EXISTENCE OF WEAK-RENORMALIZED SOLUTION FOR A NONLINEAR SYSTEM

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Abstract

We prove an existence result for a coupled system of the reaction-diffusion kind. The fact that no growth condition is assumed on some nonlinear terms motivates the search of a weak-renormalized solution.

1 Introduction. Description of the problem

This paper investigates the existence of a solution for the nonlinear system

\[\begin{align*}
-\Delta u - \nabla \cdot (\beta(v)X'(u)) &= f \quad \text{in } \Omega, \\
-\Delta v - \nabla \cdot (\beta'(v)X(u)) &= g \quad \text{in } \Omega, \\
u = 0, \quad v = 0 \quad \text{on } \partial \Omega,
\end{align*}\]

where \(\Omega\) denotes a bounded open subset of \(\mathbb{R}^N\), \(X\) is a \(C^1\) bounded \(\mathbb{R}^N\)-valued function on \(\mathbb{R}\), i.e.

\[X \in (C^1(\mathbb{R}))^N \cap (C^0_b(\mathbb{R}))^N,\]

\(\beta\) is a function whose second derivatives are bounded, i.e.

\[\beta \in W^{2,\infty}(\mathbb{R})\]

and

\[f, g \in H^{-1}(\Omega).\]

Here, the main difficulty to find a solution is that no growth restrictions are assumed on \(X'\). Since \(f\) and \(g\) belong to \(H^{-1}(\Omega)\), it is natural to look for solutions \(u\) and \(v\) belonging to \(H^1_0(\Omega)\). Thus, it is not clear how

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to give a sense to $\nabla \cdot (\beta(v)X'(u))$. This inconvenient can be overcome by introducing a weak-renormalized formulation of this problem, essentially obtained through pointwise multiplication of the first equation of (1) by $h(u)$, where $h$ belongs to $C^1_0(\mathbb{R})$, that is, $h \in C^1(\mathbb{R})$ and its support is compact.

**Remark.** We can view this system as a simplified model of a nonlinear elasticity problem characterized by a constitutive law of the form

$$\sigma = \sigma_l + Y(u),$$

where

$$(\sigma_l)_{ij} = \sum a_{ijkl} \varepsilon_{kl}(u), \quad \varepsilon_{kl}(u) = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right), \quad Y_{ij} \in C^0(\mathbb{R}^2).$$

Indeed, the conservation of momentum reads

$$\nabla \cdot \sigma = F$$

($F$ is given), which is in some sense a generalization of (1). In this paper, we study the case in which

$$Y(u) = \begin{pmatrix} \beta(u_2)X'_1(u_1) & \beta'(u_2)X_1(u_1) \\ \beta(u_2)X'_2(u_1) & \beta'(u_2)X_2(u_1) \end{pmatrix}.$$ 

### 2 The main result

**Theorem 2.1.** Under the assumptions (2), (3), (4), there exists $\{u, v\}$, with $u, v \in H_0^1(\Omega)$, such that the second equation in (1) is satisfied in the usual weak or distributional sense and the first equation holds in the following sense:

$$\begin{cases} -\nabla \cdot (h(u)\nabla u) + \nabla u \cdot \nabla h(u) - \nabla \cdot (\beta(v)h(u)X'(u)) \\ + \beta(v)X'(u) \cdot \nabla h(u) = fh(u) \text{ in } D'(\Omega) \end{cases} \quad \forall h \in C^1_0(\mathbb{R}). \tag{5}$$

A couple $\{u, v\}$ as above will be called a weak-renormalized solution to (1).
**Remark.** In (5), every term belongs to $\mathcal{D}'(\Omega)$. Indeed, $h(u)$ belongs to $H^1_0(\Omega)$, the first term is in $H^{-1}(\Omega)$. The second one is in $L^1(\Omega)$. For instance, since $h$ has a compact support, we can put

$$h(u)X'(u) = h(u)X'(T_M(u)) \quad \text{and} \quad h'(u)X'(u) = h'(u)X'(T_M(u))$$

for some $M > 0$, where $T_M$ is the usual truncation at level $M$. Thus, we see that the third term in the left belongs to $W^{-1,\infty}(\Omega)$ and the fourth term belongs to $L^2(\Omega)$.

**Remark.** Renormalized solutions to PDE’s were introduced by R. DiPerna and P.L. Lions in [4] in the framework of the Boltzmann equation. They have been used in connection with various nonlinear elliptic equations by P. Benilan et al. [2], L. Boccardo et al. [3] and P.L. Lions and F. Murat [6] (see also [7]). In the analysis of existence results for systems, weak-renormalized solutions were first considered by R. Lewandowski [5] (see also [1]).

In this paper, in order to solve (1), we will extend the techniques used in [3] in the context of a single equation.

**Remark.** With regard to uniqueness, it is an open problem. If we follow the classical argument of considering two solutions $u^i, v^i$ for $i = 1, 2$ of (1), and we compute the difference of (5) written for $u^1, v^1$ and for $u^2, v^2$, we find expressions with terms of the form $X'(\cdot)u$ that we are not able to estimate. There is another argument, due to P. L. Lions and F. Murat [7], which leads to the uniqueness of renormalized solutions, but it cannot be applied here.

# 3 The proof of theorem 2.1

**First step.** The introduction of a family of approximations. For each $\varepsilon > 0$, let us put $X^\varepsilon(s) = X(T_{1/\varepsilon}(s))$ for all $s \in \mathbb{R}$. We will introduce the following approximation to (1):

$$\begin{cases}
-\Delta u^\varepsilon - \nabla \cdot (\beta(v^\varepsilon)(X^\varepsilon)'(u^\varepsilon)) = f \quad \text{in } \Omega, \\
-\Delta v^\varepsilon - \nabla \cdot (\beta'(v^\varepsilon)X(u^\varepsilon)) = g \quad \text{in } \Omega, \\
u^\varepsilon, v^\varepsilon \in H^1_0(\Omega),
\end{cases}$$

(6)
In order to solve (6), we will apply Schauder’s theorem. Thus, for any given $\varepsilon$ and $\{u, v\} \in L^2(\Omega) \times L^2(\Omega)$, we set $R^\varepsilon(\{u, v\}) = \{u^\varepsilon, v^\varepsilon\}$, with $\{u^\varepsilon, v^\varepsilon\}$ being the unique solution to the linear system

$$
\begin{cases}
-\Delta u^\varepsilon = f + \nabla \cdot (\beta(v)(X^\varepsilon)'(u)) & \text{in } \Omega, \\
-\Delta v^\varepsilon = g + \nabla \cdot (\beta'(v)X(u)) & \text{in } \Omega, \\
u^\varepsilon, v^\varepsilon \in H^1_0(\Omega),
\end{cases}
$$

(7)

Obviously, $R^\varepsilon = R_3 \circ R_2 \circ R^\varepsilon_1$, where

- $R^\varepsilon_1 : L^2(\Omega) \times L^2(\Omega) \mapsto H^{-1}(\Omega) \times H^{-1}(\Omega)$ is the nonlinear continuous mapping given by
  $$
  R^\varepsilon_1(\{u, v\}) = \{f + \nabla \cdot (\beta(v)(X^\varepsilon)'(u)), g + \nabla \cdot (\beta'(v)X(u))\},
  $$
  \forall \{u, v\} \in L^2(\Omega) \times L^2(\Omega),

- $R_2 : H^{-1}(\Omega) \times H^{-1}(\Omega) \mapsto H^1_0(\Omega) \times H^1_0(\Omega)$ associates to each $\{f, g\} \in H^{-1}(\Omega) \times H^{-1}(\Omega)$ the unique solution $\{w, z\}$ of the following linear system
  $$
  \begin{cases}
  -\Delta w = f & \text{in } \Omega, \\
  -\Delta z = g & \text{in } \Omega, \\
w, z \in H^1_0(\Omega),
  \end{cases}
  $$

- $R_3$ is the compact embedding of $H^1_0(\Omega) \times H^1_0(\Omega)$ into $L^2(\Omega) \times L^2(\Omega)$.

Since $R^\varepsilon_1$ maps the whole space $L^2(\Omega) \times L^2(\Omega)$ inside a ball, Schauder’s theorem can be applied and (6) possesses at least one solution $\{u^\varepsilon, v^\varepsilon\}$. **Second step.** A priori estimates and weak convergence.

Choosing $u^\varepsilon$ and $v^\varepsilon$ as test functions in the first and second equation in (6) respectively, one finds:

$$
\int_{\Omega} \nabla u^\varepsilon \nabla u^\varepsilon + \int_{\Omega} \beta(u^\varepsilon)(X^\varepsilon)'(u^\varepsilon) \cdot \nabla u^\varepsilon = \langle f, u^\varepsilon \rangle_{H^{-1}, H^1_0}. 
$$

(8)

$$
\int_{\Omega} \nabla v^\varepsilon \nabla v^\varepsilon + \int_{\Omega} \beta'(v^\varepsilon)X(u^\varepsilon) \cdot \nabla v^\varepsilon = \langle g, v^\varepsilon \rangle_{H^{-1}, H^1_0}. 
$$

(9)
For $\varepsilon$ sufficiently small, $X = X \circ T_{1/\varepsilon} = X^\varepsilon$, whence we can replace $X(u^\varepsilon)$ by $X^\varepsilon(u^\varepsilon)$ in (9).

Let us introduce the function $H = (H_1, H_2, ..., H_n)$, with

$$H_i(t, s) = \int_0^s \beta(0)(X^\varepsilon_i)'(\theta)d\theta + \int_t^1 \beta'(\theta)X^\varepsilon_i(s)d\theta.$$

Then,

$$\int_\Omega \beta v^\varepsilon (X^\varepsilon)'(u^\varepsilon) \cdot \nabla u^\varepsilon + \int_\Omega \beta'(v^\varepsilon)X^\varepsilon(u^\varepsilon) \cdot \nabla v^\varepsilon = \int_\Omega \nabla \cdot H(u^\varepsilon, v^\varepsilon) = 0$$

thanks to Stokes’ theorem. Summing (8) and (9), we obtain

$$\int_\Omega |\nabla u^\varepsilon|^2 + \int_\Omega |\nabla v^\varepsilon|^2 = \langle f, u^\varepsilon \rangle_{H^{-1}, H_0^1} + \langle g, v^\varepsilon \rangle_{H^{-1}, H_0^1}$$

and

$$\|u^\varepsilon\|^2_{H_0^1} + \|v^\varepsilon\|^2_{H_0^1} \leq \|f\|^2_{H^{-1}} + \|g\|^2_{H^{-1}}.$$ 

Consequently, at least for a subsequence, still indexed by $\varepsilon$, we can conclude that

$$u^\varepsilon \to u, \quad v^\varepsilon \to v \quad \text{weakly in } H_0^1(\Omega),$$

$$u^\varepsilon \to u, \quad v^\varepsilon \to v \quad \text{strongly in } L^p(\Omega) \quad \forall p \in [1, 2^*) \text{ and a.e.} \quad (10)$$

Here, we have denoted by $2^*$ the exponent furnished by the Sobolev embedding theorem, that is

$$\begin{cases} 2^* = \frac{2N}{N-2} & \text{if } N \geq 3, \\ 2^* < +\infty \text{ arbitrarily large if } N = 2. \end{cases}$$

**Third step.** The strong convergence of $v^\varepsilon$ in $H_0^1$.

It is easy to see that $v$ is a weak solution to the problem

$$\begin{cases} -\Delta v - \nabla \cdot (\beta'(v)X(u)) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \quad (11) \end{cases}$$

Indeed, since $\beta'$ and $X$ are continuous and bounded, it is clear that $\beta'(v^\varepsilon) \to \beta'(v)$ strongly in $L^p$ for all $p \in [1, 2^*)$ and $X(u^\varepsilon) \to X(u)$.
strongly in $L^r$ for all $r \in [1, +\infty)$. This enables us to pass to the limit in the second equation in (6).

From (11), we also see that
\[ \int_{\Omega} |\nabla v|^2 = -\int_{\Omega} \beta'(v)X(u) \cdot \nabla v + \int_{\Omega} g v. \tag{12} \]
Let us use $v^\varepsilon$ as a test function in the second equation in (6). We find:
\[ \int_{\Omega} |\nabla v^\varepsilon|^2 = -\int_{\Omega} \beta'(v^\varepsilon)X(u^\varepsilon) \cdot \nabla v^\varepsilon + \int_{\Omega} g v^\varepsilon. \tag{13} \]
Arguing as above, we can pass to the limit in the right hand side in (13). Accordingly, we have:
\[ \int_{\Omega} |\nabla v|^2 \to -\int_{\Omega} \beta'(v)X(u) \cdot \nabla v + \int_{\Omega} g v. \]
This, combined with (12), gives the convergence in norm in $H^1_0$ for $v^\varepsilon$ and, consequently,
\[ v^\varepsilon \to v \text{ strongly in } H^1_0. \tag{14} \]

**Fourth step.** The strong convergence of $u^\varepsilon$ in $H^1_0$.
We will first prove that
\[ \lim_{K \to +\infty} \left( \limsup_{\varepsilon \to 0} \int_{\{|u^\varepsilon| > K\}} |\nabla u^\varepsilon|^2 \right) = 0 \tag{15} \]
Thus, let us consider the test functions $u^\varepsilon - T_K(u^\varepsilon)$ in the first equation in (6). Notice that
\[ \nabla (u^\varepsilon - T_K(u^\varepsilon)) = \begin{cases} \nabla u^\varepsilon & \text{if } |u^\varepsilon| \geq K, \\ 0 & \text{if } |u^\varepsilon| < K. \end{cases} \]
Hence,
\[ \int_{\{|u^\varepsilon| \geq K\}} |\nabla u^\varepsilon|^2 + \int_{\Omega} \beta(v^\varepsilon)(1 - T_K'(u^\varepsilon))(X^\varepsilon)'(u^\varepsilon) \cdot \nabla u^\varepsilon \]
\[ = \langle f, u^\varepsilon - T_K(u^\varepsilon) \rangle. \tag{16} \]
We can put \((1 - T_K'(u^\varepsilon))(X^\varepsilon)'(u^\varepsilon) \cdot \nabla u^\varepsilon = \nabla \cdot Y_K^\varepsilon(u^\varepsilon)\), where
\[
(Y_K^\varepsilon)_i(t) = \int_0^t (1 - T_K'(\theta))(X^\varepsilon)'(\theta) \, d\theta.
\]
Thus, the second term in the left hand side of (16) can be written in the form
\[
\int_\Omega (\nabla \cdot Y_K^\varepsilon(u^\varepsilon))\beta(u^\varepsilon) = -\int_\Omega Y_K^\varepsilon(u^\varepsilon) \cdot \nabla \beta(u^\varepsilon).
\]
Moreover,
\[
Y_K^\varepsilon(s) = \begin{cases} 
X^\varepsilon(s) - X^\varepsilon(K) & \text{if } s > K, \\
0 & \text{if } |u^\varepsilon| \leq K, \\
X^\varepsilon(s) - X^\varepsilon(-K) & \text{if } s < -K.
\end{cases}
\]
Since \(X \in C^0_b(\mathbb{R})^N\), for \(\varepsilon > 0\) sufficiently small, \(Y_K^\varepsilon\) is independent of \(\varepsilon\) and \(Y_K^\varepsilon(u^\varepsilon)\) is bounded by a constant independent of \(\varepsilon\). We also have
\[
\limsup_{\varepsilon \to 0} |Y_K^\varepsilon(u^\varepsilon)| \leq |X(u) - X(K)|1_{\{u > K\}} + |X(u) - X(-K)|1_{\{u < -K\}}
\]
for all \(K > 0\). Therefore,
\[
\limsup_{\varepsilon \to 0} \int_{\{|u^\varepsilon| > K\}} |\nabla u^\varepsilon|^2 \leq \int_\Omega |X(u) - X(K)| \cdot |\nabla \beta(v)|1_{\{u > K\}}
\]
\[
+ \int_\Omega |X(u) - X(-K)| \cdot |\nabla \beta(v)|1_{\{u < -K\}} + \langle f, u - T_K(u) \rangle,
\]
whence
\[
\lim_{K \to +\infty} \left( \limsup_{\varepsilon \to 0} \int_{\{|u^\varepsilon| > K\}} |\nabla u^\varepsilon|^2 \right) \leq \lim_{K \to +\infty} \left[ \int_\Omega |X(u) - X(K)| \cdot |\nabla \beta(v)|1_{\{u > K\}} \right.
\]
\[
+ \int_\Omega |X(u) - X(-K)| \cdot |\nabla \beta(v)|1_{\{u < -K\}} \right]
\]
\[
\left. + \lim_{K \to +\infty} \langle f, u - T_K(u) \rangle = 0. \right)
\]
This proves (15). Let us introduce the sets $F_{i,j}^\varepsilon$,

$$F_{i,j}^\varepsilon = \{ x \in \Omega : |u^\varepsilon - T_j(u)| \leq i \}.$$

We are now going to prove that

$$\lim_{j \to +\infty} \left( \limsup_{\varepsilon \to 0} \int_{F_{i,j}^\varepsilon} |\nabla (u^\varepsilon - T_j(u))|^2 \right) = 0 \quad \forall i \geq 1. \quad (19)$$

Thus, let us use $T_i(u^\varepsilon - T_j(u))$ as test function in the first equation of (6). We obtain

$$\int_{\Omega} \nabla u^\varepsilon \cdot \nabla T_i(u^\varepsilon - T_j(u)) + \int_{\Omega} \beta(v^\varepsilon) (X^\varepsilon)'(u^\varepsilon) \cdot \nabla T_i(u^\varepsilon - T_j(u))$$

$$= \langle f, T_i(u^\varepsilon - T_j(u)) \rangle. \quad (20)$$

Let us notice that

$$\nabla T_i(u^\varepsilon - T_j(u)) = 0 \text{ in } \Omega \setminus F_{i,j}^\varepsilon.$$

We can then write (20) in the form

$$\int_{F_{i,j}^\varepsilon} \nabla u^\varepsilon \cdot \nabla T_i(u^\varepsilon - T_j(u)) + \int_{F_{i,j}^\varepsilon} \beta(v^\varepsilon) (X^\varepsilon)'(u^\varepsilon) \cdot \nabla T_i(u^\varepsilon - T_j(u))$$

$$= \langle f, T_i(u^\varepsilon - T_j(u)) \rangle. \quad (21)$$

Since

$$|u^\varepsilon| \leq |u^\varepsilon - T_j(u)| + |T_j(u)| \leq i + j \quad \text{if } x \in F_{i,j}^\varepsilon,$$

we can write $T_{1/\varepsilon}(u^\varepsilon) = T_{i+j}(u^\varepsilon)$ for all $x \in F_{i,j}^\varepsilon$ whenever $\varepsilon$ is sufficiently small. This gives:

$$(X^\varepsilon)'(u^\varepsilon) = X'(T_{i+j}(u^\varepsilon))T'_{i+j}(u^\varepsilon) = X'(T_{i+j}(u^\varepsilon)) \text{ in } F_{i,j}^\varepsilon.$$

Thus, for small $\varepsilon > 0$, the second term in the left in (21) is

$$\int_{F_{i,j}^\varepsilon} \beta(v^\varepsilon) X'(T_{i+j}(u^\varepsilon)) \cdot \nabla T_i(u^\varepsilon - T_j(u))$$

and converges to

$$\int_{\Omega} \beta(v) X'(T_{i+j}(u)) \cdot \nabla T_i(u - T_j(u)) \quad (22)$$
as \( \varepsilon \to 0 \), since

\[
T_i(u^\varepsilon - T_j(u)) \to T_i(u - T_j(u)) \text{ weakly in } H_0^1
\]

and \( \beta(\varepsilon)X'(T_{i+j}(u^\varepsilon)) \) is bounded in \( (L^\infty(\Omega))^N \) and converges a.e. to \( \beta(u)X'(T_{i+j}(u)) \).

Let us introduce \( H_{i,j} = (H_{i,j}^1, H_{i,j}^2, \ldots, H_{i,j}^N) \), with

\[
H_{i,j}^s = \int_0^s T_i'(\theta - T_j(\theta))(1 - T_j'(\theta))X'(T_{i+j}(\theta)) \, d\theta.
\]

Then (22) can be rewritten in the form

\[
\int_{\Omega} (\nabla \cdot H_{K}^{i,j}(u))\beta(v) = -\int_{\Omega} H_{i,j}^{i,j}(u) \cdot \nabla \beta(v)
\]

Moreover, it is not difficult to check that

\[
H_{i,j}^{i,j}(u) = \begin{cases} 
X(i + j) - X(j) & \text{if } j < |u| < i + j, \\
0 & \text{otherwise}.
\end{cases}
\]

For any \( i \), we have \( H_{i,j}^{i,j}(u) \to 0 \) a.e. as \( j \to +\infty \). Since \( X \) is bounded, \( H_{i,j}^{i,j}(u) \) is also bounded. Thus, we obtain from Lebesgue’s theorem that

\[
\int_{\Omega} H_{i,j}^{i,j}(u) \cdot \nabla \beta(v) \to 0 \quad \text{as } j \to \infty.
\]

for all \( i \geq 1 \). Recalling (20) we see we have proved the following:

\[
\lim_{j \to +\infty} \left( \lim_{\varepsilon \to 0} \int_{F_{i,j}^\varepsilon} \nabla u^\varepsilon \cdot \nabla T_i(u^\varepsilon - T_j(u)) \right) = \lim_{j \to +\infty} \langle f, T_i(u - T_j(u)) \rangle.
\]

On the other hand,

\[
\lim_{j \to +\infty} \left( \lim_{\varepsilon \to 0} \int_{F_{i,j}^\varepsilon} \nabla T_j(u) \cdot \nabla T_i(u^\varepsilon - T_j(u)) \right) = \lim_{j \to +\infty} \int_{\Omega} \nabla T_j(u) \cdot \nabla T_i(u - T_j(u)).
\]
Consequently,

\[
\lim_{j \to +\infty} \left( \lim_{\varepsilon \to 0} \int_{F_{i,j}^\varepsilon} |\nabla(u^\varepsilon - T_j(u))|^2 \right)
= \lim_{j \to +\infty} \left( \langle f, T_i(u - T_j(u)) \rangle - \int_{\Omega} \nabla T_j(u) \cdot \nabla T_i(u - T_j(u)) \right).
\]

(24)

Notice that, the terms on the right hand side of (24) can be bounded as follows:

\[
\langle f, T_i(u - T_j(u)) \rangle - \int_{\Omega} \nabla T_j(u) \cdot \nabla T_i(u - T_j(u))
\leq (\|f\|_{H^{-1}} + \|u\|) \|u - T_j(u)\|
\]

and this converges to 0 as \( j \to +\infty \). Therefore, (19) is satisfied.

We can now prove that \( u^\varepsilon \) converges strongly in \( H^1_0 \). Indeed, observe that, if \( x \in \Omega \setminus F_{i,j}^\varepsilon \), then

\[
|u^\varepsilon| \geq |u^\varepsilon - T_j(u)| - |T_j(u)| \geq i - j,
\]

so that \( \Omega \setminus F_{i,j}^\varepsilon \subset E_{i,j}^\varepsilon \), with

\[
E_{i,j}^\varepsilon = \{ x \in \Omega : |u^\varepsilon(x)| \geq i - j \}.
\]

Therefore,

\[
\frac{1}{2} \int_{\Omega} |\nabla(u^\varepsilon - u)|^2 \leq \frac{1}{2} \int_{F_{i,j}^\varepsilon} |\nabla(u^\varepsilon - u)|^2 + \frac{1}{2} \int_{E_{i,j}^\varepsilon} |\nabla(u^\varepsilon - u)|^2
\leq \int_{F_{i,j}^\varepsilon} |\nabla(u^\varepsilon - T_j(u))|^2 + \int_{F_{i,j}^\varepsilon} |\nabla(T_j(u) - u)|^2
+ \int_{E_{i,j}^\varepsilon} |\nabla u^\varepsilon|^2 + \int_{E_{i,j}^\varepsilon} |\nabla u|^2 \leq 2(A_{ij}^\varepsilon + B_{ij}^\varepsilon + C_{ij}^\varepsilon + D_{ij}^\varepsilon).
\]

(25)

We have seen in (19) that

\[
\lim_{j \to +\infty} \limsup_{\varepsilon \to 0} A_{ij}^\varepsilon = 0 \quad \forall i \geq 1
\]

(26)

The second term \( B_{ij}^\varepsilon \) satisfies

\[
\limsup_{\varepsilon \to 0} B_{ij}^\varepsilon \leq \int_{\Omega} |\nabla(T_j(u) - u)|^2.
\]
whence we also have
\[ \lim_{j \to +\infty} \limsup_{\varepsilon \to 0} B_{ij}^\varepsilon = 0 \quad \forall i \geq 1 \]  
(27)

From (15) we know that
\[ \lim_{j \to +\infty} \limsup_{\varepsilon \to 0} C_{ij}^\varepsilon = 0 \quad \text{as } i, j \to +\infty, \ i - j \to +\infty. \]  
(28)

Finally, this is also true for \( D_{ij}^\varepsilon \), since \( u \in H_0^1 \):
\[ \lim_{j \to +\infty} \limsup_{\varepsilon \to 0} D_{ij}^\varepsilon = 0 \quad \text{as } i, j \to +\infty, \ i - j \to +\infty. \]  
(29)

From (25) and (26)–(29), we deduce at once that \( u_\varepsilon \to u \) strongly in \( H_0^1 \) as \( \varepsilon \to 0 \).

**Fifth step.** End of the proof of theorem 1.1.

Let us chose \( h \in C_0^1(\mathbb{R}) \) and \( \varphi, \psi \in \mathcal{D} \). Multiplying the first equation in (6) by \( h(u_\varepsilon) \varphi \) and the second one by \( \psi \) and integrating by parts, we obtain:
\[
\begin{aligned}
\int_\Omega (\nabla u_\varepsilon + \beta(v_\varepsilon)(X_\varepsilon)'(u_\varepsilon)) \cdot \nabla (h(u_\varepsilon) \varphi) &= \langle f, h(u_\varepsilon) \varphi \rangle \\
\int_\Omega (\nabla v_\varepsilon + \beta'(v_\varepsilon) X_\varepsilon(u_\varepsilon)) \cdot \nabla \psi &= \langle g, \psi \rangle.
\end{aligned}
\]
(30)

Since \( h \) and \( h' \) have compact support on \( \mathbb{R} \), for \( \varepsilon \) sufficiently small we have
\[ (X_\varepsilon)'(t) h(t) = X'(t) h(t), \quad (X_\varepsilon)'(t) h'(t) = X'(t) h'(t). \]

Both functions belong to \( (C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N \). Thus, we can write (30) as follows
\[
\begin{aligned}
\int_\Omega h(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi + \int_\Omega h'(u_\varepsilon) |\nabla u_\varepsilon|^2 \varphi + \int_\Omega \beta(v_\varepsilon) h(u_\varepsilon) X'(u_\varepsilon) \cdot \nabla \varphi \\
+ \int_\Omega \beta'(v_\varepsilon) h'(u_\varepsilon)(X'(u_\varepsilon) \cdot \nabla u_\varepsilon) \varphi &= \langle f, h(u_\varepsilon) \varphi \rangle \\
\int_\Omega \nabla v_\varepsilon \nabla \psi + \int_\Omega \beta'(v_\varepsilon) X(u_\varepsilon) \cdot \nabla \psi &= \langle g, \psi \rangle.
\end{aligned}
\]  
(31)
Now, using the strong convergence of $u^\varepsilon$ to $u$ in $H_0^1(\Omega)$, it is easy to pass to the limit in each term of (31); this yields

$$
\begin{aligned}
\int_\Omega h(u) \nabla u \cdot \nabla \varphi &+ \int_\Omega h'(u) [\nabla u]^2 \varphi + \int_\Omega \beta(v) h(u) X'(u) \cdot \nabla \varphi \\
&+ \int_\Omega \beta(v) h'(u) (X'(u) \cdot \nabla u) \varphi = \langle f, h(u) \varphi \rangle \\
\int_\Omega \nabla v \cdot \nabla \psi &+ \int_\Omega \beta'(v) X(u) \cdot \nabla \psi = \langle g, \psi \rangle.
\end{aligned}
$$

This completes the proof.

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