac{d}{dt}(\exp(-\lambda t) T_t) dt = id - \exp(-\lambda b) T_b. \end{aligned}$$

Differentiation under the integral yields

$$\frac{\lambda^{n+1}}{n!} R^{(n)}(\lambda) = (-1)^n \lambda^{n+1} \int_0^b \exp(-\lambda t) \frac{t^n}{n!} T_t dt$$

$T = O(1)$ on any bounded interval in $\mathbb{R}_{\geq 0}$, so

$$\frac{\lambda^{n+1}}{n!} R^{(n)}(\lambda) = O\left(\lambda^{n+1} \int_0^b \exp(-\lambda t) \frac{t^n}{n!} dt\right) = O(1)$$

for all $\lambda > 0$ and $n \in \mathbb{N}$. So the estimate and real analyticity are proved, since the remainder of the Taylor series converges to zero. ∎

Definition 3.5. *Let $a \in A$ be a given element of the convenient algebra A, a smooth map $R : \mathbb{R}_{>\omega} \to A$ is called asymptotic resolvent for $a \in A$ if*

(1) *$aR(\lambda) = R(\lambda)a$ and $R(\lambda)R(\mu) = R(\mu)R(\lambda)$ for $\lambda, \mu > \omega$.*

(2) *$(\lambda - a)R(\lambda) = e + S(\lambda)$ with $S : \mathbb{R}_{>\omega} \to A$ smooth and there is are constants $b > 0$ so that the set*

$$\{\frac{\exp{(b\lambda)}}{b^k} S^{(k)} \mid \lambda > \omega \textit{ and } k \in \mathbb{N}\}$$

is bounded in A.

Remark 3.6. The standard asymptotic resolvent family is an asymptotic resolvent. The estimate of the definition can be generalized to

$$\{\frac{\exp{(b\lambda)}}{c^k} S^{(k)} \mid \lambda > \omega \text{ and } k \in \mathbb{N}\}$$

bounded in A with $c \geq b > 0$. The following theorems stay valid, but the calculations get more complicated. The case $S = 0$ is equivalent to the choice $b = \infty$, which is not always possible, because there are semigroups with rapid growth and infinitesimal generators without classical spectral theory, respectively. For the standard asymptotic resolvent we obtain $S(\lambda) = -\exp{(\lambda b)} T_b$.

The following theorem is the generalization of the Hille-Yosida-Theorem to the convenient case (see [Ouc73] for the idea of the proof).

Theorem 3.7. *Let A be a convenient algebra and $a \in A$ an element, then a is the infinitesimal generator of a smooth semigroup T in A if and only if there is an asymptotic resolvent R for a with*

$$\{\frac{\lambda^{n+1}}{n!} R^{(n)}(\lambda) \mid \lambda > \omega \textit{ and } n \in \mathbb{N}\}$$

a bounded set in A (Hille-Yosida-condition).

Proof. If a is the infinitesimal generator of a smooth semigroup in A, then there is by prop. 2.4. an asymptotic resolvent so that the above conditions are satisfied.

Let R be an asymptotic resolvent defined on $\mathbb{R}_{>\omega}$ satisfying the hypotheses, then $\lambda R(\lambda) = e + aR(\lambda) + O(\exp(-b\lambda))$ by 2.5.2, consequently $\lim_{\lambda\to\infty} \lambda R(\lambda) = e$ is a Mackey-limit. $a_\lambda := -\lambda + \lambda^2 R(\lambda) = \lambda(-e + \lambda R(\lambda)) = \lambda R(\lambda)a + O(\lambda \exp(-b\lambda)) \to a$ as Mackey-limit for $\lambda \to \infty$. Differentiating the equation $(\lambda - a)R(\lambda) = e + S(\lambda)$ $(k+1)$-times we obtain $(\lambda-a)R^{(k+1)}(\lambda)+(k+1)R^{(k)}(\lambda) = S^{(k+1)}(\lambda)$ for $k \in \mathbb{N}$. Multiplication with $R(\lambda)$ yields

$$R^{(k+1)}(\lambda) + (k+1)R(\lambda)R^{(k)}(\lambda) = R(\lambda)S^{(k+1)}(\lambda) - S(\lambda)R^{(k+1)}(\lambda)$$

for $k \in \mathbb{N}$. Putting together the hypotheses $R(\lambda)S^{(k+1)}(\lambda) = = O(\frac{\exp(-\lambda b)b^{k+1}}{\lambda})$ and $S(\lambda)R^{(k+1)}(\lambda) = O(\exp(-\lambda b)\frac{(k+1)!}{\lambda^{k+2}})$ we arrive at

$$R^{(k+1)}(\lambda) + (k+1)R(\lambda)R^{(k)}(\lambda) = O(\exp(-\lambda b)\frac{(k+1)! + (b\lambda)^{k+1}}{\lambda^{k+2}})$$

for $k \in \mathbb{N}$ and $\lambda > \omega$. Now we try to define out of these data a smooth semigroup T. Let $t \in [0, \frac{b}{4}]$, then

$$T_t(\lambda) := \exp(-\lambda t)\left(e + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{(\lambda^2 t)^{k+1}}{(k+1)!}R^{(k)}(\lambda)\right)$$

for $\lambda > \omega$. Looking at the growth for $\lambda > \omega$ and $t \in [0, \frac{b}{4}]$ we obtain

$$T_t(\lambda) = O(\exp(-\lambda t)(1 + \sum_{k=0}^{\infty}\frac{(\lambda t)^{k+1}}{(k+1)!})) = O(1)$$

for $k \in \mathbb{N}$ and $\lambda > \omega$ by the Hille-Yosida-condition, which implies the existence of $T_t(\lambda)$ uniformly on compact intervals in λ and t as a Mackey-limit by the Cauchy condition on the convergence of infinite series. By inserting the Hille-Yosida condition in the termwise derived series we obtain the uniform convergence by the Cauchy condition on compact intervals in λ and t, which leads to smoothness of $T_t(\lambda)$ in t, even at the boundary points $t = 0$ and $t = \frac{b}{4}$: We obtain

$$\begin{aligned}
\frac{d}{dt}T_t(\lambda) =& -\lambda T_t(\lambda) + \lambda^2 \exp(-\lambda t)(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{(\lambda^2 t)^k}{k!} R^{(k)}(\lambda)) \\
=& -\lambda T_t(\lambda) + \lambda^2 \exp(-\lambda t)(R(\lambda) + \sum_{k=0}^{\infty} \frac{(-\lambda^2 t)^{k+1}}{(k+1)!(k+1)!} R^{(k+1)}(\lambda)) \\
=& -\lambda T_t(\lambda) + \\
& + \lambda^2 \exp(-\lambda t) \left(R(\lambda) + \sum_{k=0}^{\infty} (-1)^k \frac{(\lambda^2 t)^{k+1}}{(k)!(k+1)!} R(\lambda) R^{(k)}(\lambda) \right) \\
& + \lambda^2 \exp(-\lambda t) \sum_{k=0}^{\infty} \frac{(-\lambda^2 t)^{k+1}}{(k+1)!(k+1)!} S_k(\lambda)
\end{aligned}$$

with $S_k(\lambda) = R^{(k+1)}(\lambda) + (k+1)R(\lambda)R^{(k)}(\lambda)$. The last sum on the right hand side is of order

$$\lambda^2 \exp(-\lambda t) \sum_{k=0}^{\infty} \frac{(\lambda^2 t)^{k+1}}{(k+1)!(k+1)!} \exp(-\lambda b) \frac{(k+1)! + (b\lambda)^{k+1}}{\lambda^{k+2}}.$$

This term can be estimated by

$$\begin{aligned}
=& \lambda \exp(-\lambda(t+b)) \left(\sum_{k=0}^{\infty} (\frac{(\lambda^2 tb)^{k+1}}{(k+1)!(k+1)!} + \sum_{k=0}^{\infty} \frac{(\lambda t)^{k+1}}{(k+1)!} \right) \\
\leq& \lambda \left(\exp(-\lambda(\sqrt{t} - \sqrt{b})^2) + \exp(-\lambda b) \right) \\
\leq& 2\lambda \exp(-\lambda \frac{b}{4})
\end{aligned}$$

for $t \in [0, \frac{b}{4}]$, since $(\sqrt{t} - \sqrt{b})^2$ attains the minimum $\frac{b}{4}$. The middle term equals

$$\lambda^2 \exp(-\lambda t) \left(R(\lambda) + \sum_{k=0}^{\infty} (-1)^k \frac{(\lambda^2 t)^{k+1}}{(k)!(k+1)!} R(\lambda) R^{(k)}(\lambda) \right) = R(\lambda)\lambda^2 T_t(\lambda)$$

by definition. Consequently we arrive at the equation by $a_\lambda = -\lambda + \lambda^2 R(\lambda)$:

$$\frac{d}{dt}T_t(\lambda) = a_\lambda T_t(\lambda) + O(\lambda \exp(-\lambda \frac{b}{4}))$$

for $t \in [0, \frac{b}{4}]$ and $\lambda > \omega$. Finally we can calculate the difference

$$T_t(\lambda) - T_t(\mu) = \int_0^t \frac{d}{ds} \left(T_s(\lambda) T_{t-s}(\mu) \right) ds$$

because $T_0(\lambda) = e$ and so by the commutation relations

$$\begin{aligned} T_t(\lambda) - T_t(\mu) = \int_0^t & T_s(\lambda) T_{t-s}(\mu)(a_\lambda - a_\mu) ds \\ & + O(\lambda \exp(-\lambda \frac{b}{4})) + O(\mu \exp(-\mu \frac{b}{4})) \end{aligned}$$

we are led to uniform Mackey-convergence on $[0, \frac{b}{4}]$ of $T_t(\lambda)$ as $\lambda \to \infty$. We denote the limit by T_t. Due to uniform convergence on $[0, \frac{b}{4}]$ and the Mackey-property of the limits we obtain $T_t(\lambda) a_\lambda \to T_t a$, consequently the first derivatives of $T_t(\lambda)$ converge uniformly in t to $aT_t = T_t a$, which guarantees Lipschitz-differentiability of order Lip^1 of T_t with derivative aT_t. Since multiplication with a is a bounded operation we see that the first derivative is Lip^1, too. Consequently T_t is smooth on $[0, \frac{b}{4}]$. Given $t, s \in [0, \frac{b}{4}]$ with $t + s \in [0, \frac{b}{4}]$, then

$$\begin{aligned} T_{t+s} - T_t T_s &= \int_0^t \frac{d}{du} \left(T_{t-u} T_{s+u} \right) du = \\ &= \int_0^t T_{t-u} T_{s+u}(a - a) du = 0 \quad . \end{aligned}$$

So T is a smooth semigroup in A with generator a, which is the desired assertion. ■

The following theorem provides a reproduction formula, given an asymptotic resolvent, we can calculate the smooth semigroup. The formula is apparently complicated, but all the known reproduction formulas from classical theory follow (see [Ouc73] for the idea of the proof):

Theorem 3.8. *Let $a \in A$ be an element, $R : \mathbb{R}_{>\omega} \to A$ an asymptotic resolvent of a satisfying the Hille-Yosida condition, then*

$$\lim_{n \to \infty} \frac{(-1)^{n-1}}{(n-1)!} \left(\frac{n}{t}\right)^n R^{(n-1)}\left(\frac{n}{t}\right) = T_t$$

uniformly on compact intervals in $[0, b[$ as Mackey-limit, where at $t = 0$ the term is given through e.

Proof. First we show that the term can be continued by e at $t = 0$, therefore we apply the formula

$$R^{(k+1)}(\lambda) + (k+1)R(\lambda)R^{(k)}(\lambda) = O\left(\exp(-\lambda b)\frac{(k+1)! + (b\lambda)^{k+1}}{\lambda^{k+2}}\right)$$

for $k \in \mathbb{N}$ and $\lambda > \omega$ from the proof of Theorem. We prove that

$$\lim_{\lambda\to\infty} \frac{(-1)^n \lambda^{n+1}}{n!} R^{(n)}(\lambda) = e$$

is a Mackey-limit. For $n = 0$ the assertion was proved at the beginning of the previous demonstration, we assume by induction, that it is valid for some $n \geq 0$. With the formula of the proof of theorem 2.7. we obtain by applying the commutation relations

$$\frac{(-1)^{n+1}\lambda^{n+2}}{(n+1)!} R^{(n+1)}(\lambda) = \frac{(-1)^n \lambda^{n+1}}{n!} R^{(n)}(\lambda)R(\lambda)\lambda +$$
$$+ O(\exp(-\lambda b)\frac{(n+1)! + (b\lambda)^{n+1}}{(n+1)!})$$

and we can insert the hypotheses of induction. Letting λ tend to infinity we arrive at the limit result by induction (we need the case $n = 0$ and the induction hypothesis), the Mackey-property is also proved.

$$S_n(t) := \frac{(-1)^{n-1}}{(n-1)!}(\frac{n}{t})^n R^{(n-1)}(\frac{n}{t})$$

is defined for $t \in [0, b[$ and $n > b\omega$. Remark that

$$S_n(t) = O(1)$$

for $n > b\omega$ and $t \in [0, b[$ by the Hille-Yosida condition. Let $0 < t < b$ be given, then

$$\frac{d}{dt}S_n(t) = -\frac{n}{t}S_n(t) + \frac{(-1)^n}{(n-1)!}(\frac{n}{t})^n R^{(n)}(\frac{n}{t})\frac{n}{t^2} =$$
$$= -\frac{n}{t}S_n(t) + \frac{(-1)^{n-1}}{(n-1)!}(\frac{n}{t})^{n+2} R^{(n-1)}(\frac{n}{t})R(\frac{n}{t}) +$$
$$+ \frac{1}{n!}(\frac{n}{t})^{n+2} G_n(t) =$$
$$= a_{\frac{n}{t}} S_n(t) + \frac{1}{n!}(\frac{n}{t})^{n+2} G_n(t)$$

by the above formula with $G_n(t) := O(\exp(-\frac{n}{t}b)(\frac{t}{n}b^n + (\frac{t}{n})^{n+1}n!))$:

$$\frac{1}{n!}(\frac{n}{t})^{n+2}\exp(-\frac{n}{t}b)\left(\frac{t}{n}b^n + (\frac{t}{n})^{n+1}n!\right) =$$

$$= \exp(-\frac{n}{t}b)(\frac{1}{n!}(\frac{n}{t})^{n+1}b^n + \frac{n}{t}) \leq$$

$$\leq K_1\exp(-\frac{n}{t}b)(\frac{1}{n!}(\frac{nb}{t})^{n+1} + \frac{n}{t})$$

with a constant $K_1 \geq 1$. Now we apply Stirling' s formula

$$n! \sim n^n\exp(-n)\sqrt{2\pi n}$$

(see [Kno51], ch. 14, for remarkable details), consequently

$$\leq K_2\exp(-\frac{n}{t}b)(\exp(n)\sqrt{n}(\frac{b}{t})^{n+1} + \frac{n}{t}) =$$

$$= K_2\left(\exp(n(1-\frac{b}{t}))\sqrt{n}(\frac{b}{t})^{n+1} + \exp(-\frac{n}{t}b)\frac{n}{t}\right)$$

The function $f(x) = x^m\exp(-nx)$ is decreasing on the interval $[\frac{m}{n},\infty[$. Given $0 < t_0 < b$, $t \in [0,t_0]$, $n \in \mathbb{N}$ with $\beta := \frac{b}{t_0} \geq 1 + \frac{1}{n}$, then

$$\exp(-n\frac{b}{t})(\frac{b}{t})^{n+1} \leq \beta^{n+1}\exp(-n\beta)$$

$$\exp(-n\frac{b}{t})(\frac{b}{t}) \leq \beta\exp(-n\beta).$$

Inserted in our formula we arrive at

$$\leq K_3(\sqrt{n}\beta^{n+1}\exp(n(1-\beta)) + n\beta\exp(-n\beta)).$$

However, $\beta\exp(1-\beta) < 1$ for $\beta > 1$, so the term in question tends to zero as $n \to \infty$ uniformly in t on compact intervals in $[0,b[$. The following formula prepares the result:

$$S_n(t) - T_t = \int_0^t \frac{d}{ds}(S_n(s)T_{t-s})\,ds =$$

$$= \int_0^t S_n(s)T_{t-s}(a_{\frac{n}{s}} - a)ds + \int_0^t \frac{1}{n!}(\frac{n}{t})^{n+2}G_n(s)ds$$

for $n > b\omega$. Given $t \in [0,t_0]$ we obtain by boundedness of S_n and the above convergence of the perturbation $\frac{1}{n!}(\frac{n}{t})^{n+2}G_n(s)$ the result and the Mackey-property. ∎

4 Abstract Cauchy Problems

Semigroups will be denoted by $S, T, ...$, their infinitesimal generators by a, b. We use the conventions of semigroup theory: $T_t = T(t)$. The interest in semigroup theory stems from properties of the solutions of Abstract Cauchy Problems on convenient vector spaces. Let E be a convenient vector space, $a \in L(E)$ a bounded operator, then, given $x \in E$, a solution of $ACP(a)$ is a curve $x : \mathbb{R}_+ \to E$ satisfying

$$x \in Lip^1(\mathbb{R}_{\geq 0}, E) \text{ and } x(0) = x$$
$$\frac{d}{dt}x(t) = ax(t) \text{ for all } t \in \mathbb{R}_{\geq 0}$$

If $ACP(a)$ has a unique solution for every $x \in E$, one can form a semigroup T of linear mappings on E, we call such an Abstract Cauchy Problem *well-posed*. Up to webbed, Baire locally convex spaces the concepts of well-posed $ACP(a)$ and the concept of smooth semigroups are equivalent:

Proposition 4.1. *Let E be a webbed locally convex vector space, such that E_{born} is Baire, $a \in L(E)$ a bounded, linear operator, then the following assertions are equivalent:*

(1) *For any $x \in E$ the Abstract Cauchy Problem $ACP(a)$ has a unique solution with initial value x.*

(2) *The mapping $T : \mathbb{R}_{\geq 0} \to L(E)$, $t \mapsto (x \mapsto x(t))$, where $x(t)$ denotes the value of the unique solution of $ACP(a)$ with initial value x at time t, is well-defined, smooth and*

$$\frac{d}{dt}T_t = aT_t$$

for $t \in \mathbb{R}_{\geq 0}$ (and therefore a smooth semigroup in $L(E)$.)

Proof. The step from the second to the first assertion is valid in general for any locally convex space E. The other direction is a little bit more complicated:

We denote by $T_{.}x$ the unique solution with initial value $x \in E$. Remark that by definition this solution is smooth, so $T_{.}x \in C^\infty([0, \infty[, E)$.

Furthermore by uniqueness the family $\{T_t\}_{t\geq 0}$ is a semigroup of linear operators on E. We define $\eta : E_{born} \to C^\infty([0,\infty[, E_{born})$ by $\eta(x) = T_. x$, which is a linear mapping. We show that it has closed graph. Let $\{x_i\}_{i\in I}$ be a converging net with limit $x \in E_{born}$ and $\eta(x_i) \to y$ with $y \in C^\infty([0,\infty[, E_{born})$, then

$$\eta(x_i)(s) = x_i + \int_0^s a\eta(x_i)(t)dt$$

for $s \geq 0$. Passing to the limit we obtain

$$y(s) = x + \int_0^s ay(t)dt$$

for $s \geq 0$. So y is a solution of $ACP(a)$ with initial value x, consequently $y = \eta(x)$ and η has a closed graph. If E is webbed, E_{born} is webbed and Baire by assumption, consequently η continuous. $ev_t \circ \eta = T_t$ is continuous on E_{born}, so T is a smooth semigroup of bounded linear operators. ■

Proposition 3.1 justifies the introduction of the notion of a *smooth semigroup*. A smooth semigroup of bounded linear operators on E is a smooth semigroup in $L(E)$.

Next we present Holmgren's principle in the convenient setting (see [LS93] for details). The pairing will be denoted by $\langle ., . \rangle$:

Proposition 4.2. *Let E be a convenient vector space, $a \in L(E)$ a linear operator:*

(1) *If $ACP(a)$ is uniquely solvable for every initial value on E and $ACP(a')$ is uniquely solvable for every initial value on E' then the solutions determine a smooth semigroups of bounded operators on E and E', respectively. They are dual to each other.*

(2) *Let $x : \mathbb{R}_{\geq 0} \to E$ be a (nontrivial) solution of $ACP(a)$ with initial value $x(0) = 0$, then for every solution $y : \mathbb{R}_{\geq 0} \to E'$ of $ACP(a')$ we have:*
$$\forall s, t \in \mathbb{R}_{\geq 0}, n \in \mathbb{N} : \langle x^{(n)}(s), y(t) \rangle = 0$$

(3) *Let $y : \mathbb{R}_{\geq 0} \to E'$ be a (nontrivial) solution of $ACP(a')$ with initial value $y(0) = 0$, then for every solution $x : \mathbb{R}_{\geq 0} \to E$ of $ACP(a)$ we have:*
$$\forall s, t \in \mathbb{R}_{\geq 0}, n \in \mathbb{N} : \langle x(s), y^{(n)}(t) \rangle = 0$$

In other words the non-uniqueness of the ACP associated to a or a', respectively, determines forbidden zones for the dual problem, that means subspaces where solutions of the dual problems cannot pass by.

Proof. The first assertion follows from the observation that the semigroups of (possibly unbounded) operators T on E and S on E' are dual to each other, consequently bounded. For the proof we look at the following curve. Let $t > 0$, $x \in E$, $y \in E'$ be fixed, then

$$c(s) := \langle T_{t-s}x, S_s y\rangle \text{ for } s \in [0,t]$$

is a smooth curve with derivative zero, because of the boundedness of the pairing, so $c(0) = \langle T_t x, y\rangle = c(t) = \langle x, S_t y\rangle$. A mapping is bounded if the composition with all bounded functionals is bounded, consequently the given semigroups are semigroups of bounded linear maps.

For the last two assertions we have to examine a classical object, the shift semigroup on $C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ given by

$$(S_s f)(t) = f(t+s) \text{ for all } t, s \geq 0 \quad .$$

This is a smooth semigroup for the convenient topology on $C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ associated to $\frac{\partial}{\partial t}$, so the solutions of the associated abstract Cauchy problem are unique. Taking the data from point ii. we can define $f(t,s) := \langle x(s), y(t)\rangle$:

$$\frac{\partial}{\partial s} f(t,s) = \langle ax(s), y(t)\rangle = \langle x(s), a'y(t)\rangle = \frac{\partial}{\partial t} f(t,s)$$

for $s, t \geq 0$. Using cartesian closedness we obtain that $f_s := f(s,.)$ for $s \geq 0$ is a solution of the abstract Cauchy problem given above. This means that

$$f(t,s) = (S_s f_0)(t) = f_0(t+s) = 0$$

because $x(0) = 0$. Derivations in s-direction give the desired assertion. Taking the data of 3. we can proceed in the same manner by interchanging the roles. ∎

Corollary 4.3. *Let E be a convenient vector space, $a \in L(E)$ a linear operator. If $ACP(a)$ is solvable for every initial value on E and $ACP(a')$*

is solvable for every initial value on E', then the solutions determine a smooth semigroups of bounded operators on E and E', respectively.

Proof. By Proposition 3.2.2 and 3.2.3 we obtain that the solutions have to be unique, because they exist for all initial values. Therefore we can apply Prop. 3.2.1. ∎

Example 4.4. (rapid growth) Let E be the space of entire functions on the complex plane $\mathcal{H}(\mathbb{C})$ and define a to be the multiplication operator by the function id, then the $(ACP)(a)$

$$\frac{\partial}{\partial t} f(t,z) = z f(t,z)$$

is solvable. Nevertheless the solution $\exp(zt)f(z)$ grows faster in t than any exponential in the natural topology of uniform convergence on compact sets on $\mathcal{H}(\mathbb{C})$ for any non-zero initial value. So only asymptotic resolevnts exist!

Nevertheless there are vector spaces, where all Abstract Cauchy Problems are solvable, but not uniquely. The following theorem treats some infinite products of real or complex lines, which have this property, even more. Let $a \in L(E)$ be some bounded linear operator, $f \in C^\infty(\mathbb{R}_{\geq 0}, E)$ some function, then a Lip^1-curve $x : \mathbb{R}_{\geq 0} \to E$ is the solution of the *inhomogeneous Abstract Cauchy Problem* $ACP(a,f)$ with initial value x if $x(0) = x$ and $\frac{d}{dt}x(t) = ax(t) + f(t)$. Remark that such a solution has to be smooth.

Proposition 4.5. *Let B be a non-empty set, $a \in L(\mathbb{K}^B)$ a bounded linear operator,*
$\mathbf{f} \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{K}^B)$*, then there is a solution of* $ACP(a, \mathbf{f})$ *for any initial value* $\mathbf{x} \in \mathbb{K}^B$*.*

Proof. See [Shk92] for the idea of the proof, but without matrix formulation. For the matrix formulation see [Tei99]. ∎

This is one extreme case of solvability on a special type of locally convex spaces. Remark that this restricts solvability on the dual space, because the solutions maybe non-unique. Another extreme case is given by so called LN-spaces (see [LS93], p.148-155 for the proof in the case $X = \mathbb{R}$): This indicates a class of nuclear Fréchet spaces E, where for all

$a \in L(E)$ the exponential $\exp(at)$ exists for all times t, which is a surprising fact. Even more surprising is the fact that even nonlinear equations can be solved uniquely on these spaces. Remark that many classical problems of analysis can be formulated in this context, for example the evolution of the heat equation in negative time direction.

5 Infinite products of semigroups

We are going to prove a simple approximation theorem:

Proposition 5.1. *Let A be a convenient algebra, $\{T_n\}_{n\in\mathbb{N}}$ a sequence of smooth semigroups with infinitesimal generators $\{a_n\}_{n\in\mathbb{N}}$. If $\{a_n\}_{n\in\mathbb{N}}$ is a Mackey-Cauchy sequence and $\{T_n(t)|0 \le t \le s\}$ is bounded in A (which is equivalent to boundedness in $C^\infty(\mathbb{R}_{\ge 0}, A)$), then there is a semigroup T with infinitesimal generator $a := \lim_{n\to\infty} a_n$ and*

$$\lim_{n\to\infty} T_n = T$$

in $C^\infty(\mathbb{R}_{\ge 0}, A)$.

Proof. We show that $\{T_n\}_{n\in\mathbb{N}}$ is a Mackey-Cauchy sequence in $C^\infty(\mathbb{R}_{\ge 0}, A)$. To do this we show that all derivatives converge uniformly on compact subsets of $\mathbb{R}_{\ge 0}$ in A (see [KM97], ch.1). Let $I \subset \mathbb{R}_{\ge 0}$ be compact, then we obtain

$$\begin{aligned} T_n^{(k)}(t) - T_m^{(k)}(t) =& a_n^k T_n(t) - a_m^k T_m(t) = (a_n^k - a_m^k) T_n(t) + \\ &+ a_m^k (T_n(t) - T_m(t)) = (a_n^k - a_m^k) B + tC(a_n - a_m) D \end{aligned}$$

for $k, n, m \in \mathbb{N}$, $t \in I$, where B, C, D denote appropriately chosen absolutely convex, closed bounded sets, depending on k, t, but not on m, n. By the Mackey-Cauchy-property we obtain that

$$a_n^k - a_m^k = \sum_{i=0}^{k} a_n^{i-1}(a_n - a_m) a_m^{k-i} \in t_{nm} D'$$

for a bounded, absolutely convex and closed subset of A and the given double sequence $\{t_{nm}\}_{n,m\in\mathbb{N}}$ measuring the convergence of $\{a_n\}_{n\in\mathbb{N}}$. So it is a Mackey-Cauchy sequence, too, for $k \in \mathbb{N}$.

The given sequence of smooth semigroups is consequently a Mackey-Cauchy sequence, so there is a smooth curve in the convenient space

$C^\infty(\mathbb{R}_{\geq 0}, A)$ being the limit. A fortiori this is a smooth semigroup by boundedness of the multiplication. ■

Let A be a unital convenient algebra, $\{T_n\}_{n\in\mathbb{N}}$ be a commuting sequence of smooth semigroups with infinitesimal generators $\{a_n\}_{n\in\mathbb{N}}$, such that

$$U_n(t) = \prod_{i=0}^{n} T_i(t)$$

for $t \in \mathbb{R}_{\geq 0}$ satisfies the boundedness-hypotheses of the above convergence theorem and $b_n = \sum_{i=0}^{n} a_i$ for $n \in N$ is a Mackey-Cauchy sequence, then the infinite product $\prod_{i=0}^{\infty} T_i := \lim_{n\to\infty} U_n$ of the sequence of semigroups exists in $C^\infty(\mathbb{R}_{\geq 0}, A)$ and is a smooth semigroup with infinitesimal generator $\sum_{i=0}^{\infty} a_i$.

This simple observation can be applied to the following situation. Let T be a smooth bounded group with infinitesimal generator a in a complex unital convenient algebra A, this means that we can find a closed absolutely convex bounded subset B of the convenient algebra A, so that $T_t \in B$ for all $t \in \mathbb{R}$. In classical theory of C_0-semigroups there is a beautiful formula calculating a new semigroup from the given one:

$$S(\lambda) = \frac{1}{\sqrt{4\pi\lambda}} \int_{\mathbb{R}} e^{-\frac{s^2}{\lambda}} T(s) ds$$

for $\lambda \in \Sigma_{\frac{\pi}{2}} \setminus \{0\} := \{-\pi < \arg\lambda < \pi\}$. The proof is remarkably simple: Take $l \in A'$ a bounded linear functional to investigate analyticity (see [KM97], ch.3), then

$$\frac{d}{d\lambda} l \circ S(\lambda)T(t) = \frac{d}{d\lambda}\Big(\frac{1}{\sqrt{4\pi\lambda}} \int_{\mathbb{R}} e^{-\frac{s^2}{\lambda}} l \circ T(t-s) ds\Big) = l \circ (a^2 T(t))$$

by the symmetry of the integral and the integral representation of the one-dimensional Gaussian semigroup for $\lambda \in \Sigma_{\frac{\pi}{2}} \setminus \{0\}$ (see for example [EN00]). So the integral defines a holomorphic semigroup on the given sector with generator a^2 (see [Tei99]).

The above observations can be applied in the following theorem, which generalizes an already known theorem about infinite products of a commuting family of C_0-semigroups.

Theorem 5.2. *Let E be a complex Fréchet space. Let $\{T_n\}_{n\in\mathbb{N}}$ be a commuting sequence of bounded smooth groups, so that*

$$\prod_{i=0}^{\infty} T_i =: T$$

exists in $C^\infty(\mathbb{R}, L(E))$ and is a smooth group with generator $\sum_{i=0}^{\infty} a_i$, where the sum converges absolutely in $L(E)$ (so the order of the product can be chosen arbitrarily). Denote by S_n the associated bounded holomorphic semigroup generated by a_n^2. If there is a bounded, closed and absolutely convex subset, where all the finite products $\prod_{i=0}^{n} S_i$ for $n \in N$ lie, then the infinite product

$$\prod_{i=0}^{\infty} S_i =: S$$

exists in $C^\infty(\mathbb{R}_{\geq 0}, L(E))$ and the infinitesimal generator is $\sum_{i=0}^{\infty} a_i^2$.

Proof. The only thing to prove is the (absolute) convergence of the series $s_n := \sum_{i=0}^{n} a_i^2$. Let p be a continuous seminorm on E, then

$$p((s_n - s_{n+k})(x)) \leq \sum_{i=n+1}^{n+k} p(a_i^2(x)) \leq \sum_{i=n+1}^{n+k} q(a_i(x)) \overset{n,k\to\infty}{\to} 0$$

where q denotes a continuous seminorm on E. The existence of q follows from the fact, that $\{a_i\}_{i\in\mathbb{N}}$ is bounded in $L(E)$, consequently equicontinuous, because E is barreled, so for every continuous seminorm p there is a continuous seminorm q, so that $p(a_i(x)) \leq q(x)$ for $x \in E$. So we obtain that the above series converges pointwisely absolutely. By the uniform boundedness principle the convergence is uniform to the bounded limit in $L(E)$ (see for example [Jar81]). ∎

Let X be a complex Banach space, $\{T_n\}_{n\in\mathbb{N}}$ a commuting family of bounded C_0-groups on X with infinitesimal generators $\{A_n\}_{n\in\mathbb{N}}$. The norm shall be denoted by p, remark that bounded strongly continuous groups are contraction groups. The linear space

$$\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} D(A_n^k)$$

is on the one hand dense in X by the abstract version of the Mittag-Leffler-Theorem (see [Est84] and [ADEM98]), on the other hand a Fréchet space with obvious seminorms $p_{n,k}(x) := \sum_{i=0}^{k} \|A_n^i x\|$. On this Fréchet space all the groups T_n are smooth and bounded. We associate the semigroups S_n to T_n with generator A_n^2 and obtain bounded C_0-semigroups on X and smooth bounded semigroups on $\mathfrak{F}$, respectively. If we assume that the series $\sum_{i=0}^{\infty} A_i$ converges absolutely on $\mathfrak{F}$, then the infinite product

$$\prod_{i=0}^{\infty} T_i =: T$$

exists in $C^\infty(\mathbb{R}, L(\mathfrak{F})$ and is a bounded smooth group, because

$$p_{n,k}(T_i(t)(x)) \leq p_{n,k}(x)$$

and $p(T_i(t)(x)) \leq p(x)$ for all $x \in \mathfrak{F}$ (they commute!). In [ADEM98], lemma 2.6. the authors assume that

$$D_1 = \{x \in \bigcap_{n\in\mathbb{N}} D(A_n) \mid \sum_{n=0}^{\infty} p(A_k x) < \infty\}$$

is dense in X and conclude that the infinite product of groups $\prod_{k=0}^{\infty} T_k$ exists strongly on X as strongly continuous bounded group. Consequently the sum $\sum_{k=0}^{\infty} A_k$ on D_1 is the restriction of the infinitesimal generator A of the infinite product on the core D_1. By the abstract version of the Mittag-Leffler-theorem $\mathfrak{F} \cap \bigcap_{n\in\mathbb{N}} D(A^n)$ is dense in X and there the sum converges absolutely in the Fréchet space topology to the restriction of A (look at the sequence of generators of bounded C_0-groups $A, -A, A_0, A_1, ...$). Hence it is no restriction to assume absolute convergence on $\mathfrak{F}$.

Consequently T can be extended to X by density and the given estimates as a bounded C_0-group with infinitesimal generator the closure of $\sum_{i=0}^{\infty} A_i$ on X, because $\mathfrak{F}$ is a core of the infinitesimal generator. By the above theorem we obtain that

$$\prod_{i=0}^{\infty} S_i = S$$

exists in $C^\infty(\mathbb{R}_{\geq 0}, L(\mathfrak{F})$, because

$$\begin{aligned} p(S_i(t)(x)) &= \lim_{n\to\infty} p((id - \frac{t}{n}A_i^2)^{-n}) = \\ &= \lim_{n\to\infty} p((id - \frac{\sqrt{t}}{\sqrt{n}}A_i)^{-n}(id + \frac{\sqrt{t}}{\sqrt{n}}A_i)^{-n}(x)) \leq p(x) \end{aligned}$$

for $t \geq 0$ and $x \in X$ by the classical Hille-Yosida-theorem (see [EN00]), where from the other necessary estimates for the boundedness of the finite products in $L(\mathfrak{F})$ follow. This semigroup can be extended to a C_0-semigroup on X with infinitesimal generator the closure of $\sum_{i=0}^{\infty} A_i^2$ on X by the same argument. In some cases this semigroup is referred to as infinite dimensional Gaussian semigroup, taking translation-groups in different directions on appropriate spaces as groups T_n. This consideration solves a problem raised by [ADEM98] on page 525, which forces them to consider some slightly stronger notion of convergence.

6 Remarks to existing literature

A smooth semigroup T in a convenient algebra A is called exponentially bounded if

$$T_t = O(\exp(\omega t))$$

on $\mathbb{R}_{\geq 0}$ for a given $\omega > 0$. Exponentially bounded smooth semigroups can be easily treated by the following methods. First the classical resolvent exists for $\lambda > \omega$, consequently we obtain an asymptotic resolvent with $S = 0$. The exponential formula is therefore valid

$$\lim(e - \frac{ta}{n})^{-n} = T_t$$

in all derivatives on compact subsets of $\mathbb{R}_{\geq 0}$. This problem was treated by several authors with similar approaches motivated by differing interests. Isao Miyadera [Miy59] was the first to deal strongly continuous semigroups on Fréchet spaces assuming exponential boundedness, Sunao Ouchi [Ouc73] generalized the theory to sequentially complete locally convex spaces. In both cases the convenient setting applies and yields the results.

In [Jef86], [Jef87] weakly integrable semigroups of continuous linear operators are dealt with, which is a very weak concept of one-parameter semigroups. Nevertheless as far as generators are concerned we can apply the given ideas. A semigroup of linear continuous operators $S : \mathbb{R}_{\geq 0} \to L(E)$, where E denotes a locally convex space is called weakly integrable if there is a S'-invariant, point separating subspace F of the continuous dual E' with the property that for a given $\omega > 0$ the functions $t \longmapsto \exp(-\lambda t)\langle S(t)x, \xi\rangle$ are integrable for $\lambda > \omega$, $x \in E$, $\xi \in F$ on $\mathbb{R}_{\geq 0}$ such that the operators $R(\lambda) : E \to E$ with

$$\langle R(\lambda)x, \xi\rangle = \int_0^\infty \exp(-\lambda t)\langle S(t)x, \xi\rangle dt$$

exist. Applying our method one should look at $\sigma(E, F)$, the mapping $S : \mathbb{R}_{\geq 0} \to L(E^{\sigma(E,F)})$ is a semigroup of linear continuous operators because of invariance. We assume $E^{\sigma(E,F)}$ to be convenient, but we do not need to assume the existence of the above resolvents. Passing to the C_0-subspace we have to assume that the smooth vectors exist in the given locally convex topology. Remark that this subspace is closed with respect to Mackey-sequences, so convenient. Consequently we can pass to the subspace of smooth vectors by proposition 2.2 and apply the result.

In [Hug77] semigroups of unbounded operators on Banach spaces are investigated. They can by definition be reduced to strongly continuous semigroups on a Fréchet space. The author assumes exponential boundedness in his article, consequently the above theory applies.

In [Kom68] a distributional approach towards the problem is chosen, which could be reformulated in the convenient setting. Another interesting reference in this direction is [Dem74]. The conditions on asymptotic resolvents are realized to be some Palais-Wiener conditions for distributional Laplace transforms to stem from a smooth semigroup. Anyway the calculations in the smooth setting are simpler and yield the same or even better results, since convenience is in fact much weaker than sequential completeness.

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