An Ostrowski like inequality for convex functions and applications

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ABSTRACT

In this paper we point out an Ostrowski type inequality for convex functions which complement in a sense the recent results for functions of bounded variation and absolutely continuous functions. Applications in connection with the Hermite-Hadamard inequality are also considered.

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1. Introduction

In 1938, A. Ostrowski [9] proved the following integral inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{4} \left[ \frac{x - a + b}{b-a} \right]^2 (b-a) \| f' \|_{\infty}$$

provided $f$ is differentiable and $\| f' \|_{\infty} = \sup_{t \in (a,b)} |f'(t)| < \infty$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In the last 5 years, many authors have concentrated their efforts in generalising (1.1) and have applied the obtained results in different fields, including Numerical Integration, Probability Theory and Statistics, Information Theory, etc. For a comprehensive approach in the field, see the recent book [5] where many other references may be found.
One direction of generalising (1.1) was pointed out by the author in [2] – [4]. Let us recall here a couple of the main results obtained in the above papers.

**Theorem 1.** Let $I_k: a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and $\alpha_i$ ($i = 0, \ldots, k + 1$) be $k + 2$ points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ ($i = 1, \ldots, k$) and $\alpha_{k+1} = b$. If $f: [a, b] \rightarrow \mathbb{R}$ is of bounded variation on $[a, b]$, then we have the inequality:

$$
\left| \int_a^b f(x) \, dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \leq \frac{1}{2} \nu(h) + \max \left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|, \quad i = 0, \ldots, k - 1 \right\} \sqrt{f},
$$

where $\nu(h) := \max \{ h_i | i = 0, \ldots, k - 1 \}$, $h_i := x_{i+1} - x_i$ ($i = 0, \ldots, k - 1$) and $\sqrt{f}$ is the total variation of $f$ on $[a, b]$.

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

If one would assume more for the function $f$, for example, absolute continuity, then the following result holds.

**Theorem 2.** Under the assumptions of Theorem 1 for $I_k$ and $\alpha_i$ ($i = 0, \ldots, k + 1$) and if $f: [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$, then

$$
\left| \int_a^b f(x) \, dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \leq \left\{ \begin{array}{ll}
\left[ \frac{1}{4} \sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left( \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right) \right] && \text{if } f' \in L_{\infty} [a, b]; \\
\frac{1}{(q+1) \frac{q}{2}} \left[ \sum_{i=0}^{k-1} \left( (\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1} \right) \right]^{\frac{1}{q+1}} \| f' \|_p && \text{if } f' \in L_p [a, b], \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1;
\end{array} \right.
$$

where $\| \cdot \|_p$ ($p \in [1, \infty]$) are the Lebesgue norms, i.e.,

$$
\| h \|_\infty := \text{ess sup}_{t \in [a, b]} |h(t)|,
$$

$$
\| h \|_p := \left( \int_a^b |h(t)|^p \, dt \right)^{\frac{1}{p}}, \quad p \in [1, \infty).
$$
The constants $\frac{1}{4}$, $\frac{1}{(q+1)^{\frac{1}{2}}}$ and $\frac{1}{2}$ are best in the sense mentioned above.

In this paper, the case of convex functions $f : [a, b] \to \mathbb{R}$ is examined. Some particular cases in connection with the well known Hermite-Hadamard inequality for convex functions are also considered.

2. The Results

The following result holds.

**Theorem 3.** Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and $\alpha_i$ $(i = 0, \ldots, k + 1)$ be $k + 2$ points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$, then we have the inequality:

$$
\frac{1}{2} \sum_{i=0}^{k-1} \left[ (x_{i+1} - \alpha_{i+1})^2 f'_+ (\alpha_{i+1}) - (\alpha_{i+1} - x_i)^2 f'_- (\alpha_{i+1}) \right] \quad (2.1)
$$

$$
\leq \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f (x_{i+1}) - \int_a^b f (t) \, dt
$$

$$
\leq \frac{1}{2} \sum_{i=0}^{k-1} \left[ (x_{i+1} - \alpha_{i+1})^2 f'_- (x_{i+1}) - (\alpha_{i+1} - x_i)^2 f'_+ (x_i) \right].
$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

**Proof.** Using the integration by parts formula, we may prove the equality (see for example [3]):

$$
\sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f (x_{i+1}) - \int_a^b f (t) \, dt = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (t - \alpha_{i+1}) f' (t) \, dt \quad (2.2)
$$

for any locally absolutely continuous function $f : (a, b) \to \mathbb{R}$.

Since $f$ is convex, then it is locally Lipschitzian on $(a, b)$ and thus the above equality holds. Also, we have

$$
f'_+ (x_i) \leq f' (t) \leq f'_- (\alpha_{i+1}) \quad \text{for a.e. } t \in [x_i, \alpha_{i+1}] \quad (2.3)
$$

and

$$
f'_+ (\alpha_{i+1}) \leq f' (t) \leq f'_- (x_{i+1}) \quad \text{for a.e. } t \in [\alpha_{i+1}, x_{i+1}] \quad (2.4)
$$

Using (2.3) and (2.4), we may write that

$$
f'_- (\alpha_{i+1}) \int_{x_i}^{\alpha_{i+1}} (t - \alpha_{i+1}) \, dt \leq \int_{x_i}^{\alpha_{i+1}} f' (t) (t - \alpha_{i+1}) \, dt \leq f'_+ (x_i) \int_{x_i}^{\alpha_{i+1}} (t - \alpha_{i+1}) \, dt \quad (2.5)
$$
and
\[ f_+^{'} (\alpha_{i+1}) \int_{\alpha_{i+1}}^{x_{i+1}} (t - \alpha_{i+1}) \, dt \leq \int_{\alpha_{i+1}}^{x_{i+1}} f^{'} (t) (t - \alpha_{i+1}) \, dt \leq f_-^{'} (x_{i+1}) \int_{\alpha_{i+1}}^{x_{i+1}} (t - \alpha_{i+1}) \, dt. \] (2.6)

Adding (2.5) and (2.6) and taking into account that
\[ \int_{x_{i+1}}^{\alpha_{i+1}} (t - \alpha_{i+1}) \, dt = -\frac{1}{2} (\alpha_{i+1} - x_i)^2 \]
and
\[ \int_{\alpha_{i+1}}^{x_{i+1}} (t - \alpha_{i+1}) \, dt = \frac{1}{2} (x_{i+1} - \alpha_{i+1})^2, \]
we get
\[ \frac{1}{2} \left[ (x_{i+1} - \alpha_{i+1})^2 f_+^{'} (\alpha_{i+1}) - (\alpha_{i+1} - x_i)^2 f_-^{'} (\alpha_{i+1}) \right] \] (2.7)
\[ \leq \int_{x_{i+1}}^{\alpha_{i+1}} f^{'} (t) \, dt \]
\[ \leq \frac{1}{2} \left[ (x_{i+1} - \alpha_{i+1})^2 f_-^{'} (x_{i+1}) - (\alpha_{i+1} - x_i)^2 f_+^{'} (x_i) \right] \]
for any \( i = 0, \ldots, k - 1 \).

If we sum (2.7) over \( i \) from 0 to \( k - 1 \) and use the identity (2.2), we deduce the desired result (2.1).

The sharpness will be proved in what follows for a particular case.

It is natural to consider the following particular case.

**Corollary 1.** Let \( L_k \) and \( f \) be as in the above theorem. Then we have the inequality
\[
0 \leq \frac{1}{8} \sum_{i=0}^{k-1} \left[ f_+^{'} \left( \frac{x_i + x_{i+1}}{2} \right) - f_-^{'} \left( \frac{x_i + x_{i+1}}{2} \right) \right] (x_{i+1} - x_i)^2 \] (2.8)
\[
\leq \frac{1}{2} \left[ (x_1 - a) f (a) + \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f (x_i) + (b - x_{k-1}) f (b) \right]
- \int_a^b f (t) \, dt \]
\[
\leq \frac{1}{8} \sum_{i=0}^{k-1} \left[ f_-^{'} (x_{i+1}) - f_+^{'} (x_i) \right] (x_{i+1} - x_i)^2 .
\]
The constant \( \frac{1}{8} \) in both inequalities is sharp.
The proof follows by the above theorem choosing $\alpha_i = \frac{x_{i+1} + x_i}{2}$, $i = 1, \ldots, k$ and taking into account that (see also [2])

$$
\sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i)
$$

The following corollary for equidistant partitioning also holds.

**Corollary 2.** Let

$$
I_k : x_i := a + (b - a) \cdot \frac{i}{k} \quad (i = 0, \ldots, k)
$$

be an equidistant partitioning of $[a, b]$. If $f : [a, b] \to \mathbb{R}$ is convex on $[a, b]$, then we have the inequalities

$$
0 \leq \frac{(b - a)^2}{8n^2} \sum_{i=0}^{k-1} \left\{ f'\left[ a + \left( i + \frac{1}{2} \right) \frac{b - a}{n} \right] - f'\left[ a + \left( i + \frac{1}{2} \right) \frac{b - a}{n} \right] \right\}
$$

$$
\leq \frac{1}{k} \frac{f(a) + f(b)}{2} (b - a)
$$

$$
+ \frac{b - a}{k} \sum_{i=1}^{k-1} \int_a^b f(t) \, dt - \int_a^b f(t) \, dt
$$

$$
\leq \frac{(b - a)^2}{8n^2} \sum_{i=0}^{k-1} \left\{ f'\left[ a + (i + 1) \cdot \frac{b - a}{n} \right] - f'\left[ a + i \cdot \frac{b - a}{n} \right] \right\}.
$$

The following particular cases which hold when we assume differentiability conditions may be stated.

**Corollary 3.** If $\alpha_i \in (a, b)$ for $i = 1, \ldots, k$ are points of differentiability for $f$, then we have the inequality

$$
\sum_{i=0}^{k-1} (x_{i+1} - x_i) \left( \frac{x_i + x_{i+1}}{2} - \alpha_{i+1} \right) f'(\alpha_{i+1})
$$

$$
\leq \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_{i+1}) - \int_a^b f(t) \, dt.
$$
If we denote by \( \nu(I_n) := \max \{x_{i+1} - x_i | i = 0, \ldots, k - 1\} \), then the following corollary also holds.

**Corollary 4.** If \( x_i (i = 1, \ldots, k - 1) \) are points of differentiability for \( f \) then

\[
\frac{1}{2} \left[ (x_1 - a) f(a) + \sum_{i=0}^{k-1} (x_{i+1} - x_i) f(x_i) + (b - x_k) f(b) \right] - \int_a^b f(t) \, dt \\
\leq \frac{1}{8} \left[ \nu(I_n) \right]^2 \left[ f'_-(b) - f'_+(a) \right].
\]  

(2.12)

### 3. Some Particular Inequalities

(1) If we choose \( x_0 = a, x_1 = b, \alpha_0 = a, \alpha_1 = x \in (a, b), \alpha_2 = b \), then from (2.1) we deduce (see also [6])

\[
\frac{1}{2} \left[ (b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \\
\leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t) \, dt \\
\leq \frac{1}{2} \left[ (b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right].
\]  

(3.1)

The constant \( \frac{1}{2} \) is sharp in both inequalities (see for example [6]).

If \( x = \frac{a+b}{2} \), then by (3.1) one deduces (see also [6])

\[
0 \leq \frac{1}{8} (b-a)^2 \left[ f'_+ \left( \frac{a+b}{2} \right) - f'_- \left( \frac{a+b}{2} \right) \right] \\
\leq \frac{f(a) + f(b)}{2} \cdot (b-a) - \int_a^b f(t) \, dt \\
\leq \frac{1}{8} (b-a)^2 \left[ f'_- (b) - f'_+ (a) \right]
\]  

(3.2)

and the constant \( \frac{1}{8} \) in both inequalities is sharp (see for example [6]).

If one would assume that \( x \in (a, b) \) is a point of differentiability, then

\[
(b-a) \left( \frac{a+b}{2} - x \right) f'(x) \leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t) \, dt.
\]  

(3.3)
(2) If we choose $a = x_0 < x < x_2 = b$ and the numbers $\alpha_0 = a$, $\alpha \in (a, x]$, $\beta \in [x, b)$ and $\alpha_3 = b$, then by Theorem 3, we deduce

\[
\frac{1}{2} \left[ (x - \alpha)^2 f'_+ (\alpha) - (\alpha - a)^2 f'_- (\alpha) + (b - \beta)^2 f'_+ (\beta) - (\beta - x)^2 f'_- (\beta) \right] \leq (\alpha - a) f (\alpha) + (\beta - \alpha) f (x) + (b - \beta) f (b) - \int_a^b f (t) \, dt \tag{3.4}
\]

\[
\leq \frac{1}{2} \left[ (x - \alpha)^2 f'_- (x) - (\alpha - a)^2 f'_+ (a) + (b - \beta)^2 f'_- (\beta) - (\beta - x)^2 f'_+ (x) \right].
\]

The constant $\frac{1}{2}$ is sharp in both inequalities.

(a) Note that if we let $\alpha \to a+$ and $\beta \to b-$, then from (3.4), by taking into account firstly that $(x - \alpha)^2 f'_+ (a) \leq (x - \alpha)^2 f'_+ (\alpha)$ and $-(\beta - x)^2 f'_- (b) \leq -(\beta - x)^2 f'_- (\beta)$, we may deduce the inequality obtained in [7]:

\[
\frac{1}{2} \left[ (b - x)^2 f'_+ (x) - (x - a)^2 f'_- (x) \right] \leq \int_a^b f (t) \, dt - (b - a) f (x) \leq \frac{1}{2} \left[ (\beta - x)^2 f'_- (b) + (x - a)^2 f'_+ (a) \right]. \tag{3.5}
\]

The constant $\frac{1}{2}$ is sharp in both inequalities (see for example [7]).

If in (3.5) we choose $x = \frac{a + b}{2}$, then (see also [7])

\[
0 \leq \frac{1}{8} (b - a)^2 \left[ f'_+ \left( \frac{a + b}{2} \right) - f'_- \left( \frac{a + b}{2} \right) \right] \tag{3.6}
\]

\[
\leq \int_a^b f (t) \, dt - (b - a) f \left( \frac{a + b}{2} \right)
\]

\[
\leq \frac{1}{8} (b - a)^2 \left[ f'_- (b) - f'_+ (a) \right]
\]

and the constant $\frac{1}{8}$ is sharp in both inequalities.

We may state now the following result for convex functions improving Hermite-Hadamard integral inequalities.
Proposition 1. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a convex function on \([a, b]\). Then

\[
0 \leq \frac{1}{8} (b - a) \left[ f_+ \left( \frac{a + b}{2} \right) - f_- \left( \frac{a + b}{2} \right) \right] - \frac{1}{2a - b} \int_a^b f(t) \, dt - f \left( \frac{a + b}{2} \right) \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t) \, dt \leq \frac{1}{8} (b - a) \left[ f_+ (b) - f_+ (a) \right].
\]  

(3.7)

The constant \( \frac{1}{8} \) is sharp in both parts.

If one would assume that \( x \in (a, b) \) is a differentiability point for \( f \), then we have the inequality [7]

\[
(b - a) \left( \frac{a + b}{2} - x \right) f'(x) \leq \int_a^b f(t) \, dt - (b - a) f(x).
\]  

(3.8)

(b) If we choose \( \alpha = \frac{a + x}{2} \) and \( \beta = \frac{x + b}{2} \), then by (3.4) we have the three point inequality:

\[
0 \leq \frac{1}{8} \left\{ (x - a)^2 \left[ f_+ \left( \frac{a + x}{2} \right) - f_- \left( \frac{a + x}{2} \right) \right] + (b - x)^2 \left[ f_+ \left( \frac{x + b}{2} \right) - f_- \left( \frac{x + b}{2} \right) \right] \right\} \leq \frac{1}{2} \left\{ (x - a) f(a) + f(x) (b - a) + (b - x) f(b) \right\} - \int_a^b f(t) \, dt \leq \frac{1}{8} \left\{ (x - a)^2 \left[ f_+ (x) - f_- (a) \right] + (b - x)^2 \left[ f_+ (b) - f_+ (x) \right] \right\}
\]  

(3.9)

for any \( x \in (a, b) \). The constant \( \frac{1}{8} \) is sharp in both parts.

If in (3.9) we choose \( x = \frac{a + b}{2} \), then we get

\[
0 \leq \frac{1}{32} (b - a)^2 \left[ f_+ \left( \frac{3a + b}{4} \right) - f_- \left( \frac{3a + b}{4} \right) \right] + f_+ \left( \frac{a + 3b}{4} \right) - f_- \left( \frac{a + 3b}{4} \right) \leq \frac{1}{2} \left\{ f(a) + f(b) \right\} (b - a) - \int_a^b f(t) \, dt \leq \frac{1}{32} (b - a)^2 \left[ f_+ (b) - f_+ \left( \frac{a + b}{2} \right) \right] + f_+ \left( \frac{a + b}{2} \right) - f_+ (a)
\]  

(3.10)
If one would assume that $f$ is differentiable in $\frac{a+b}{2}$, then we get the following reverse of Bullen’s inequality

$$0 \leq \frac{1}{2} \left[ f\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) \right] (b-a) - \int_a^b f(t) \, dt \quad (3.11)$$

$$\leq \frac{1}{32} (b-a)^2 \left[ f'_-(b) - f'_+(a) \right].$$

The constant $\frac{1}{32}$ is sharp.

(c) Now, if we choose $\alpha = \frac{5a+b}{6}$, $\beta = \frac{a+5b}{6}$ and $x \in \left[ \frac{5a+b}{6}, \frac{a+5b}{6} \right]$ in (3.4), then we have the inequalities

$$\frac{1}{2} \left[ \left( x - \frac{5a+b}{6} \right)^2 f'_+ \left( \frac{5a+b}{6} \right) - \frac{(b-a)^2}{36} f'_- \left( \frac{5a+b}{6} \right) \right] + \frac{(b-a)^2}{36} f'_+ \left( \frac{a+5b}{6} \right) - \left( \frac{a+5b}{6} - x \right)^2 f'_- \left( \frac{a+5b}{6} \right) \leq \frac{b-a}{3} \left[ f\left(\frac{a+b}{2}\right) + 2f(x) \right] - \int_a^b f(t) \, dt$$

$$\leq \frac{1}{2} \left[ \left( x - \frac{5a+b}{6} \right)^2 f'_- (x) - \frac{(b-a)^2}{36} f'_+ (a) + \frac{(b-a)^2}{36} f'_- (b) - \left( \frac{a+5b}{6} - x \right)^2 f'_+ (x) \right].$$

If in (3.12) we choose $x = \frac{a+b}{2}$, then we get the Simpson’s inequality

$$\frac{1}{18} (b-a)^2 \left[ f'_+ \left( \frac{5a+b}{6} \right) - \frac{1}{4} f'_- \left( \frac{5a+b}{6} \right) + \frac{1}{4} f'_+ \left( \frac{a+5b}{6} \right) - f'_- \left( \frac{a+5b}{6} \right) \right] \leq \frac{b-a}{3} \left[ f\left(\frac{a+b}{2}\right) + 2f \left( \frac{a+b}{2} \right) \right] - \int_a^b f(t) \, dt$$

$$\leq \frac{1}{18} (b-a)^2 \left[ f'_- \left( \frac{a+b}{2} \right) - \frac{1}{4} f'_+ (a) + \frac{1}{4} f'_- (b) - f'_+ \left( \frac{a+b}{2} \right) \right].$$
If the function is differentiable on \((a, b)\), then we get
\[
-\frac{1}{24} (b - a)^2 \left[ f' \left( \frac{a + 5b}{6} \right) - f' \left( \frac{5a + b}{6} \right) \right] \leq \frac{b - a}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] - \int_a^b f(t) \, dt \leq \frac{1}{72} (b - a)^2 \left[ f'-(b) - f'-(a) \right]
\] (3.14)

References