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FUNCTION SPACES IN LIPSCHITZ DOMAINS AND ON LIPSCHITZ MANIFOLDS. CHARACTERISTIC FUNCTIONS AS POINTWISE MULTIPLIERS

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Abstract

Function spaces of type B^s_{pq} and F^s_{pq} cover as special cases classical and fractional Sobolev spaces, classical Besov spaces, Hölder-Zygmund spaces and inhomogeneous Hardy spaces. In the last 2 or 3 decades they haven been studied preferably on $\mathbb{R}^n$ and in smooth bounded domains in $\mathbb{R}^n$ including numerous applications to pseudodifferential operators, elliptic boundary value problems etc. To a lesser extent spaces of this type have been considered in Lipschitz domains. But in recent times there is a growing interest to study and to use spaces of this type in Lipschitz domains and on their boundaries. It is the aim of this paper to deal with function spaces of B^s_{pq} and F^s_{pq} type in Lipschitz domains and on Lipschitz manifolds in a systematic (although not comprehensive) way: We describe and comment on known results, seal some gaps, give new proofs, and add a few new results of relevant aspects.

1 Introduction

Let $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ be the nowadays well-established function spaces on a domain Ω in $\mathbb{R}^n$ with

$$s \in \mathbb{R}, \quad 0 < p \leq \infty, \quad 0 < q \leq \infty, \quad (p < \infty \text{ for } F\text{-spaces}), \qquad (1)$$

defined by restriction of corresponding spaces on $\mathbb{R}^n$ to Ω. Recall that these two scales cover as special cases, classical and fractional Sobolev spaces, classical Besov spaces, Hölder-Zygmund spaces and (inhomogeneous) Hardy spaces. Let Ω be either $\mathbb{R}^n$ or $\mathbb{R}^n_+$ (half-space) or a

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bounded C^∞ domain in $\mathbb{R}^n$. The theory of the corresponding spaces in its full extent has been developed in the last decades systematically. We refer to the books [26], [37], [38], [10], [39], [7], [29], [1], [40]. A more complete bibliography of books and surveys dealing with special cases of the above spaces and including in particular the work of the Russian school may be found in [41], pp. 1-4. There is also a short list of the classical spaces mentioned above.

Again let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. Then the theory of the spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ is fully developed. Here are a few key-words: embeddings, compactness, degree of compactness expressed in terms of entropy numbers and diverse types of widths, traces (on the boundary), interpolation, pointwise multipliers, scales, equivalent (quasi-)norms, intrinsic descriptions etc. Technically one developes first a corresponding theory for spaces on $\mathbb{R}^n_+$ and reduces afterwards problems for spaces on bounded C^∞ domains via local charts (resolution of unity combined with local diffeomorphisms) to this standard situation. If Ω is a bounded domain in $\mathbb{R}^n$ but not necessarily smooth then this method of local charts does not work (at least not without severe restrictions for the parameters involved). The question arises to which extent the well-established theory of the function spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ can be carried over from bounded C^∞ domains Ω in $\mathbb{R}^n$ to bounded non-smooth domains in $\mathbb{R}^n$. Especially bounded Lipschitz domains attracted a lot of attention in recent times. One reason is the study of (elliptic) PDE's in such domains. Relevant references may be found in [41], 20.13, p. 308, and also in [36], Ch. 4. Furthermore there is a growing interest in numerical solutions of (elliptic) problems in Lipschitz domains using wavelets and splines. However as far as we know there is no comprehensive treatment of function spaces in bounded Lipschitz domains comparable with what is available for bounded C^∞ domains. But there are several papers dealing with specific aspects. It is beyond the scope of this paper to seal this gap. But we wish to touch on a few of the topics listed above and to discuss which of these assertions remain valid for bounded Lipschitz domains and to which extent. Comparing properties of function spaces in bounded C^∞ domains and in bounded Lipschitz domains there are 3 types of assertions:

I There is no difference neither in the formulation nor in the proof of corresponding properties. This applies to all embeddings between function spaces, including compactness and the degree of compactness expressed in terms of entropy numbers.

II The respective properties are the same, but new arguments are needed in case of bounded Lipschitz domains. This applies, for example, to extension problems and to interpolation assertions, but also to duality.

III The assertions must be modified when bounded C^∞ domains are replaced by bounded Lipschitz domains. Typical examples are traces on, say, the boundary, or scales and lifting properties.

The plan of the paper is the following.
Section 2 deals with basic properties of function spaces in (special and bounded) Lipschitz domains covering assertions of types I and II, including extensions and interpolation.
Section 3 covers a few more special properties, including subspaces, extensions by zero, duality and scales.
Section 4 concentrates on Lipschitz diffeomorphisms in $\mathbb{R}^n$ and its consequences for function spaces in Lipschitz domains. As said above in case of bounded C^∞ domains Ω in $\mathbb{R}^n$, properties of respective function spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ are derived via the method of local charts, based on resolutions of unity and C^∞- diffeomorphisms on $\mathbb{R}^n$. We clarify under which restrictions for the parameters s, p, q involved this method can be applied in case of Lipschitz domains. Non-smooth atomic decompositions of function spaces will play a decisive role.
Section 5 might be considered as a digression. On the one hand we use the assertions of Section 4 to formulate a few results concerning pointwise multipliers related to Lipschitz domains. But on the other hand we add some remarks dealing with characteristic functions of irregular domains as pointwise multipliers in $F^s_{pq}(\mathbb{R}^n)$ somewhat outside of the main line of this paper.
Section 6 contains a second application of the results of Section 4. We introduce (concrete and abstract) n-dimensional Lipschitz manifolds M (for example the boundary $\partial\Omega$ of a bounded Lipschitz domain Ω in $\mathbb{R}^{n+1}$). With the help of the well-known method of local charts the theory of function spaces F^s_{pq} and B^s_{pq} can be transferred from $\mathbb{R}^n$ to M under natural restrictions for s, p, q.

2 Definitions and basic properties

2.1 Basic notation

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be euclidean n-space, where $n \in \mathbb{N}$; put $\mathbb{R} = \mathbb{R}^1$; whereas $\mathbb{C}$ is the complex plane. Let

$$\mathbb{R}^n_+ = \big\{ x \in \mathbb{R}^n : \quad x = (x', x_n) \text{ with } x' \in \mathbb{R}^{n-1} \text{ and } x_n > 0 \big\} \qquad (2)$$

be the half-space ($n \geq 2$). As usual, $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, ..., m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Furthermore, $a_+ = \max(a, 0)$ if $a \in \mathbb{R}$.

Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote the topological dual, the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \leq \infty$, is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\| f \,| L_p(\mathbb{R}^n) \| = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

with the obvious modification if $p = \infty$. If $\varphi \in S(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} \, \varphi(x) \, dx, \quad x \in \mathbb{R}^n, \qquad (3)$$

denotes the Fourier transform of φ. As usual, $F^{-1}\varphi$ or $\varphi^\vee$, stands for the inverse Fourier transform, given by the right-hand side of (3) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi \in S(\mathbb{R}^n)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \leq 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \geq \frac{3}{2}. \qquad (4)$$

We put $\varphi_0 = \varphi$; $\varphi_1(x) = \varphi(\frac{x}{2}) - \varphi(x)$; and

$$\varphi_k(x) = \varphi_1 \left(2^{-k+1} x \right), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}.$$

Then, since

$$\sum_{k=0}^{\infty} \varphi_k(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n,$$

the φ_k form a dyadic resolution of unity in $\mathbb{R}^n$. Recall that $(\varphi_k \widehat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$. In particular, $(\varphi_k \widehat{f})^\vee(x)$ makes sense pointwise.

Definition 2.1. (i) *Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that*

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \, \|(\varphi_j \widehat{f})^\vee \,|L_p(\mathbb{R}^n)\|^q \right)^{\frac{1}{q}} \tag{5}$$

(with the usual modification if $q = \infty$) is finite.

(ii) *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$. Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that*

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{\frac{1}{q}} \,|L_p(\mathbb{R}^n) \right\| \tag{6}$$

(with the usual modification if $q = \infty$) is finite.

Remark 2.2. These spaces, including their forerunners and special cases, have a long history. We refer to the books mentioned in the Introduction and to the more complete list given in [41], pp. 1-2. In particular, both $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces which are independent of the function φ according to (4), in the sense of equivalent quasi-norms. This justifies our omission of the subscript φ in (5) and (6), in what follows. If $p \geq 1$ and $q \geq 1$ then both $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are Banach spaces. Otherwise we assume that the reader is familiar with the basic assertions of these spaces. In [39], Ch. 1, one finds a historically-minded survey. For sake of completeness we give a short list of special cases without further comments. Some more details, in particular a description in terms of classical norms, may also be found in [7], 2.2.2; [40], 10.5; and [41], 1.2; including some references.

(i) Let $1 < p < \infty$. Then

$$F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$$

(Littlewood-Paley property).

(ii) Let $1 < p < \infty$ and $s \in \mathbb{N}_0$. Then

$$F^s_{p,2}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n)$$

are the classical Sobolev spaces.

(iii) Let $1 < p < \infty$ and $s \in \mathbb{R}$. Then

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$$

are the Sobolev spaces (sometimes denoted as fractional Sobolev spaces or as Bessel-potential spaces).

(iv) Let $s > 0$. Then

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n)$$

are the classical Hölder-Zygmund spaces.

(v) Let $s > 0$, $1 \le p \le \infty$ and $1 \le q \le \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ are the classical Besov spaces (including the limiting cases $p = 1$ and $p = \infty$).

(vi) Let $0 < p < \infty$. Then

$$h_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)$$

are the (inhomogeneous) Hardy spaces.

2.2 Spaces on domains

Let Ω be a domain (i.e., open set) in $\mathbb{R}^n$. Its boundary is denoted by $\partial\Omega$. Let $0 < p \le \infty$. Then $L_p(\Omega)$ is the quasi-Banach space of all complex-valued Lebesgue integrable functions in Ω such that

$$\|f\,|L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty$$

(with the usual modification if $p = \infty$). As usual, $D(\Omega) = C_0^\infty(\Omega)$ stands for the collection of all complex infinitely differentiable functions in $\mathbb{R}^n$ with compact support in Ω. Let $D'(\Omega)$ be the dual space of distributions on Ω. Let $g \in S'(\mathbb{R}^n)$. Then we denote by $g|\Omega$ its restriction to Ω,

$$g|\Omega \in D'(\Omega) : \quad (g|\Omega)(\varphi) = g(\varphi) \quad \text{for} \quad \varphi \in D(\Omega).$$

Definition 2.3. *Let Ω be a domain in $\mathbb{R}^n$. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Let A^s_{pq} stand either for B^s_{pq} or F^s_{pq} (with $p < \infty$ in the F-case).*

(i) *$A^s_{pq}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a $g \in A^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$. Furthermore,*

$$\|f \,|A^s_{pq}(\Omega)\| = \inf \|g \,|A^s_{pq}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^n)$ such that its restriction $g|\Omega$ to Ω coincides in $D'(\Omega)$ with f.

(ii) *$\mathring{A}^s_{pq}(\Omega)$ is the completion of $D(\Omega)$ in $A^s_{pq}(\Omega)$.*

(iii) *$\widetilde{A}^s_{pq}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a*

$$g \in A^s_{pq}(\mathbb{R}^n) \quad \textit{with} \quad g|\Omega = f \quad \textit{and} \quad \textit{supp}\, g \subset \overline{\Omega}. \tag{7}$$

Furthermore,

$$\|f \,|\widetilde{A}^s_{pq}(\Omega)\| = \inf \|g \,|A^s_{pq}(\mathbb{R}^n)\|,$$

where the infimum is taken over all g with (7).

Remark 2.4. All spaces are quasi-Banach spaces (Banach spaces if $p \ge 1$ and $q \ge 1$). A detailed discussion of the spaces $\mathring{A}^s_{pq}(\Omega)$ and $\widetilde{A}^s_{pq}(\Omega)$ will be given in Section 3. We concentrate here on the spaces $A^s_{pq}(\Omega)$. The above definition coincides essentially with [41], Definition 5.3, p. 44 (extended to unbounded domains). We are mainly interested in Lipschitz domains Ω. But we postpone the definition of what is meant by a Lipschitz domain to 2.4 and formulate first some assertions of the spaces $A^s_{pq}(\Omega)$ which are independent of the quality of Ω and its boundary $\partial\Omega$.

2.3 Embeddings and entropy numbers

Embedding theorems for spaces $A^s_{pq}(\mathbb{R}^n)$ and their special cases have a long history. We studied assertions of this type in [38], 2.3.2 and 2.7. Short surveys including more recent assertions may be found in [7], 2.3.3, and in [29], 2.2. In [41], Ch. 2, we dealt in detail with limiting embeddings. All these embedding theorems can be carried over from $\mathbb{R}^n$ to arbitrary domains. This follows immediately from the definition of $A^s_{pq}(\Omega)$ as restriction of $A^s_{pq}(\mathbb{R}^n)$. We describe an example without further explanations. Let Ω be an arbitrary domain in $\mathbb{R}^n$. Let

$$s_0 \in \mathbb{R}, \quad s_1 \in \mathbb{R}, \quad 0 < p_0 < p_1 < \infty, \quad s_1 - \frac{n}{p_1} \le s_0 - \frac{n}{p_0},$$

and $0 < q_0 \le \infty$, $0 < q_1 \le \infty$. Then

$$F^{s_0}_{p_0 q_0}(\Omega) \subset F^{s_1}_{p_1 q_1}(\Omega), \tag{8}$$

where '$\subset$' always means that the respective embedding operator is linear and bounded. This assertion with $\mathbb{R}^n$ in place of Ω may be found, for example, in [38], 2.7.1, p. 129. As said, then (8) is an immediate consequence of the definition of the spaces on Ω by restriction. This applies to all embedding assertions of this type.

If Ω is a bounded domain in $\mathbb{R}^n$ then it makes sense to ask under which conditions embeddings of type (8) are compact. Entropy numbers are the natural quantities to measure the degree of compactness. We repeat first the abstract definition of entropy numbers and describe afterwards a typical example of compact embeddings between function spaces in bounded domains. If A and B are quasi-Banach spaces, then $L(A,B)$ is the space of linear and bounded operators from A into B. Let U_A, U_B be the unit ball in A, B, respectively.

Definition 2.5. *Let A, B be quasi-Banach spaces and let $T \in L(A,B)$. Then for all $k \in \mathbb{N}$ the kth entropy number $e_k(T)$ is defined as the infimum of all positive numbers ε such that*

$$T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \quad \textit{for some } b_1, ..., b_{2^{k-1}} \in B.$$

Remark 2.6. As for the abstract theory of entropy numbers in Banach spaces we refer to [27], Sect. 12, and [6]. A short description of some aspects (interpolation theory in Banach spaces) on an abstract level may also be found in [37], 1.16. An extension of this theory to quasi-Banach spaces has been given in [7], Ch. 1. We refer also to [40], Sect. 6. There one finds also applications to spectral theory of diverse types of partial differential operators. This is the point where the interest in entropy numbers of compact embeddings between function spaces comes from. We formulate a typical result.

Theorem 2.7. *Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$. Let*

$$-\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \leq \infty, \quad 0 < p_2 \leq \infty, \tag{9}$$

$0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ *and*

$$\delta_+ = s_1 - s_2 + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0\,. \tag{10}$$

Then the embedding of $B^{s_1}_{p_1q_1}(\Omega)$ into $B^{s_2}_{p_2q_2}(\Omega)$ is compact and for the related entropy numbers we have

$$e_k\left(id:\ B^{s_1}_{p_1q_1}(\Omega) \mapsto B^{s_2}_{p_2q_2}(\Omega)\right) \sim k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N}. \tag{11}$$

Remark 2.8. The equivalence in (11) and also equivalences at later occasions must be understood as follows: there are two positive numbers c_1 and c_2 such that

$$c_1 e_k \leq k^{-\frac{s_1-s_2}{n}} \leq c_2 e_k \quad \text{for all } k \in \mathbb{N}.$$

A proof of this theorem may be found in [40], Sect. 23. There are also further references. This can be extended to other embeddings as far as compactness is expressed in terms of entropy numbers. On the other hand, if one replaces entropy numbers by some types of widths, for example approximation numbers, then the method used in [40] to prove the above theorem for arbitrary bounded domains Ω in $\mathbb{R}^n$ does not work. One needs additional restrictions for Ω such that the function spaces defined on Ω have the extension property. It comes out that this is the case if Ω is a bounded Lipschitz domain.

2.4 Lipschitz domains and extensions

Let $n-1 \in \mathbb{N}$. Recall that

$$x' \in \mathbb{R}^{n-1} \mapsto h(x') \in \mathbb{R} \tag{12}$$

is called a Lipschitz function (on $\mathbb{R}^{n-1}$) if there is a number $c > 0$ such that

$$|h(x') - h(y')| \le c\,|x' - y'| \quad \text{for all } x' \in \mathbb{R}^{n-1},\, y' \in \mathbb{R}^{n-1}. \tag{13}$$

Definition 2.9. *Let $n-1 \in \mathbb{N}$.*
(i) *A special Lipschitz domain in $\mathbb{R}^n$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ such that*

$$h(x') < x_n < \infty,$$

where $h(x')$ is a Lipschitz function according to (12), (13).

(ii) *A bounded Lipschitz domain in $\mathbb{R}^n$ is a bounded domain Ω in $\mathbb{R}^n$ where $\partial\Omega$ can be covered by finitely many open balls B_j in $\mathbb{R}^n$; $j = 1, ..., J$; centred at $\partial\Omega$ such that*

$$B_j \cap \Omega = B_j \cap \Omega_j \quad \textit{with} \quad j = 1, ..., J, \tag{14}$$

where Ω_j are rotations of suitable special Lipschitz domains in $\mathbb{R}^n$.

Remark 2.10. (*Localization method*) If Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, then $\partial\Omega$ can be covered by J open balls $B_j = B(x^j, r)$, centred at $x^j \in \partial\Omega$ and of radius $r > 0$. Let $\{\varphi_j\}_{j \in J}$ be a subordinated resolution of unity,

$$\varphi_j \in S(\mathbb{R}^n), \quad 0 \le \varphi_j \le 1;$$

and

$$\mathit{supp}\, \varphi_j \subset B_j, \quad \sum_{j=1}^{J} \varphi_j(x) = 1 \quad \text{in a neighbourhood of } \partial\Omega.$$

Let $A^s_{pq}(\Omega)$ be a space according to Definition 2.3. Then $\varphi_j f \in A^s_{pq}(\Omega)$ if $f \in A^s_{pq}(\Omega)$ and there is a number $c > 0$ such that

$$\|\varphi_j f \,|A^s_{pq}(\Omega)\| \le c\, \|f\,|A^s_{pq}(\Omega)\| \quad \text{for all } f \in A^s_{pq}(\Omega).$$

This follows from a corresponding property for the spaces $A^s_{pq}(\mathbb{R}^n)$ and Definition 2.3. Many properties for spaces of type A^s_{pq} in domains are reduced with the help of this *localization method* to local investigations. This applies in particular to the above special and bounded Lipschitz domains. But it is obvious that this method works also for arbitrary (not necessarily bounded) Lipschitz domains defined in the same way as in part (ii) of the above definition, now with respect to balls $B_j = B(x^j, r)$, $j \in \mathbb{N}$, and under the assumption that the Lipschitz constants according to (13) and related to the special Lipschitz domains Ω_j are uniformly bounded. We will not stress this point in the sequel.

The extension problem. Let Ω be an (arbitrary) domain in $\mathbb{R}^n$. Then, by Definition 2.3, the *restriction operator re*,

$$re\,(g) = g|\Omega \;:\quad S'(\mathbb{R}^n) \mapsto D'(\Omega), \tag{15}$$

generates a linear and bounded operator

$$re \;:\quad A^s_{pq}(\mathbb{R}^n) \mapsto A^s_{pq}(\Omega), \tag{16}$$

for all admitted $A = B$, $A = F$, and s, p, q. Of course, *re* in (16) is the restriction of the operator in (15) to $A^s_{pq}(\mathbb{R}^n)$. But as usual this will not be indicated by additional marks. This tacit agreement applies also to other related operators such as extension operators and the identity acting in spaces or between different spaces, where (11) may serve as an example. The extension problem is characterized by the question of whether there is a linear and bounded *extension operator ext*, such that

$$ext \;:\quad A^s_{pq}(\Omega) \mapsto A^s_{pq}(\mathbb{R}^n) \tag{17}$$

with

$$re \circ ext = id \quad (\text{identity in } A^s_{pq}(\Omega)). \tag{18}$$

If Ω is a bounded C^∞ domain in $\mathbb{R}^n$ then this problem has been solved satisfactory in [39], 4.5, with many forerunners; [37], [38], as far as our own contributions are concerned. There one finds also the necessary references and historical comments. It is, or, as far as smooth bounded domains are concerned, it was one of the key problems of the theory of the spaces $A^s_{pq}(\Omega)$. If $\partial\Omega$ is smooth then the basic idea to construct operators of type (17), (18), is to localize first the problem according to Remark 2.10, to reduce afterwards the problem via local C^∞ diffeomorphic maps $y = \psi(x)$ to $\mathbb{R}^n_+$, to construct there explicitly extension operators from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ and to return afterwards to the bounded C^∞ domain Ω. By [39], Corollary 4.5.2, p. 225, one finds in this way for any $\varepsilon > 0$ a common extension operator ext^ε,

$$ext^\varepsilon \ : A^s_{pq}(\Omega) \mapsto A^s_{pq}(\mathbb{R}^n) \ \text{ where } \ |s| < \varepsilon^{-1}, \varepsilon < p < \infty, \varepsilon < q \le \infty, \quad (19)$$

(with $\varepsilon < p \le \infty$ and $0 < q \le \infty$ if $A = B$). We use the notation *common extension operator*, if the operator in question, in our case ext^ε, is defined on the union of all admitted spaces, and if the restriction of this operator to each member of this union results in a linear and bounded operator, (19) in our case. If Ω is a (special or bounded) Lipschitz domain then one has the localization method as described in Remark 2.10, but the reduction of the extension problem to $\mathbb{R}^n_+$ works only under severe restrictions of the parameters s, p, q. This will be the subject of Section 4. But in general one needs other arguments. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. By Calderón's extension method (1960/61) combined with some interpolation one gets the following assertion: For any $N \in \mathbb{N}$, there is a common extension operator ext^N for all spaces

$$H^s_p(\Omega) \quad \text{and} \quad B^s_{pq}(\Omega), \quad 0 < s < N, 1 < p < \infty, 0 < q \le \infty.$$

We refer to [37], 4.2.3, p. 314, where one finds also the necessary references. This result was extended by E. M. Stein in [35], p. 181, combined with some interpolation, by constructing a common extension operator ext^∞ for all spaces

$$H^s_p(\Omega) \quad \text{and} \quad B^s_{pq}(\Omega), \quad s > 0, 1 < p < \infty, 0 < q \le \infty,$$

(including even Soblev spaces $W^k_1(\Omega)$, $W^k_\infty(\Omega)$, $k \in \mathbb{N}_0$). The next step is due to G. A. Kalyabin. He proved in [16], Theorem 1, that Stein's extension operator ext^∞ is also a common extension operator for all spaces

$$F^s_{pq}(\Omega), \quad s > 0, 1 < p < \infty, 1 < q < \infty,$$

(and for more general spaces of F-type considered there). We refer also to [15] and [17], Sect. 7. The final step is due to V. S. Rychkov in [31]. We call an extension operator *universal* if it is a common extension operator for all spaces considered,

$$A^s_{pq}(\Omega), \quad s \in \mathbb{R}, 0 < p \le \infty, 0 < q \le \infty, \quad (20)$$

with $p < \infty$ if $A = F$.

Theorem 2.11. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there is a universal extension operator.*

Remark 2.12. (*Rough domains, intrinsic characterizations*) The universal extension operator of the above theorem is constructed explicitly in [31], Theorem 4.1, p. 253, based on an adapted Calderón reproducing formula. This gives also the possibility to decide intrinsically to which spaces $A^s_{pq}(\Omega)$ a given element $f \in D'(\Omega)$ or $f \in re\, S'(\mathbb{R}^n)$ belongs: for the related quasi-norms one needs only the knowledge of f in Ω (and not of some $g \in S'(\mathbb{R}^n)$ with $g|\Omega = f$). We refer in this context also to [30]. If a bounded domain Ω is not Lipschitz then it is not clear of whether there are linear extension operators of the above type for the spaces in (20). A partial result concerning some Sobolev spaces $H^s_p(\Omega)$ with $s > 0$, $1 < p < \infty$, may be found in [32]. Fortunately enough, by Theorem 2.7 assertions about entropy numbers do not depend on the quality of the bounded domain Ω. Furthermore there are intrinsic characterizations of the spaces $A^s_{pq}(\Omega)$ with (20) for large classes of bounded domains Ω in $\mathbb{R}^n$ which need not to be Lipschitz in terms of atomic representations. We refer to [42]. A short description may be found in [7], 2.5, pp. 57-65.

2.5 Interpolation

We assume that the reader is familiar with the basic assertions of interpolation theory. Let $\{A_0, A_1\}$ ba an interpolation couple of complex quasi-Banach spaces. Then

$$(A_0, A_1)_{\theta,q}\,, \quad 0 < \theta < 1\,, \quad 0 < q \leq \infty, \tag{21}$$

denotes, as usual, the real interpolation method based on Peetre's K-functional. The original complex interpolation method,

$$[A_0, A_1]_\theta\,, \quad 0 < \theta < 1, \tag{22}$$

as introduced by Calderón, is restricted to complex Banach spaces. We refer to [2] and [37]. Several attemps have been made to extend the complex method to quasi-Banach spaces. We refer to [38], 2.4 and 3.3.6, as far as references and our own contributions are concerned. The resulting interpolation formulas for the B-spaces and F-spaces have the expected form, but the interpolation property is not satisfied automatically, it must be checked case by case (what is always possible when used). Fortunately enough, O. Mendez and M. Mitrea extended in [24] Calderón's original complex interpolation method from complex Banach spaces to the larger class of so-called A-convex (analytically convex) complex quasi-Banach spaces. Restricted to this sub-class of quasi-Banach spaces the interpolation property is always valid. It is one of the main aims of [24], Sect. 4, to prove that all spaces

$$A^s_{pq}(\mathbb{R}^n), \quad s \in \mathbb{R},\, 0 < p \le \infty,\, 0 < q \le \infty, \tag{23}$$

with $p < \infty$ if $A = F$, are A-convex quasi-Banach spaces. This extension of Calderón's complex interpolation method is again denoted by (22). For example, let

$$s_0 \in \mathbb{R},\, s_1 \in \mathbb{R},\, 0 < p_0 < \infty,\, 0 < p_1 < \infty,\, 0 < q_0 \le \infty,\, 0 < q_1 \le \infty. \tag{24}$$

Let $0 < \theta < 1$ and

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{25}$$

Then

$$\left[F^{s_0}_{p_0q_0}(\mathbb{R}^n), F^{s_1}_{p_1q_1}(\mathbb{R}^n)\right]_\theta = F^s_{pq}(\mathbb{R}^n). \tag{26}$$

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let ext be the respective extension operator according to Theorem 2.11. Then we have (17), (18) for all admitted spaces (20). Furthermore,

$$P = ext \circ re\,; \quad A^s_{pq}(\mathbb{R}^n) \mapsto A^s_{pq}(\mathbb{R}^n)\,, \tag{27}$$

is a universal projection operator of $A^s_{pq}(\mathbb{R}^n)$ onto a complemented subspace of $A^s_{pq}(\mathbb{R}^n)$, denoted by $PA^s_{pq}(\mathbb{R}^n)$, and ext is an isomorphic map of $A^s_{pq}(\Omega)$ onto $PA^s_{pq}(\mathbb{R}^n)$. By the definition of A-convex quasi-Banach spaces in [24] (with a reference to [18]) it follows that closed subspaces of A-convex quasi-Banach spaces are again A-convex quasi-Banach spaces. Furthermore, a quasi-Banach space which is isomorphic to an A-convex quasi-Banach space is also A-convex. By the above remarks this applies to all spaces $A^s_{pq}(\Omega)$ in (20) where Ω is a bounded Lipschitz domain. Then the interpolation theory for complemented subspaces in

[37], 1.17.1, p. 118, can be extended from Banach spaces to these classes of A-convex quasi-Banach spaces. We formulate the outcome for the spaces in (20). We give also a short proof based on the interpolation property which is now available both for the real and the complex interpolation method within the classes $A^s_{pq}(\mathbb{R}^n)$ and $A^s_{pq}(\Omega)$ for all admitted parameters s, p, q, according to (23) and (20).

Theorem 2.13. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $A^s_{pq}(\mathbb{R}^n)$ and $A^s_{pq}(\Omega)$ be the spaces according to Definition 2.3. Then any real or complex interpolation formula for A-spaces on $\mathbb{R}^n$ which results again in an A-space on $\mathbb{R}^n$ remains valid if on replaces $\mathbb{R}^n$ by Ω. For example,*

$$\left[F^{s_0}_{p_0q_0}(\Omega), F^{s_1}_{p_1q_1}(\Omega)\right]_\theta = F^s_{pq}(\Omega) \tag{28}$$

with (24), (25).

Proof. Based on the above explanations the proof of the theorem is not very complicated. As an example we justify (28). Let, for brevity,

$$F^k(\mathbb{R}^n) = F^{s_k}_{p_kq_k}(\mathbb{R}^n), \quad \text{where } k = 0 \text{ or } k = 1,$$

and similarly $F^k(\Omega)$. Let, as a notation,

$$F^\theta(\Omega) = \left[F^0(\Omega), F^1(\Omega)\right]_\theta, \quad 0 < \theta < 1.$$

Then we have to prove that

$$F^\theta(\Omega) = F^s_{pq}(\Omega).$$

By (25) and the interpolation property for the F-spaces both on $\mathbb{R}^n$ and on Ω it follows that

$$\|f\,|F^\theta(\Omega)\| = \|re \circ ext\, f\,|F^\theta(\Omega)\| \le c_1\,\|ext\, f|\,F^\theta(\mathbb{R}^n)\| \le c_2\,\|f\,|F^\theta(\Omega)\|.$$

Recall that ext is an universal extension operator. Then we get by (26) that

$$\|f\,|F^\theta(\Omega)\| \sim \|ext\, f\,|F^s_{pq}(\mathbb{R}^n)\| \sim \|f\,|F^s_{pq}(\Omega)\|.$$

This proves (28).

2.6 s-numbers and widths

According to Definition 2.5 one can describe the degree of compactness of an operator $T \in L(A,B)$ in terms of entropy numbers $e_k(T)$. However in the abstract theory of Banach spaces there are several other quantities, called s-numbers and widths, including approximation numbers, Kolmogorov numbers, Gelfand numbers and Weyl numbers measuring the quality of operators $T \in L(A,B)$. We refer to [27], especially Sect. 11; [28], especially Ch. 2, and [20]. A short description of some aspects (interpolation theory in Banach spaces) on an abstract level may also be found in [37], 1.16. At least for some of these numbers it makes sense to extend the abstract theory from Banach spaces to quasi-Banach spaces. We give an example.

Let A, B be quasi-Banach spaces and let $T \in L(A,B)$. Then given any $k \in \mathbb{N}$, the kth approximation number $a_k(T)$ of T is defined by

$$a_k(T) = \inf\{\|T - L\| \, : \, L \in L(A,B), \, rank\, L < k\}, \tag{29}$$

where $rank\, L$ is the dimension of the range of L.

These approximation numbers, but also other s-numbers $\{s_k(T) : k \in \mathbb{N}\}$, extendable to quasi-Banach spaces, have the following decisive properties,

$$s_{l+k-1}(S+T) \le c\,\left(s_l(T) + s_k(T)\right), \quad l \in \mathbb{N}, \quad k \in \mathbb{N}; \tag{30}$$

and

$$s_k(T_1 \circ T \circ T_2) \le \|T_1\| \cdot s_k(T) \cdot \|T_2\|, \quad k \in \mathbb{N}, \tag{31}$$

where

$$S \in L(A,B), \quad T \in L(A,B), \quad T_1 \in L(B,C), \quad T_2 \in L(D,A),$$

and A, B, C, D, are complex quasi-Banach spaces. Here $c \ge 1$ (with $c = 1$ in case of Banach spaces). A lot has been done over the years to study these numbers in connection with function spaces. Of peculiar interest is the situation as described in Theorem 2.7. As for the classical theory in smooth domains (up to the middle of the seventies) one may consult [37], 4.10. More recent results and references can be found in [7]. Let Ω be a bounded domain in $\mathbb{R}^n$. Assume that the parameters are the same as in Theorem 2.7, in particular they are restricted by (9), (10). Then we denote the corresponding embedding by

$$id_\Omega : \quad A^{s_1}_{p_1 q_1}(\Omega) \mapsto A^{s_2}_{p_2 q_2}(\Omega). \tag{32}$$

(It does not matter very much whether we choose $A = B$ or $A = F$). In case of bounded C^∞ domains Ω we studied in [7], Theorem 3.3.4, p. 119, in detail the behaviour of the approximation numbers $a_k(id_\Omega)$. The outcome is more complicated than the corresponding behaviour (11) of the related entropy numbers. A case left open was solved afterwards in [5]. Let $K = \{y : \ |y| < 1\}$ be the unit ball and let id_K be the corresponding embedding operator (32).

Theorem 2.14. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.*

(i) *Let $a_k(id_\Omega)$ be the approximation numbers of the embedding (32). Then*

$$a_k(id_\Omega) \sim a_k(id_K), \quad k \in \mathbb{N}, \tag{33}$$

where K is the unit ball in $\mathbb{R}^n$ and where the equivalence constants are independent of k.

(ii) *Let s_k be fixed s-numbers satisfying (30), (31), and well-defined for the considered couple of function spaces. Then*

$$s_k(id_\Omega) \sim s_k(id_K), \quad k \in \mathbb{N}. \tag{34}$$

Proof. Step 1. Let K_1 and K_2 be two open balls with

$$\overline{K}_1 \subset \Omega \quad \text{and} \quad \overline{\Omega} \subset K_2\,.$$

Let ψ be a cut-off function,

$$\psi \in D(K_2) \quad \text{and} \quad \psi(x) = 1 \quad \text{if} \quad x \in \Omega_\varepsilon\,,$$

where Ω_ε is an ε-neighbourhood of Ω. Let *ext* be the extension operator according to Theorem 2.11. Then

$$id_\Omega = re \circ id_{K_2} \circ \psi \circ ext.$$

By (31) with $s_k = a_k$ the left-hand side of (33) can be estimated from above by the right-hand side. In the same way one gets the converse assertion using now K_1 and Ω.

Step 2. In part (ii) we assume in addition that $s_k(id_\Omega)$ and $s_k(id_K)$ make sense. Then the proof is the same as in part (i)

Remark 2.15. In the above proof one does not really need the cut-off function ψ (at least not in Step 1). It simply makes clear that the behaviour near $\partial\Omega$ of functions belonging to some function spaces is immaterial for assertions of type (33), (34). Furthermore it emphasizes that $\psi \circ ext$ is also an universal extension operator with the additional property that the extended functions vanish outside of a given neighbourhood of $\overline{\Omega}$. The additional restriction in part (ii) must be checked in dependence of the considered s-numbers. Sometimes it might mean that the considered function spaces must be Banach spaces. In a concrete situation one can simply check whether the above reasoning is applicable. We describe an example. Let $a_k(T)$ with $T \in L(A, B)$ be the approximation numbers as introduced in (29). Then

$$x_k(T) = \sup \{a_k(T \circ Q) \,:\, Q \in L(\ell_2, A),\ \|Q\| = 1\}\,, \quad k \in \mathbb{N},$$

are the Weyl numbers. In [20], 3.3.c.5, 3.3.c.7, pp. 186, 189, one finds estimates of $x_k(id)$ for the embedding operator id from $B^s_{pq}(\Omega)$ into $L_r(\Omega)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $s > 0$, and p, q, r, are between 1 and ∞. This assertion has been extended in [3] and [4] to all spaces $A^s_{pq}(\Omega)$ considered in this paper under the assumption that Ω is a bounded C^∞ domain in $\mathbb{R}^n$. By part (ii) of the above theorem these estimates remain valid for arbitrary bounded Lipschitz domains in $\mathbb{R}^n$.

Remark 2.16. Although part (ii) of the above theorem is little bit cryptical, it might be of some help in concrete situations where assertions of this type are needed. This happens, for example, if one deals with quadratic forms in $L_2(\Omega)$ or with weak (variational) solutions of elliptic differential operators of second order with, maybe, rough coefficients. Then one gets via duality theory or by the Lax-Milgram theorem (which may be found in [45], p. 92) inverse operators T of type

$$T: \quad H^{-1}(\Omega) \mapsto H^1(\Omega),$$

where, in our notation $H^\sigma(\Omega) = F^\sigma_{2,2}(\Omega)$. Combined with the embedding of $H^1(\Omega)$ into $H^{-1}(\Omega)$ one wishes to get sharp assertions for diverse s-numbers. Also the complexity of some numerical problems such as the computation of integrals or the solution of some PDE's is sometimes expressed in terms of some widths and s-numbers in function spaces of positive and negative smoothness. The underlying domain might be L-shaped, a polyhedron, or simply a cube.

3 Further properties, subspaces

3.1 Preliminaries

It is the main aim of this section to have a closer look at the spaces $\mathring{A}^s_{pq}(\Omega)$ and $\widetilde{A}^s_{pq}(\Omega)$, introduced in Definition 2.3, and their connections with the spaces $A^s_{pq}(\Omega)$. This has a long tradition, at least as far as the classical spaces are concerned, from the very beginning of the theory of function spaces. Such subspaces and their interrelations have not only be considered for their own sake but also in connection with diverse applications, for example to boundary value problems of elliptic partial differential equations. Related key words characterizing problems to be treated in this context are: duality, liftings, scale properties, relations to spaces on $\mathbb{R}^n$, extensions by zero, interpolation. A corresponding theory, restricted mainly to bounded C^∞ domains Ω in $\mathbb{R}^n$ and to the classical spaces

$$B^s_{pq}(\Omega) \quad \text{and} \quad H^s_p(\Omega) \quad \text{with} \quad s \in \mathbb{R}, 1 < p < \infty, 1 \le q \le \infty, \tag{35}$$

and their subspaces may be found in [37], Ch. 4, especially 4.2, 4.3, 4.8, 4.9. We returned to this subject in [41], Sect. 5, now for the full scale of the spaces $A^s_{pq}(\Omega)$, but again restricted to bounded C^∞ domains Ω in $\mathbb{R}^n$. It is the aim of this section to investigate to which extent at least a few key assertions remain valid in case of bounded Lipschitz domains Ω in $\mathbb{R}^n$. The expected outcome is the same as indicated in the introduction: some assertions can simply be carried over from bounded C^∞ domains to bounded Lipschitz domains without changing the proofs; other assertions remain valid but new proofs are needed; in some cases it is at least doubtful whether properties for C^∞ domains can be extended to Lipschitz domains.

3.2 Classes of subspaces

First we complement Definition 2.3 as follows. Let Ω be a domain in $\mathbb{R}^n$ and let s, p, q as in Definition 2.3. Then we wish to compare the space $\widetilde{A}^s_{pq}(\Omega)$ according to Definition 2.3(iii) with the closed subspace of $A^s_{pq}(\mathbb{R}^n)$ (with the $\mathbb{R}^n$-quasi-norm) given by

$$\widetilde{A}^s_{pq}(\overline{\Omega}) = \left\{ f \in A^s_{pq}(\mathbb{R}^n) \ : \quad supp\, f \subset \overline{\Omega} \right\}.$$

Although $\widetilde{A}^s_{pq}(\Omega)$ is a subspace of $D'(\Omega)$ and $\widetilde{A}^s_{pq}(\overline{\Omega})$ is a subspace of $S'(\mathbb{R}^n)$ they can be compared and one gets (in obvious interpretation)

$$\widetilde{A}^s_{pq}(\Omega) = \widetilde{A}^s_{pq}(\overline{\Omega}) / \left\{ h \in A^s_{pq}(\mathbb{R}^n) \ : \ supp\, h \subset \partial\Omega \right\} \tag{36}$$

as a factor space. This makes clear what is meant by equality of $\widetilde{A}^s_{pq}(\Omega)$ and $\widetilde{A}^s_{pq}(\overline{\Omega})$.

Proposition 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let*

$$s \in \mathbb{R}, \quad 0 < p \leq \infty, \ (p < \infty \text{ for the } F\text{-spaces}), \quad 0 < q \leq \infty.$$

(i) *Let, in addition,*

$$\max\left(\frac{1}{p} - 1, n(\frac{1}{p} - 1)\right) < s < \infty.$$

Then

$$\widetilde{A}^s_{pq}(\Omega) = \widetilde{A}^s_{pq}(\overline{\Omega}). \tag{37}$$

(ii) *Let, in addition,*

$$0 < p < \infty, \quad 0 < q < \infty, \quad \max\left(\frac{1}{p} - 1, n(\frac{1}{p} - 1)\right) < s < \frac{1}{p}, \tag{38}$$

and $q \geq \min(p, 1)$ for the F-spaces. Then

$$\mathring{A}^s_{pq}(\Omega) = A^s_{pq}(\Omega) = \widetilde{A}^s_{pq}(\Omega). \tag{39}$$

Proof. Step 1. By (36), the proof of (37) reduces to

$$\left\{ h \in A^s_{pq}(\mathbb{R}^n) \ : \quad supp\, h \subset \partial\Omega \right\} = \{0\}$$

under the given restrictions for s, p, q. But the corresponding proof in [41], pp. 45-46, with respect to bounded C^∞ domains, can be taken over without any changes.

Step 2. By [9], Corollary 13.6, the characteristic function χ_Ω of a bounded Lipschitz domain is a pointwise multiplier in $F^s_{pq}(\mathbb{R}^n)$ under the condition (38) and, in addition, $q \geq \min(1,p)$. (We return later on in Sect. 5 in greater detail to this subject). By real interpolation, χ_Ω is a pointwise multiplier for $B^s_{pq}(\mathbb{R}^n)$ with (38). Then we are in similar (but slightly different) situation as in [41], 5.11, p. 58-59. Together with (37) one gets the second equality in (39).

Step 3. It remains to prove the first equality in (39). But this can be done by standard arguments using (37) and the second equality in (39): localization according to Remark 2.10, reduction locally to special Lipschitz domains as in (14), approximation via local translation and mollification (where $p < \infty$, $q < \infty$ are needed).

Remark 3.2. If Ω is a bounded C^∞ domain then the restriction $q \geq \min(1,p)$ in (ii) for the F-spaces is not necessary. This comes from the better pointwise multiplier properties for bounded C^∞ domains, [41], p. 58, known so far. (We return to this question in greater detail in Sect. 5 where we give references to the original papers).

Definition 3.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let*

$$s \in \mathbb{R}, \quad 1 < p < \infty, \quad 1 < q < \infty. \tag{40}$$

Again let either $A = B$ or $A = F$. Then

$$\overline{A}^s_{pq}(\Omega) = A^s_{pq}(\Omega) \quad \textit{if} \quad s < \frac{1}{p} \tag{41}$$

and

$$\overline{A}^s_{pq}(\Omega) = \widetilde{A}^s_{pq}(\Omega) \quad \textit{if} \quad s > \frac{1}{p} - 1. \tag{42}$$

Remark 3.4. The overlap in (41), (42) is justified by (39) with (38). Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and let A^s_{pq} be the classical spaces in (35). Then the above definition coincides with the scales considered in [37], 4.9.2, p. 335. It comes out that these spaces $\overline{A}^s_{pq}(\Omega)$ with $s \in \mathbb{R}$ and fixed p, q are scales which are the right substitutes of the corresponding spaces $A^s_{pq}(\mathbb{R}^n)$. There are satisfactory duality assertions, interpolation formulas and extension properties. As will be shown, a substantial part of this theory remains valid for the above spaces in bounded Lipschitz domains.

3.3 Main assertions

Recall that duality always must be understood in the framework of the dual pairing $(D(\Omega), D'(\Omega))$. As above $(\cdot,\cdot)_{\theta,q}$ and $[\cdot,\cdot]_\theta$ are the real and (classical) complex interpolation method, respectively.

Theorem 3.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let again be either $A = B$ or $A = F$. Let $\overline{A}^s_{pq}(\Omega)$ and similarly $\overline{A}^{s_0}_{p_0q_0}(\Omega)$, $\overline{A}^{s_1}_{p_1q_1}(\Omega)$, be the spaces introduced in Definition 3.3.*

(i) *Let s, p, q be given by (40). Then $\overline{A}^s_{pq}(\Omega)$ is a reflexive Banach space; $D(\Omega)$ is dense in it and (dual spaces)*

$$\big(\overline{A}^s_{pq}(\Omega)\big)' = \overline{A}^{-s}_{p'q'}(\Omega) \quad \textit{with} \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1\,. \tag{43}$$

(ii) *Let*

$$1 < p < \infty, \quad 1 < q < \infty, \quad s > \frac{1}{p} - 1.$$

Then Ext,

$$(Ext\, f)(x) = f(x) \text{ if } x \in \Omega; \quad (Ext\, f)(x) = 0 \text{ if } x \in \mathbb{R}^n\backslash\Omega; \tag{44}$$

is a (linear and bounded) extension operator from $\overline{A}^s_{pq}(\Omega)$ into $A^s_{pq}(\mathbb{R}^n)$.

(iii) *Let $0 < \theta < 1$,*

$$s_0 \in \mathbb{R},\ s_1 \in \mathbb{R}; \quad 1 < p_0 < \infty,\ 1 < p_1 < \infty; \quad 1 < q_0 < \infty,\ 1 < q_1 < \infty;$$

and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad s = (1-\theta)s_0 + \theta s_1\,.$$

Then

$$\big[\,\overline{A}^{s_0}_{p_0q_0}(\Omega), \overline{A}^{s_1}_{p_1q_1}(\Omega)\big]_\theta = \overline{A}^s_{pq}(\Omega), \tag{45}$$

where either all A-spaces are B-spaces or all A-spaces are F-spaces (hence (45) must be read either with $A = B$ or with $A = F$).

(iv) *Let $0 < \theta < 1$,*

$$-\infty < s_0 < s_1 < \infty; \quad 1 < p < \infty;$$

$$1 < q < \infty, \quad 1 < q_0 < \infty,\ 1 < q_1 < \infty;$$

and let $s = (1-\theta)s_0 + \theta s_1$. *Then*

$$\left(\overline{A}^{s_0}_{pq_0}(\Omega), \overline{A}^{s_1}_{pq_1}(\Omega) \right)_{\theta,q} = \overline{B}^s_{pq}(\Omega), \tag{46}$$

where the two A*-spaces might be independently* B*-spaces or* F*-spaces.*

Proof. Step 1. First we prove that $D(\Omega)$ is dense in $\overline{A}^s_{pq}(\Omega)$. Let $s > \frac{1}{p} - 1$. By (42), (37), and the standard arguments indicated at the end of Step 3 of the proof of Proposition 3.1 it follows that $D(\Omega)$ is dense in $\overline{A}^s_{pq}(\Omega)$. Let $s < 0$. Then $L_p(\Omega)$ is dense in $A^s_{pq}(\Omega)$. Since $D(\Omega)$ is dense in $L_p(\Omega)$ it follows that $D(\Omega)$ is also dense in $\overline{A}^s_{pq}(\Omega) = A^s_{pq}(\Omega)$ if $s < 0$. This covers all cases.

Step 2. We prove the remaining assertions of (i). Recall

$$\left(A^s_{pq}(\mathbb{R}^n) \right)' = A^{-s}_{p'q'}(\mathbb{R}^n)\,; \quad s,\, p,\, q \text{ as in (40), (43).} \tag{47}$$

We refer to [37], 2.11.2, p. 178, always based on the interpretation of dual spaces on $\mathbb{R}^n$ within the dual pairing $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$. In particular, all these spaces are reflexive Banach spaces. Let $s > \frac{1}{p} - 1$. Then by (42), (37) the space $\overline{A}^s_{pq}(\Omega)$ can be identified with a closed subspace of $A^s_{pq}(\mathbb{R}^n)$ (equivalent norms). Hence by the Hahn-Banach and Eberlein-Shmulyan theorems, [45], p. 141, these spaces are also reflexive Banach spaces. By (27) and the arguments after all spaces $A^s_{pq}(\Omega)$ in question are isomorphic to complemented subspaces of $A^s_{pq}(\mathbb{R}^n)$. Hence by the above reasoning, they are also reflexive. This covers all cases. It remains to prove (43) as far as part (i) is concerned. First we observe

$$\left(\overline{A}^s_{pq}(\Omega) \right)' = A^{-s}_{p'q'}(\Omega), \quad s > \frac{1}{p} - 1\,, \tag{48}$$

where duality is always interpreted within the dual pairing $(D(\Omega), D'(\Omega))$. This follows from (47) and the definition of the spaces on the right-hand side of (48) as restriction of the spaces on the right-hand side of (47) on Ω. We refer for details to [37], 4.8.1. The remaining cases follow now by reflexity and density of $D(\Omega)$ in all spaces considered.

Step 3. We prove part (ii). Let *ext* be the extension operator according to Theorem 2.11. As usual, *re* stands for the restriction operator. Using the notation (41) we have

$$re \circ ext = id \quad \text{in} \quad \overline{A}^s_{pq}(\Omega) \quad \text{where} \quad s < \frac{1}{p} \,.$$

By (43) we obtain that

$$(ext)' \circ (re)' = id \quad \text{in} \quad \overline{A}^s_{pq}(\Omega) \quad \text{where} \quad s > \frac{1}{p} - 1 \,. \tag{49}$$

We claim

$$(re)' = Ext \, : \quad \widetilde{A}^s_{pq}(\Omega) \mapsto A^s_{pq}(\mathbb{R}^n),$$

where Ext is given by (44). This follows from (in obvious notation as dual pairings of suitable smooth functions)

$$(re\, f, g)_\Omega = \int\limits_\Omega (re\, f)(x)\, g(x)\, dx = \int\limits_{\mathbb{R}^n} f(x)\, (Ext\, g)(x)\, dx = (f, Ext\, g)_{\mathbb{R}^n} \,.$$

Now part (ii) is a consequence of (49).

Step 4. We prove parts (iii) and (iv). If both $s_0 < \frac{1}{p_0}$ and $s_1 < \frac{1}{p_1}$ then all interpolation formulas are covered by Theorem 2.13. If both $s_0 > \frac{1}{p_0} - 1$ and $s_1 > \frac{1}{p_1} - 1$ then we have by (ii) the common extension operator Ext. By the same arguments as in connection with the proof of Theorem 2.13 any interpolation formula in $\mathbb{R}^n$ can be carried over to Ω. In particular we have (45), (46) in these two separate s_0 - s_1-regions. Fortunately there is an overlap of these two regions in the strip

$$\left\{ \left(\frac{1}{p}, \sigma \right) : \quad 1 < p < \infty, \; \frac{1}{p} - 1 < \sigma < \frac{1}{p} \right\}.$$

Then one can apply Wolff's interpolation theorems, [44]: Let, for example, A_1, A_2, A_3 and A_4 be 4 of the above Banach spaces (recall that $D(\Omega)$ is dense in all these spaces) with

$$A_1 \subset A_2 \subset A_3 \subset A_4, \quad A_2 = (A_1, A_3)_{\theta_1, q_1} \,, \quad A_3 = (A_2, A_4)_{\theta_2, q_2} \,.$$

Then

$$A_2 = (A_1, A_4)_{\theta_3, q_3} \quad \text{and} \quad A_3 = (A_1, A_4)_{\theta_4, q_4} \tag{50}$$

with naturally calculated θ_3, θ_4, q_3 and q_4 according to the reiteration theorem of interpolation theory. This applies to (46) with respect to the two overlapping s_0 - s_1-regions and results finally in a full proof of part (iv). There is a corresponding assertion in [44] with respect to the classical complex interpolation method without the assumed monotonicity of the spaces A_j involved in (50). A slightly more general version of Wolff's theorem can be found in [13], 1.3.

Remark 3.6. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. For fixed $1 < p < \infty$, $1 < q < \infty$, the scale

$$\left\{\overline{A}^s_{pq}(\Omega) : \quad s \in \mathbb{R}\right\}$$

is the correct substitute of the corresponding scale

$$\left\{A^s_{pq}(\mathbb{R}^n) : \quad s \in \mathbb{R}\right\}$$

in $\mathbb{R}^n$. This follows from our detailed studies in [37], Ch. 4. By the above theorem it comes out that at least some assertions remain valid for bounded Lipschitz domains: extension, interpolation, duality.

4 Lipschitz diffeomorphisms and atomic representations

4.1 Preliminaries

Let

$$y = \Phi(x) : \quad \mathbb{R}^n \mapsto \mathbb{R}^n, \tag{51}$$

be a C^∞ diffeomorphism of $\mathbb{R}^n$ onto itself. Then

$$f \mapsto f \circ \Phi \tag{52}$$

is an isomorphic map of any space $A^s_{pq}(\mathbb{R}^n)$ with $A = B$ or $A = F$ according to Definition 2.1 onto itself. Assertions of this type have a substantial history. After a long battle a complete proof of this assertion was given in [39], 4.3.2, p. 209. There, and also in its forerunner [38], 2.10, one finds the history of this subject and related references. It is the main aim of this section to discuss the question for which spaces $A^s_{pq}(\mathbb{R}^n)$ the mapping (52) makes sense and remains to be an isomorphism of $A^s_{pq}(\mathbb{R}^n)$ onto itself if (51) is only a Lipschitz diffeomorphism. Assuming that this is the case for *some* space $A^s_{pq}(\mathbb{R}^n)$ then one is in the same comfortable position for the respective space $A^s_{pq}(\Omega)$ in bounded Lipschitz domains as for *all* spaces $A^s_{pq}(\Omega)$ if Ω is a bounded C^∞ domain: By localization as described in Remark 2.10 and local diffeomorphisms one can reduce many problems for spaces in Ω via local charts to corresponding related problems (locally) in $\mathbb{R}^n_+$. It is just the lack of this possibility if one switches from bounded C^∞ domains to bounded Lipschitz domains which causes additional difficulties and which are the subject of the two preceding sections. In Sections 5 and 6 we give two applications of the main results of this section: Characteristic functions of bounded Lipschitz domains as pointwise multipliers in some function spaces (resulting in an 1-line-proof of a known result) and, maybe more significantly, function spaces on Lipschitz manifolds.

In this section we rely on atomic decompositions of some function spaces $A^s_{pq}(\mathbb{R}^n)$ in terms of Lipschitz atoms. On the one hand we restrict ourselves to the description of the bare minimum needed for our arguments. On the other hand we developed in [40] and more systematically in [41], the theory of quarkonial (or subatomic) decompositions of function spaces. Although not needed for our main applications we outline how to develop a corresponding theory for function spaces in bounded Lipschitz domains.

4.2 Lipschitz atomic representations

Let $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then Q_{jm} denotes a closed cube in $\mathbb{R}^n$ with sides parallel to the axes centred at $2^{-j}m$ and with side length 2^{-j}. Let Q be a cube in $\mathbb{R}^n$ and $r > 0$; then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length r times that of Q. As usual we introduce the numbers

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ \tag{53}$$

where $0 < p \le \infty$, $0 < q \le \infty$.

We are interested in atomic representations

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\, a_{jm} \tag{54}$$

in the spaces $A^s_{pq}(\mathbb{R}^n)$ where

$$0 < p \leq \infty, \quad 0 < q \leq \infty, \quad \sigma_p < s < 1 \quad \text{if} \quad A = B \tag{55}$$

and

$$0 < p < \infty, \quad 0 < q \leq \infty, \quad \sigma_{pq} < s < 1 \quad \text{if} \quad A = F. \tag{56}$$

Here

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : \quad j \in \mathbb{N}_0, \; m \in \mathbb{Z}^n\} \tag{57}$$

are sequences of complex numbers belonging to the sequence spaces b_{pq} and f_{pq}, respectively which are defined as follows. Let

$$\chi^{(p)}_{jm}(x) = 2^{\frac{jn}{p}} \text{ if } x \in Q_{jm} \quad \text{and} \quad \chi^{(p)}_{jm}(x) = 0 \text{ if } x \in \mathbb{R}^n \backslash Q_{jm} \tag{58}$$

be the p-normalized characteristic function of Q_{jm}. Then $\lambda \in f_{pq}$ and $\lambda \in b_{pq}$ if

$$\|\lambda \,|f_{pq}\| = \left\| \left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \left| \lambda_{jm} \, \chi^{(p)}_{jm}(\cdot) \right|^q \right)^{\frac{1}{q}} |L_p(\mathbb{R}^n) \right\| < \infty \tag{59}$$

and

$$\|\lambda \,|b_{pq}\| = \left(\sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty, \tag{60}$$

respectively (with the usual modification if $p = \infty$ and/or $q = \infty$). Furthermore when $r > 1$ then $a_{jm}(x)$ are atoms with

$$supp\, a_{jm} \subset rQ_{jm}, \quad j \in \mathbb{N}_0 \quad \text{and} \quad m \in \mathbb{Z}^n, \tag{61}$$

and

$$|a_{jm}(x)| \leq 2^{-j(s-\frac{n}{p})}, \quad |a_{jm}(x) - a_{jm}(y)| \leq 2^{-j(s-\frac{n}{p}-1)}|x-y|. \tag{62}$$

In other words, a_{jm} are normalized (s,p)-Lip atoms. It comes out that $f \in S'(\mathbb{R}^n)$ belongs to $A^s_{pq}(\mathbb{R}^n)$ (again $A = B$ or $A = F$ with the restrictions indicated above) if, and only if, it can be represented by (54) with (57), (59), (60), and (61), (62). Furthermore,

$$\|f \,|A^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|\mathfrak{a}_{pq}\|, \tag{63}$$

where temporarily $\mathfrak{a}_{pq} = b_{pq}$ if $A = B$ and $\mathfrak{a}_{pq} = f_{pq}$ if $A = F$ (not to be mixed with the atoms a_{jm}) and where the infimum is taken over all admitted representations. If the atoms are smooth then this well-known atomic representation (no moment conditions are required) may be found in [40], Sect. 13, especially Theorem 13.8 on p. 75. There are references to the literature. The above non-smooth version is covered by [42] and [7], Theorem 2.2.3 on pp. 31/32.

4.3 Lipschitz diffeomorphisms

The one-to-one mapping (51) of $\mathbb{R}^n$ onto itself is called a Lipschitz diffeomorphism if the components $\Phi_k(x)$ of $\Phi(x) = (\Phi_1(x), ..., \Phi_n(x))$ are Lipschitz functions on $\mathbb{R}^n$ and

$$|\Phi(x) - \Phi(y)| \sim |x - y|, \quad x \in \mathbb{R}^n,\, y \in \mathbb{R}^n; \quad |x - y| \le 1,$$

where the equivalence constants are independent of x and y. Of course the inverse $\Phi^{-1}(x)$ of $\Phi(x)$ is also a Lipschitz diffeomorphism on $\mathbb{R}^n$.

Proposition 4.1. *Let Φ be a Lipschitz diffeomorphism in $\mathbb{R}^n$. Let $A^s_{pq}(\mathbb{R}^n)$ be the above spaces with (55) and (56), respectively. Then $f \mapsto f \circ \Phi$ is an isomorphic map of $A^s_{pq}(\mathbb{R}^n)$ onto itself.*

Proof. Let f be given by (54), where a_{jm} are (s,p)-Lip atoms. Then we have

$$f \circ \Phi = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\, a_{jm} \circ \Phi \tag{64}$$

with

$$\begin{aligned} &|(a_{jm} \circ \Phi)(x) - (a_{jm} \circ \Phi)(y)| \\ &\le 2^{-j(s-\frac{n}{p}-1)}\, |\Phi(x) - \Phi(y)| \le c\, 2^{-j(s-\frac{n}{p}-1)}\, |x - y| \end{aligned}$$

and an adequate substitute of the localization requirement (61). Hence, ignoring unimportant constants, $a_{jm} \circ \Phi$ are again (s,p)-Lip atoms. In particular, (64) is an atomic decomposition and

$$\|f \circ \Phi \,|A^s_{pq}(\mathbb{R}^n)\| \le c\, \|f \,|A^s_{pq}(\mathbb{R}^n)\|.$$

As mentioned, Φ^{-1} is also a Lipschitz diffeomorphism. This proves the converse.

Remark 4.2. It is the question of whether the abvoe proposition can be extended to other spaces $A^s_{pq}(\mathbb{R}^n)$. If $s < 0$, $1 < p < \infty$, $1 < q < \infty$, then one might think to get the desired assertion by duality, starting from $A^{-s}_{p'q'}(\mathbb{R}^n)$. But it is not so clear whether the resulting Radon-Nikodym derivative is a pointwise multiplier in the respective spaces. Fortunately enough in order to establish the method of local charts for bounded Lipschitz domains as described in 4.1 it is sufficient to deal with the special Lipschitz diffeomorphisms (51) given by

$$\Phi : \quad y_j = x_j \text{ if } j = 1, ..., n-1 \quad \text{and} \quad y_n = x_n - h(x'), \tag{65}$$

where $h(x')$ is a Lipschitz function on $\mathbb{R}^{n-1}$ according to (12), (13). Then the Radon-Nikodym derivative (distortion factor) is 1. Let

$$\mathbf{R} = \left\{ \left(\frac{1}{p}, s\right) : \quad 0 < p \le \infty, \quad -1 + 2n\,(\frac{1}{p} - 1)_+ < s < 1 \right\}$$

as indicated in Fig. 1.

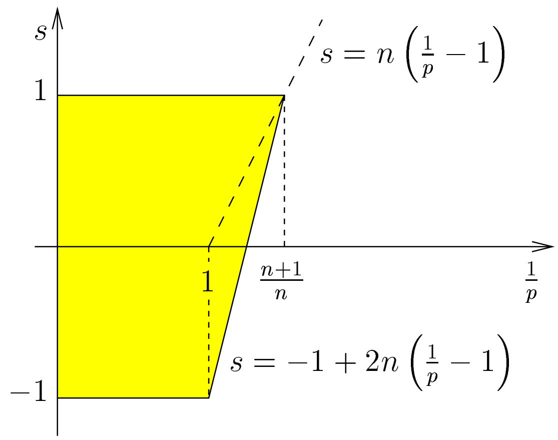

Fig. 1

Theorem 4.3. *Let $n \in \mathbb{N}$ with $n \ge 2$. Let $h(x')$ be a Lipschitz function in $\mathbb{R}^{n-1}$ according to (12), (13). Let*

$$\left(\frac{1}{p}, s\right) \in \mathbf{R} \quad \text{and} \quad 0 < q \le \infty \tag{66}$$

(with $p < \infty$ and $q \ge \min(1, p)$ in case of the F-spaces). Let Φ be the Lipschitz diffeomorphism given by (65). Then $f \mapsto f \circ \Phi$ is an isomorphic map of $A^s_{pq}(\mathbb{R}^n)$ onto itself.

Proof. First, by Proposition 4.1 it follows that $f \mapsto f \circ \Phi$ is an isomorphic mapping from $A^s_{pq}(\mathbb{R}^n)$ onto itself if

$$0 < p \leq \infty, \quad 0 < q \leq \infty, \quad \sigma_p < s < 1,$$

(with $p < \infty$ and $q \geq \min(p, 1)$ if $A = F$). Secondly, the duality formula (47) with (40) can be extended to $q = 1$ both for $A = B$ and $A = F$ and it can also be extended to $p = 1$ for $A = B$ (again with $q = 1$ as an admitted choice). We refer to [38], 2.11.2, p. 178, and [29], p. 20, and the references given there as far as the included limiting cases are concerned. Obviously, the inverse of Φ in (65) is of the same type (with h in place of $-h$) and the Radon-Nikodym derivative (volume distortion) of $y = \Phi(x)$ is 1. Hence by Proposition 4.1 and duality one gets the desired assertion for the spaces $A^s_{pq}(\mathbb{R}^n)$ with

$$1 < p \leq \infty \quad (p < \infty \text{ if } A = F), \quad 1 < q \leq \infty, \quad -1 < s < 0.$$

The rest is a matter of complex interpolation as described in 2.5 and real interpolation.

Remark 4.4. In case of $A = B$ the restriction (66) looks natural. In case of $A = F$ the restriction $q \geq \min(p, 1)$ is disturbing although both Proposition 4.1 and also the above complex interpolation cover in addition a few cases with $q < \min(p, 1)$. The above theorem applies in particular to some (inhomogeneous) Hardy spaces

$$h_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n), \quad \frac{2n}{2n+1} < p \leq 1, \quad n \geq 2.$$

4.4 Quarkonial decompositions of functions in Lipschitz domains

Let $\mathbb{N}^n_0$, where $n \in \mathbb{N}$, be the set of all multi-indices

$$\alpha = (\alpha_1, ..., \alpha_n) \text{ with } \alpha_j \in \mathbb{N}_0 \quad \text{and} \quad |\alpha| = \sum_{j=1}^n \alpha_j \,;$$

and let

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \quad \text{where} \quad x = (x_1, ..., x_n) \in \mathbb{R}^n \quad \text{and} \quad \beta \in \mathbb{N}^n_0 \,.$$

Let $\psi \in D(\mathbb{R}^n)$ be a non-negative function with

$$\sum_{m \in \mathbb{Z}^n} \psi(x-m) = 1 \quad \text{if} \quad x \in \mathbb{R}^n .$$

Let $s \in \mathbb{R}$, $0 < p \le \infty$, $\beta \in \mathbb{N}_0^n$ and $\psi^\beta(x) = x^\beta \, \psi(x)$. Then

$$(\beta qu)_{jm}(x) = 2^{-j(s-\frac{n}{p})} \, \psi^\beta(2^j x - m), \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n ,$$

is called an (s,p)-β-quark related to the cube Q_{jm} as introduced at the beginning of 4.2. We developed in [41], Sect. 2, the theory of quarkonial (subatomic) decompositions of the spaces $A^s_{pq}(\mathbb{R}^n)$ under the restrictions (55), (56), respectively (regular case) and extended this theory in [41], Sect. 3, to all spaces $A^s_{pq}(\mathbb{R}^n)$ (general case). In the regular case it came out that $f \in S'(\mathbb{R}^n)$ is an element of $A^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{\beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \lambda^\beta_{jm} \, (\beta qu)_{jm} \tag{67}$$

(unconditional convergence in $S'(\mathbb{R}^n)$ or, likewise, absolute and, hence, unconditional convergence in $L_1^{loc}(\mathbb{R}^n)$) where $(\beta qu)_{jm}(x)$ are the above (s,p)-β-quarks, and

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta|} \, \|\lambda^\beta \, |\mathfrak{a}_{pq}\| < \infty, \tag{68}$$

for some $\varrho > 0$ where we use $\mathfrak{a}_{pq}$ as in (63) and λ^β is given by (57) with λ^β, λ^β_{jm} in place of λ, λ_{jm}, respectively. Furthermore,

$$\|f \, |A^s_{pq}(\mathbb{R}^n)\| \sim \inf \left[\sup_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta|} \, \|\lambda^\beta \, |\mathfrak{a}_{pq}\| \right], \tag{69}$$

where the infimum is taken over all admissible representations (67), (68). Furthermore there are optimal coefficients

$$\lambda^\beta_{jm}(f) = \int_{\mathbb{R}^n} f(x) \, \Lambda^\beta_{jm}(x) \, dx, \quad \Lambda^\beta_{jm}(x) \in S(\mathbb{R}^n), \tag{70}$$

which depend linearly on f and

$$\|f \, |A^s_{pq}(\mathbb{R}^n)\| \sim \sup_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta|} \, \|\lambda^\beta(f) \, |\mathfrak{a}_{pq}\| \tag{71}$$

(equivalent quasi-norm). The functions Λ^{β}_{jm} can be constructed explicitly. There is a complete counterpart for the general case, covering all spaces $A^s_{pq}(\mathbb{R}^n)$. We refer for details and additional explanations to [41], Sections 2 and 3.

Combining these results with Theorem 2.11 one gets for bounded Lipschitz domains Ω constructive quarkonial characterizations for all spaces $A^s_{pq}(\Omega)$. We give a brief description restricting ourselves again to the regular case. Let χ_Ω be the characteristic function of Ω. Let $\sum_\Omega$ be the sum over those $\beta \in \mathbb{N}^n_0$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ in (67) with

$$\overline{\Omega} \cap supp\, \psi(2^j \cdot -m) \neq \emptyset.$$

Similarly we write $\mathfrak{a}^{\Omega}_{pq}$ when the summation in the sequence spaces $\mathfrak{a}_{pq}$ is restricted in the same way. Let

$$(\beta qu)^{\Omega}_{jm}(x) = (\beta qu)_{jm}(x) \cdot \chi_\Omega(x)$$

Theorem 4.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Let $A^s_{pq}(\Omega)$ be the spaces according to Definition 2.3(i) restricted by (55) and (56), respectively. Then $A^s_{pq}(\Omega)$ is the collection of all $f \in L_1(\Omega)$ which can be represented as*

$$f = \sum_{\Omega} \lambda^{\beta}_{jm}\, (\beta qu)^{\Omega}_{jm} \tag{72}$$

(absolute convergence in $L_1(\Omega)$) such that

$$\sup_{\beta \in \mathbb{N}^n_0} 2^{\varrho|\beta|}\, \|\lambda^\beta \,|\mathfrak{a}^{\Omega}_{pq}\| < \infty, \tag{73}$$

where $\varrho > 0$ and λ^β have the above meaning. Furthermore,

$$\|f \,|A^s_{pq}(\Omega)\| \sim \inf \left[\sup_{\beta \in \mathbb{N}^n_0} 2^{\varrho|\beta|}\, \|\lambda^\beta \,|\mathfrak{a}^{\Omega}_{pq}\| \right]$$

(equivalent quasi-norms) where the infimum is taken over all admissible representations (72), (73). Furthermore there are optimal coefficients $\lambda^{\beta}_{jm}(f)$ which depend linearly on f such that

$$\|f \,|A^s_{pq}(\Omega)\| \sim \sup_{\beta \in \mathbb{N}^n_0} 2^{\varrho|\beta|} \|\lambda^\beta(f) \,|\mathfrak{a}^{\Omega}_{pq}\|.$$

Proof. Let *ext* be the extension operator according to Theorem 2.11. One can expand $ext\, f$ by (67) - (71). Restriction to Ω gives the desired result.

Remark 4.6. In [41], Ch. 1, we dealt with several types of quarkonial decompositions of function spaces, also in domains. We managed to overlook the above simple application of corresponding expansions in $\mathbb{R}^n$, even in case of bounded C^∞ domains. If (55) or (56) is not satisfied, then $(\beta qu)_{jm}$ and $(\beta qu)^{\Omega}_{jm}$ must be complemented in $\mathbb{R}^n$ and in Ω, respectively, by

$$\left(\Delta^N \psi^\beta\right)(2^j x - m) \quad \text{and} \quad \left(\Delta^N \psi^\beta\right)(2^j x - m) \cdot \chi_\Omega(x), \tag{74}$$

with some $N \in \mathbb{N}$ (in dependence of s, p, q) and suitably normalized. This is the simplest way to incorporate moment conditions. If the support of $\psi(2^j x - m)$ intersects $\partial\Omega$ then the second building blocks in (74) do not satisfy moment conditions. Nevertheless they remain to be the correct building blocks. This effect on the level of so-called boundary atoms is known: We refer to [42] and [7], 2.5, pp. 57-65.

5 Characteristic functions as pointwise multipliers

5.1 Pointwise multipliers

Let $A^s_{pq}(\mathbb{R}^n)$ with $A = B$ or $A = F$ be the spaces introduced in Definition 2.1. A function $m \in L_\infty(\mathbb{R}^n)$ is called a pointwise multiplier for $A^s_{pq}(\mathbb{R}^n)$ if

$$f \mapsto m\,f \quad \text{generates a bounded map in } A^s_{pq}(\mathbb{R}^n).$$

One must say what this means. But for this technical side we refer to [29], 4.2. Pointwise multipliers in general and characteristic functions of domains as pointwise multipliers in particular attracted a lot of attention. As far as classical Besov spaces and (fractional) Sobolev spaces are concerned we refer to [22] and [23]. Pointwise multipliers in general spaces $A^s_{pq}(\mathbb{R}^n)$ have been studied in [29] (where one finds also many references) and in the most recent and most advanced papers [33], [34]. As for our own contributions we refer to [38], 2.8, and [39], 4.2.

5.2 Characteristic functions of Lipschitz domains

Let χ_Ω be the characteristic function of an arbitrary domain in $\mathbb{R}^n$. We write χ_+ if Ω is the half-space $\mathbb{R}^n_+$ according to (2). Recall the following definitive result.

Proposition 5.1. *Let s, p, q be as in (1). Then χ_+ is a pointwise multiplier for $A^s_{pq}(\mathbb{R}^n)$ if, and only if,*

$$\max\left(n(\frac{1}{p}-1), \frac{1}{p}-1\right) < s < \frac{1}{p}, \tag{75}$$

Fig. 2

Remark 5.2. A complete proof may be found in [29], 4.6.3, p. 208. First results of this type had been obtained in [38], 2.8.7, p. 158. This covers the case $A = B$ (with exception of some limiting cases). If $A = F$ then there are some curious restrictions for q.

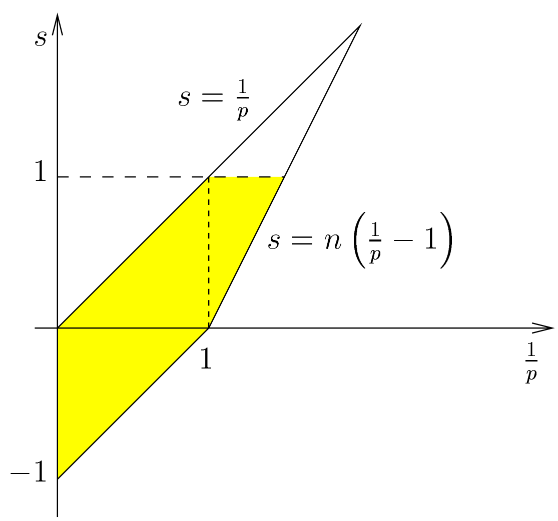

Fig. 2

The proof of the independence of the pointwise multiplier assertion for χ_+ on q in case of $A = F$ and hence the first proof of Proposition 5.1 is due to J. Franke, [8]. Further references may be found in [38], 2.8.5, Remark 4, p. 154, and in [29], p. 258.

Proposition 5.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let p, q, s be restricted as in Theorem 4.3. Then χ_Ω is a pointwise multiplier for $A^s_{pq}(\mathbb{R}^n)$ if, and only if, χ_+ is a pointwise multiplier for $A^s_{pq}(\mathbb{R}^n)$.*

Proof. By Theorem 4.3 and Remark 2.10 one can apply the method of local charts which results in the above assertion.

Remark 5.4. This is the first application of Theorem 4.3 we have in mind. By Proposition 5.1 it follows that χ_Ω is a pointwise multiplier for $A^s_{pq}(\mathbb{R}^n)$ if $0 < p \le \infty$,

$$\max\left(n(\frac{1}{p}-1), \frac{1}{p}-1\right) < s < \min(\frac{1}{p}, 1) \quad \text{and} \quad 0 < q \le \infty$$

(with $p < \infty$ and $q \ge \min(1, p)$ if $A = F$), Fig. 2. By Proposition 4.1 and Remark 4.4 one can extend this proof to a few other cases but the outcome is not satisfactory. The above results including the extension just indicated are known. One can ask the following question:

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Is χ_Ω a pointwise multiplier for $A^s_{pq}(\mathbb{R}^n)$ if, and only if, χ_+ is a pointwise multiplier for $A^s_{pq}(\mathbb{R}^n)$?

The best result known so far may be found in [9], Corollary 13.6. It is the direct extension of [38], 2.8.7, p. 158, from χ_+ to χ_Ω. In particular it covers the above Proposition 5.3 and it gives an affirmative answer of the above question if $A = B$. However in case of $A = F$ there remains a dependence on q and the above question seems to be open for F-spaces.

5.3 Characteristic functions of arbitrary domains

Pointwise multiplier properties of χ_Ω with respect to arbitrary domains Ω in $\mathbb{R}^n$ for $F^s_{pq}(\mathbb{R}^n)$ have been considered in [29], Theorem 3 on p. 216, [33], Theorem 1 on p. 227, Corollary 1 on p. 230, Theorem 6 on p. 240, Remark 22 on p. 241 and in [34]. There are sufficient conditions which ensure that χ_Ω is a pointwise multiplier for $F^s_{pq}(\mathbb{R}^n)$ depending on q. In these papers one finds also additional references describing the state of art. It is not our aim to deal with problems of this type here in detail. There is only one point which we wish to discuss briefly and which applies to bounded Lipschitz domains but also to more general domains:

The interrelation of geometric properties of $\partial\Omega$ (expressed in terms of the so-called ball condition) and the q-dependence of the pointwise multiplier property of χ_Ω for $F^s_{pq}(\mathbb{R}^n)$ (if there is any).

Definition 5.5. *A non-empty Borel set Γ in $\mathbb{R}^n$ is said to satisfy the ball condition if there is a number $0 < \eta < 1$ with the following property: For any ball $B(x,t)$ centred at $x \in \mathbb{R}^n$ and of radius $0 < t < 1$ there is a ball $B(y, \eta t)$ (centred at $y \in \mathbb{R}^n$ and of radius ηt) with*

$$B(y,\eta t) \subset B(x,t) \quad \textit{and} \quad B(y,\eta t) \cap \overline{\Gamma} = \emptyset.$$

Remark 5.6. This definition coincides with [41], 9.16, p. 138, and, in turn, with [40], 18.10, p. 142. But this notation appears also in several other contexts, for example, fractal geometry and physics (porous media). It plays also a remarkable role in connection with traces of function spaces $A^s_{pq}(\mathbb{R}^n)$ on compact sets Γ with Lebesgue measure $|\Gamma| = 0$. We refer to [41], 9.16-9.22, pp. 138-143. In particular, by [41], Theorem 9.21 on pp. 141-142, traces of spaces $F^s_{pq}(\mathbb{R}^n)$ (with $1 < p < \infty$) on compact sets Γ satisfying the ball condition, if exist, are independent of q with $0 < q \leq \infty$. The independence of traces of spaces $F^s_{pq}(\mathbb{R}^n)$ on sets satisfying the ball condition has also been observed by Netrusov in [25], assertion 1.4, p. 193. This suggests to discuss the following situation: Let Ω be a, say, bounded domain in $\mathbb{R}^n$ such that $\Gamma = \partial\Omega$ satisfies the ball condition. Let p, q, s be restricted as in Proposition 3.1(i). Is there an alternative such that *either* the trace of $F^s_{pq}(\mathbb{R}^n)$ on Γ exists (and is independent of q) *or* χ_Ω is a pointwise multiplier for $F^s_{pq}(\mathbb{R}^n)$ (independent of q)? It might well be the case that some limiting spaces in this context must be discussed separately and (as a consequence) excluded from this alternative. Let, for example, $\Gamma = \partial\Omega$ be a d-set. We refer for definition and generalization to [41], 9.1, p. 120. We have $d \geq n-1$ with $d = n-1$ in case of bounded Lipschitz domains. If $n - 1 \leq d < n$, then $\Gamma = \partial\Omega$ satisfies the ball condition, [41], Remark 9.19, pp. 140-141, and $F^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ are limiting spaces in the above context, [40], Corollary 18.12, p. 142. Concerning the independence of q of pointwise multipliers χ_Ω for $F^s_{pq}(\mathbb{R}^n)$ (if exists) we have no definitive results comparable with correponding assertions for traces as indicated above. But we wish to complement the existing literature by a few related results.

Proposition 5.7. *Let Ω be a domain in $\mathbb{R}^n$ such that its boundary $\Gamma = \partial\Omega$ satisfies the ball condition. Let p and s be restricted by (75), Fig. 2.*

(i) *Let χ_Ω be a pointwise multiplier for $F^s_{pq_0}(\mathbb{R}^n)$ for some $0 < q_0 \le \infty$. Then χ_Ω is a pointwise multiplier for all spaces $F^s_{pq}(\mathbb{R}^n)$ with $q_0 \le q \le \infty$.*

(ii) *Let, in addition, $1 < p < \infty$ and let χ_Ω be a pointwise multiplier for $F^s_{pq_0}(\mathbb{R}^n)$ for some $1 \le q_0 < \infty$. Then χ_Ω is a pointwise multiplier for all spaces $F^s_{pq}(\mathbb{R}^n)$ with $1 \le q \le \infty$,*

(iii) *Let, in addition, $1 < p < \infty$, $s = 0$ and $1 \le q \le \infty$. Then χ_Ω is a pointwise multiplier for all spaces $F^0_{pq}(\mathbb{R}^n)$.*

Proof. Step 1. By Proposition 5.1 characteristic functions of cubes Q are pointwise multipliers for $F^s_{pq}(\mathbb{R}^n)$, where the corresponding multiplier constants are independent of Q. Then, by the localization principle for F-spaces, [39], 2.4.7, p. 124, we may assume, in addition, that Ω is bounded. We prove part (i). Let $f \in F^s_{pq}(\mathbb{R}^n)$. We use the same quarkonial decompositions as in Step 2 of the proof in [41], Theorem 9.21, p. 142, without further explanations,

$$f = \sum_{\beta,j,m} \left(\eta^\beta_{jm} \, (\beta qu)_{jm} + \lambda^\beta_{jm} \, (\beta qu)^L_{jm} \right) = f_1 + f_2, \tag{76}$$

where f_1 collects all those (j,m) with

$$dQ_{jm} \cap \Gamma \neq \emptyset$$

for a suitable $d > 0$, and f_2 the remaining terms. Then it follows $\chi_\Omega f_2 \in F^s_{pq}(\mathbb{R}^n)$ and

$$\|\chi_\Omega f_2 \,|F^s_{pq}(\mathbb{R}^n)\| \le c \, \|f \,|F^s_{pq}(\mathbb{R}^n)\|. \tag{77}$$

As for f_1 we have

$$\begin{aligned} \|\chi_\Omega f_1 \,|F^s_{pq}(\mathbb{R}^n)\| &\le c_1 \, \|\chi_\Omega f_1 \,|F^s_{pq_0}(\mathbb{R}^n)\| \le c_2 \, \|f_1 \,|F^s_{pq_0}(\mathbb{R}^n)\| \\ &\le c_3 \, \|f_1 \,|F^s_{pq}(\mathbb{R}^n)\| \le c_4 \, \|f \,|F^s_{pq}(\mathbb{R}^n)\| \,. \end{aligned} \tag{78}$$

The first inequality comes from the monotonicity of the quasi-norms involved (recall $q \ge q_0$). The second inequality is the assumption that χ_Ω is a pointwise multiplier for $F^s_{pq_0}(\mathbb{R}^n)$. The main observation is the third inequality. Here we use that Γ satisfies the ball condition. We refer for details to [41], especially formula (9.99) on p. 143. The final estimate comes from the quarkonial decomposition using that the above coefficients η^β_{jm} and λ^β_{jm} in (76) are universal. Now part (i) follows from (77), (78).

Step 2. We prove part (ii). By part (i), χ_Ω is a pointwise multiplier for all spaces $F^s_{pq}(\mathbb{R}^n)$ with $q_0 < q \le \infty$. We use the duality (47), (40) with $A = F$. Then χ_Ω is also a pointwise multiplier for all spaces

$$F^{-s}_{p'q'}(\mathbb{R}^n) \quad \text{with} \quad 1 < q' < q_0'\,.$$

Application of part (i) shows that χ_Ω is a pointwise multiplier for all spaces

$$F^{-s}_{p',u}(\mathbb{R}^n) \quad \text{with} \quad 1 < u < \infty.$$

A second application of duality proves part (ii) with $q > 1$. It remains the case $q = 1 < q_0$. Let $\overset{\circ}{F}{}^s_{p\infty}(\mathbb{R}^n)$ be the completion of $S(\mathbb{R}^n)$ in $F^s_{p\infty}(\mathbb{R}^n)$. Then the duality (47) can be complemented by

$$\left(\overset{\circ}{F}{}^s_{p,\infty}(\mathbb{R}^n)\right)' = F^{-s}_{p',1}(\mathbb{R}^n), \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad s \in \mathbb{R},$$

[29], p. 20, with a reference to [21]. Furthermore by the above quarkonial arguments χ_Ω is also a pointwise multiplier for $\overset{\circ}{F}{}^s_{p,\infty}(\mathbb{R}^n)$. Then one can argue as before. In particular, χ_Ω is also a pointwise multiplier for $F^s_{p,1}(\mathbb{R}^n)$.

Step 3. Part (iii) follows from part (i), the Littlewood-Paley assertion

$$F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n), \quad 1 < p < \infty,$$

and the obvious observation that any characteristic function χ_Ω is a pointwise multiplier in $L_p(\mathbb{R}^n)$.

Remark 5.8. The a-priori restriction that p and s satisfy (75) is natural but not necessary. We used it at the beginning of Step 1 since (76) is applied in [41] to compact Γ. But it can be extended to closed Γ. Furthermore we remark that if

$$s > 0, \quad 1 \le p < \infty, \quad 1 \le q_0 \le q_1 \le \infty,$$

then any pointwise multiplier m for $F^s_{pq_0}(\mathbb{R}^n)$ is also a pointwise multiplier for $F^s_{pq_1}(\mathbb{R}^n)$. We refer to [34], Lemma 2.14, p. 300. Hence, part (i) complements this result, restricted to $m = \chi_\Omega$. By Proposition 5.1 the restriction $p > 1$ in part (iii) is natural. Pointwise multipliers for Besov spaces $B^0_{pq}(\mathbb{R}^n)$ with $1 \le p \le \infty$ and $1 \le q \le \infty$ have been treated recently in [19], concentrating preferably on the limiting cases $B^0_{\infty,1}(\mathbb{R}^n)$ and $B^0_{\infty,\infty}(\mathbb{R}^n)$.

6 Spaces on Lipschitz manifolds

6.1 Preliminaries

Function spaces on other structures than euclidean n-spaces and domains in euclidean n-spaces have been studied since a long time with great intensity based on a large variety of different methods. We mention a few of them. In [14] Jonsson and Wallin introduced function spaces of Besov type and potential type (fractional Sobolev spaces) on some sets in $\mathbb{R}^n$ using differences of functions and approximation techniques. Our own approach in [40] and in [41], especially Sect. 9, to spaces of type $B^s_{pq}(M)$ on compact d-sets M in $\mathbb{R}^n$ and on more general sets M in $\mathbb{R}^n$ is characterized by traces of suitable spaces of type B^s_{pq} and F^s_{pq} on $\mathbb{R}^n$ and intrinsic descriptions in terms of quarkonial representations. Next we mention the substantial theory of function spaces of type $F^s_{pq}(M)$ and $B^s_{pq}(M)$ on so-called spaces of homogeneous type (M, ϱ, μ) consisting of an abstract connected set M furnished with a quasi-metric ϱ and a suitable measure μ. It is based on so-called Calderón reproducing formulas and results in a remarkable analysis on these spaces of homogeneous type. We refer to the two surveys [11] and [12]. There, and also in the more recent paper [43] one finds further references and applications. The Lipschitz manifolds M we have in mind are special spaces of homogeneous type (X, ϱ, μ). But our approach is rather different. In particular we do not rely on the techniques developed in [11], [12], and the related papers. We use the method of local charts and follow closely the approach presented in [39], Ch. 7, where we studied spaces $F^s_{pq}(M)$ and $B^s_{pq}(M)$ on Lie groups and on C^∞-Riemannian manifolds with bounded geometry and positive injectivity radius. As a consequence, the spaces $F^s_{pq}(M)$ have priority. They are defined directly by the method of local charts. The spaces $B^s_{pq}(M)$ are introduced afterwards via real interpolation. We are doing here the same but now based on the Lipschitz diffeomorphisms according to Proposition 4.1.

6.2 Definitions

Let (M, ϱ, μ) be a connected set, equipped with a metric ϱ, generating a topology, and a Borel measure μ. A ball centred at $y \in M$ and of radius $t > 0$ is denoted by $B(y,t)$. Let J be either $J = \{1, ..., j_0\}$ for some $j_0 \in \mathbb{N}$ (compact case) or $J = \mathbb{N}$ (non-compact case). We assume that there are an $n \in \mathbb{N}$, two positive numbers c and Λ, grids $\{y^j\}_{j\in J} \subset M$ and bi-Lipschitzian maps

$$\Phi_j \ : \quad B(y^j, 2\Lambda) \mapsto \mathbb{R}^n \quad \text{where} \quad j \in J, \tag{79}$$

such that

$$\varrho(y^j, y^k) \geq c \quad \text{if } j \neq k, \quad j \in J, \quad k \in J \ ; \tag{80}$$

$$\bigcup_{j\in J} B(y^j, \Lambda) = M; \tag{81}$$

$$\Phi_j \left(B(y^j, 2\Lambda) \right) = W_j \quad \text{bounded neighbourhood of } 0 \in \mathbb{R}^n \ ; \tag{82}$$

$$\left| \Phi_j(z^1) - \Phi_j(z^2) \right| \sim \varrho(z^1, z^2); \quad z^1,\, z^2 \in B(y^j, 2\Lambda); \tag{83}$$

with equivalence constants in (83) which are independent of z^1, z^2 and j. We call

$$\left\{ B\left(y^j, 2\Lambda\right), \Phi_j \right\}_{j\in J} \tag{84}$$

an atlas. Obviously any notation introduced on (M, ϱ, μ) furnished with an atlas according to (84) must be invariant with respect to all admissible atlases of the described type.

Definition 6.1. *Let $n \in \mathbb{N}$ and let (M, ϱ, μ) be a connected set, equipped with a metric ϱ, generating a topology, and a Borel measure μ, furnished with an atlas according to (84) with the properties (79) - (83). Then (M, ϱ, μ) is called an n-dimensional Lipschitz manifold if*

$$\mu\left(B(y,r)\right) \sim r^n \quad \text{with } y \in M \text{ and } 0 < r < 1,$$

where the equivalence constants are independent of y and r.

Remark 6.2. The measure μ is locally the pullback measure of the Lebesgue measure $|\cdot|$ on $\mathbb{R}^n$,

$$\mu(E) \sim |\Phi_j(E)|\,, \quad E \subset B(y^j, 2\Lambda).$$

Concrete examples of abstract n-dimensional Lipschitz manifolds are the boundaries $M = \partial\Omega$ of special Lipschitz domains in $\mathbb{R}^{n+1}$ (non-compact case) and of bounded Lipschitz domains in $\mathbb{R}^{n+1}$ (compact case) according to Definition 2.9. In these cases $\mu \sim \mathcal{H}^n|M$, the restriction of the n-dimensional Hausdorff measure $\mathcal{H}^n$ in $\mathbb{R}^{n+1}$ to M.

6.3 Function spaces on Lipschitz manifolds

Let M be a n-dimensional Lipschitz manifold according to Definition 6.1. We always assume that n, ϱ, μ are fixed in what follows. As usual, $L_p(M)$ with $0 < p \le \infty$ are the quasi-Banach spaces of complex-valued μ-measurable functions f on M such that

$$\|f\,|L_p(M)\| = \left(\int_M |f(x)|^p \,\mu(dx) \right)^{\frac{1}{p}} < \infty\,, \quad 0 < p < \infty,$$

(with the usual modification if $p = \infty$). Furthermore, $Lip\,(M)$ is the usual Lipschitz spaces on M consisting of all complex-valued continuous functions on M such that

$$\|f\,|Lip(M)\| = \sup_{x \in M} |f(x)| + \sup_{x \ne y, \varrho(x,y) \le \Lambda} \frac{|f(x) - f(y)|}{\varrho(x,y)} < \infty\,.$$

Of course, $Lip(M)$ is a Banach space.

Next we introduce resolutions of unity on M using standard arguments. Let $\Lambda > 0$ be as above and let φ be a non-negative C^∞ function on $\mathbb{R}$ such that

$$supp\,\varphi \subset (-2\Lambda, 2\Lambda) \quad \text{and} \quad \varphi(t) = 1 \quad \text{if} \quad |t| \le \Lambda.$$

Let $\{y^j\}_{j \in J} \subset M$ be the grid used in the atlas according to (84) with the properties (79) - (83). Let

$$\varphi_j(y) = \varphi\left(\varrho(y^j, y)\right) \quad \text{and} \quad \psi_j(y) = \frac{\varphi_j(y)}{\sum_{k \in J} \varphi_k(y)}\,, \quad j \in J. \qquad (85)$$

Then $\psi_j \in Lip(M)$, and

$$supp\, \psi_j \subset B(y^j, 2\Lambda)\,, \quad \sum_{j\in J} \psi_j(y) = 1 \quad \text{where} \quad y \in M. \tag{86}$$

Hence, $\{\psi_j\}_{j\in J}$ is a resolution of unity on M, consisting of Lip-functions and adapted to the given atlas (84). The numbers σ_p and σ_{pq} have the same meaning as in (53). We use real interpolation according to (21).

Definition 6.3. *Let M be an n-dimensional Lipschitz manifold according to Definition 6.1. Let $\{\psi_j\}_{j\in J}$ be the above resolution of unity.*

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$ (with $q = \infty$ if $p = \infty$) and $\sigma_{pq} < s < 1$. Then $F^s_{pq}(M)$ is the collection of all $f \in L_1^{loc}(M)$ with*

$$(\psi_j f) \circ \Phi_j^{-1} \in F^s_{pq}(\mathbb{R}^n)\,, \quad j \in J\,, \tag{87}$$

and

$$\|f \,| F^s_{pq}(M)\| = \left(\sum_{j\in J} \left\| (\psi_j f) \circ \Phi_j^{-1} \,| F^s_{pq}(\mathbb{R}^n) \right\|^p \right)^{\frac{1}{p}} < \infty \tag{88}$$

(with the usual modification if $p = q = \infty$).

(ii) *Let $0 < p \le \infty$, $0 < q \le \infty$ and $\sigma_p < s < 1$. Let*

$$\sigma_p < s_1 < s < s_0 < 1 \quad \textit{and} \quad s = (1-\theta)s_0 + \theta s_1\,.$$

Then

$$B^s_{pq}(M) = \left(F^{s_0}_{pp}(M), F^{s_1}_{pp}(M) \right)_{\theta,q}\,. \tag{89}$$

Remark 6.4. In part (i) we put $F^s_{\infty\infty} = B^s_{\infty\infty}$ for obvious reasons in connection with part (ii). Of course, Φ_j^{-1} is the inverse of the mapping Φ_j with the properties (82), (83). In (87) we assume that the function $(\psi_j f) \circ \Phi_j^{-1}$, originally defined on W_j, is extended by zero to $\mathbb{R}^n$. In case of $M = \mathbb{R}^n$, the equality (88) reflects the so-called *localization principle* for the spaces $F^s_{pq}(\mathbb{R}^n)$, [39], 2.4.7, p. 124. In particular $F^s_{pq}(M)$ with $M = \mathbb{R}^n$ coincides with $F^s_{pq}(\mathbb{R}^n)$. This property was taken in [39], Ch. 7, to introduce the spaces $F^s_{pq}(M)$ on C^∞-Riemannian manifolds and Lie groups for all s, p, q with (1). Now we are doing the same but with the restrictions indicated. The main point is to prove that these spaces are independent of the admitted atlases and resolutions of unity. This comes out easily (equivalent quasi-norms) and may justify our omission to indicate the given atlas and the given resolution of unity on the left-hand side of (88). Furthermore, (89) is a well-known interpolation formula in case of $M = \mathbb{R}^n$. Again, both in [39], 7.3, p. 309, and also here we rely on this property to define the spaces $B^s_{pq}(M)$ on more general structures. Then we are precisely in the same position as in [39], Ch. 7, and the related arguments concerning the spaces $B^s_{pq}(M)$ will not be repeated here.

Theorem 6.5. *Let M be an n-dimensional Lipschitz manifold according to Definition 6.1 furnished with an atlas (84). Let $\{\psi_j\}_{j\in J}$ be a subordinated resolution of unity, (85), (86).*

(i) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad \sigma_{pq} < s < 1.$$

Then $F^s_{pq}(M)$ according to Definition 6.3(i) is a quasi-Banach space (Banach space if $p \ge 1$, $q \ge 1$). It is independent of admissible atlases and admissible resolutions of unity.

(ii) *Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad \sigma_p < s < 1.$$

Then $B^s_{pq}(M)$ according to Definition 6.3(ii) is a quasi-Banach space (Banach space if $p \ge 1$, $q \ge 1$). It is independent of s_0, s_1.

Proof. Let $\psi \in Lip(\mathbb{R}^n)$. By the atomic representations (54) for the admitted spaces $A^s_{pq}(\mathbb{R}^n)$ with (55), (56) it follows that

$$\|\psi f \,|A^s_{pq}(\mathbb{R}^n)\| \le c\,\|\psi \,|Lip(\mathbb{R}^n)\| \cdot \|f \,|A^s_{pq}(\mathbb{R}^n)\|$$

for some $c > 0$ which is independent of ψ and f. This pointwise multiplier property is well-known, [39], Corollary 4.2.2, p. 205. This observation and Proposition 4.1 prove part (i) by standard arguments. As for part (ii) we have the same situation as in [39], Theorem 7.3.1, p. 309, including

$$F^s_{pp}(M) = B^s_{pp}(M)\,, \quad 0 < p \le \infty, \quad \sigma_p < s < 1,$$

as it should be. We do not repeat the details.

6.4 Properties, atomic representations

By Definition 6.3 and Theorem 6.5 we have now the same situation as in [39], Ch. 7. On the basic of the technique developed there one can derive diverse properties, for example embedding theorems and interpolation formulas. We do not repeat these assertions here and refer in particular to [39], 7.4.2, 7.4.4, pp. 314-316, 317-318. On the other hand, intrinsic characterizations of spaces $F^s_{pq}(M)$ and $B^s_{pq}(M)$ on C^∞ Riemannian manifolds (with bounded geometry and positive injectivity radius) and Lie groups in terms of local means and geodesics have no immediate counterpart. However atomic and quarkonial representations can be intrinsically transferred from $\mathbb{R}^n$ to Lipschitz manifolds. We describe the outcome in case of atomic representations and indicate the necessary modifications.

To simplify the presentation we assume that M is non-compact (the modifications in case of compact M are obvious). We cover M with dyadic grids

$$\left\{x^j_m \,:\, m \in \mathbb{N}\right\}, \quad j \in \mathbb{N}_0; \tag{90}$$

such that there are two numbers $c > 0$ and $C > 0$ with

$$\varrho\left(x^j_m, x^j_l\right) \geq c\, 2^{-j}, \quad j \in \mathbb{N}_0, \quad m \neq l;$$

$$\bigcup_{m\in\mathbb{N}} B\left(x^j_m, C\, 2^{-j}\right) = M, \quad j \in \mathbb{N}_0.$$

Then, in analogy to (61), (62), the function $a_{jm}(x) \in Lip(M)$ is called an (s,p)-Lip atom if for $j \in \mathbb{N}_0$, $m \in \mathbb{N}$,

$$supp\, a_{jm} \subset B\left(x^j_m,\, C\, 2^{-j+1}\right), \tag{91}$$

$$|a_{jm}(x)| \leq 2^{-j(s-\frac{n}{p})}, \quad |a_{jm}(x) - a_{jm}(y)| \leq 2^{-j(s-\frac{n}{p}-1)}\, \varrho(x,y), \tag{92}$$

where $x \in M$, $y \in M$. We need also the counterparts of (57) - (60). Let

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : \quad j \in \mathbb{N}_0, \quad m \in \mathbb{N}\}, \tag{93}$$

$$\chi^{(p)}_{jm}(x) = 2^{\frac{jn}{p}}\, \chi_{jm}(x), \quad x \in M, \quad 0 < p \leq \infty,$$

where χ_{jm} are the characteristic functions of the balls $B(x^j_m, C2^{-j})$. Then $\lambda \in f^M_{pq}$ and $\lambda \in b^M_{pq}$ if

$$\|\lambda \,| f^M_{pq}\| = \left\| \left(\sum_{j=0}^{\infty} \sum_{m=1}^{\infty} \left| \lambda_{jm}\, \chi^{(p)}_{jm}(\cdot) \right|^q \right)^{\frac{1}{q}} | L_p(M) \right\| < \infty \tag{94}$$

and

$$\|\lambda \,| b^M_{pq}\| = \left(\sum_{j=0}^{\infty} \left(\sum_{m=1}^{\infty} |\lambda_{jm}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty,$$

respectively (with the usual modification if $p = \infty$ and/or $q = \infty$).

Theorem 6.6. *Let M be a (non-compact) n-dimensional Lipschitz manifold according to Definition 6.1.*

(i) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad \sigma_{pq} < s < 1.$$

Then $F^s_{pq}(M)$ consists of all $f \in L^{loc}_1(M)$ which can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} \lambda_{jm}\, a_{jm}, \quad \textit{absolute convergence in } L^{loc}_1(M), \tag{95}$$

where a_{jm} are (s,p)-Lip atoms according to (91), (92) and $\lambda \in f^M_{pq}$. Furthermore,

$$\|f \,| F^s_{pq}(M)\| \sim \inf \|\lambda \,| f^M_{pq}\|,$$

where the infimum is taken over all representations (95) with $\lambda \in f^M_{pq}$.

(ii) *Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad \sigma_p < s < 1. \tag{96}$$

Then $B^s_{pq}(M)$ consists of all $f \in L^{loc}_1(M)$ which can be represented by (95), where again a_{jm} are (s,p)-Lip atoms and $\lambda \in b^M_{pq}$. Furthermore,

$$\|f \,| B^s_{pq}(M)\| \sim \inf \|\lambda \,| b^M_{pq}\|,$$

where the infimum is taken over all representations (95) with $\lambda \in b^M_{pq}$.

Proof. Step 1. First we remark that $f \in L_1^{loc}(M)$ and the absolute convergence in $L_1^{loc}(M)$ in (95) are no additional restrictions but consequences of the assumptions: Recall that we have for p, q, s given by (96),

$$B^s_{pq}(\mathbb{R}^n) \subset L_r(\mathbb{R}^n) \quad \text{with} \quad s - \frac{n}{p} > -\frac{n}{r} \quad \text{and} \quad \infty \geq r \geq \max(p,1).$$

This is a well-known embedding theorem, [38], 2.3.3, 2.7.1. In particular $f \in L_1^{loc}(M)$, and under the above assumption, (95) converges absolutely in $L_1^{loc}(M)$. Secondly, by 4.2 we must carry over the atomic representations from $\mathbb{R}^n$ to M via local charts. This will be done for the spaces $F^s_{pq}(M)$ in the next step. As far as the spaces $B^s_{pq}(M)$ are concerned we are afterwards very much in the same situation as in case of corresponding function spaces on Riemannian manifolds and on Lie groups. We developed in [39], 7.3.2, technical instruments which can be applied also to the above situation. In this way one can prove part (ii) of the theorem under the assumption that part (i) holds.

Step 2. We prove the atomic representation for the spaces $F^s_{pq}(M)$. Let $g_l = (\psi_l\, f) \circ \Phi_l^{-1}$ according to Definition 6.3(i). Then g_l can be expanded in $F^s_{pq}(\mathbb{R}^n)$ in the desired way, where we may assume that all atoms a_{jm} (in $\mathbb{R}^n$) involved are supported in W_l given by (82). This follows from the atomic representations as described in 4.2 where one may assume that $j = j_0, j_0+1, \ldots$ and $j_0 \in \mathbb{N}$ can be chosen arbitrarily large (instead of $j \in \mathbb{N}_0$). These atoms can be transferred from $\mathbb{R}^n$ to M via local charts. Then one gets atoms of type (91), (92). It remains to clip together these local atomic representations. Let $B(y^l, \Lambda)$ be the balls according to (81), now with $l \in \mathbb{N}$, and let $\{x^j_m\}$ be the dyadic grids in (90). Then we decompose the grids in (93), (94) by

$$G_l = \Big\{ (j,m) : \quad j \in \mathbb{N}_0, \quad m \in \mathbb{N}, \quad x^j_m \in B(y^l, \Lambda) \Big\}, \quad l \in \mathbb{N}.$$

(There might be some overlaps, but this is immaterial for what follows). By (94) we have

$$\|\lambda \,|f^M_{pq}\|^p \sim \sum_{l=1}^{\infty} \int_M \left(\sum_{(j,m)\in G_l} |\lambda_{jm}\, \chi^{(p)}_{jm}(x)|^q \right)^{\frac{p}{q}} \mu(dx).$$

Hence one has again a localization principle for the spaces $F^s_{pq}(M)$ as described in case of $\mathbb{R}^n$ in [39], 2.4.7. This results in the desired atomic decomposition for $F^s_{pq}(M)$. We refer also to [41], Sect. 6, where we used this decomposition technique extensively and in greater details.

Remark 6.7. We mentioned in 6.1 several other possibilities to introduce function spaces on more general structures. This applies also to Lipschitz manifolds and in particular to the concrete Lipschitz manifolds according to Remark 6.2. One can expect that for given p, q, s the diverse possibilities to say what is meant by B^s_{pq} or F^s_{pq} always result in the same spaces, hopefully.

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