On the L^2-pointwise Regularity of Functions in Critical Besov Spaces

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ABSTRACT

We show that the functions in $L^2(\mathbf{R}^n)$ given by the sum of infinitely sparse wavelet expansions are regular, *i.e.* belong to $C^\infty_{L^2}(x_0)$, for all $x_0 \in \mathbf{R}^n$ which is outside a set of vanishing Hausdorff dimension.

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1. Introduction

Let $2^{nj/2}\psi(2^j x - k)$, $\psi \in F$, $k \in \mathbf{Z}^n$, $j \in \mathbf{Z}$ be an orthonormal wavelet basis, where F is a finite set of $2^n - 1$ wavelets in the Schwartz class.

The purpose of this paper is to study the pointwise regularity of functions

$$f(x) = \sum_{\psi\in F}\sum_j\sum_k \alpha_{j,k}2^{nj/2}\psi(2^j x - k), \quad \text{with} \quad \sum_j\sum_k |\alpha_{j,k}|^p < \infty \quad \forall p > 0. \tag{1}$$

This assumption on the sequence $(\alpha_{j,k})$ $(j \in \mathbf{Z},\ k \in \mathbf{Z}^n)$ can be expressed by saying that its non-increasing rearrangement $\alpha_0^* \geq \alpha_1^* \geq \cdots$ has a fast decay at infinity: $\alpha_\ell^* \leq C_q \ell^{-q}$ for $q = 0, 1, 2, \ldots$

Condition (1) is equivalent to

$$f \in \bigcap_{p>0} \dot{\mathbf{B}}_p^{n(1/p-1/2),p}$$

(see [1], [8]), where $\dot{\mathbf{B}}_q^{s,p}$ ($s \in \mathbf{R}$, $0 < p, q \leq \infty$) denotes the homogeneous Besov space on $\mathbf{R}^n$.

As it is well known (see [6], [7]) the smoothness of a function is closely related to the decay speed of its rearranged wavelet coefficients. In a different context, it was shown in [6] that if $g(x) = \sum_{\psi \in F} \sum_j \sum_k \alpha_{j,k} \psi(2^j x - k)$, where F and $(\alpha_{j,k})$ are as above, then the pointwise Hölder exponent of g at x is $+\infty$, for all $x \in \mathbf{R}^n$ and outside a set of vanishing Hausdorff dimension.Observe that here wavelets are normalized in $\mathbf{L}^\infty$ norm. In terms of Besov spaces, we have in this case $g \in \bigcap_{p>0} \dot{\mathbf{B}}_p^{n/p,p}$. As it is easily checked, this implies that g belongs to the Wiener algebra and, in particular, g is continuous in $\mathbf{R}^n$ and vanishes at infinity.

The situation is quite different in our case. Indeed, we have the following theorem:

Theorem 1.1. *There exists a sparse wavelet series $f(x)$ which is nowhere continuous and fulfils* (1)*.*

The proof is quite simple. Let $(x_j)_{j \in \mathbf{N}}$, be dense in $\mathbf{R}^n$ with $x_j = k_j 2^{-j}$, $k_j \in \mathbf{N}^n$. Let

$$f(x) = \sum_1^\infty \exp(-(\log j)^2) 2^{nj/2} \psi(2^j(x - x_j)).$$

Then f is nowhere continuous. Indeed, if f were continuous at y, one would have $|\alpha_{j,k}| = o(1)$ as $2^{-j} + |y - k2^{-j}| \to 0$. Here $\alpha_{j,k} = \exp(-(\log j)^2) 2^{nj/2}$ tends to $+\infty$ as $j \to +\infty$.

For that reason, we cannot expect to prove regularity in a sense which is governed by L^∞ norms. One should instead use L^2 norms (this is quite natural, since the wavelet are normalized with respect to this norm).

We denote by $[\cdot]$ the entire part and $|B(x_0, r)|$ the volume of the ball of radius r centered at x_0. For all $s \in \mathbf{R}$ and $1 \leq q \leq \infty$ we denote by $T_s^q(x_0)$ the space of functions f which satisfy the following conditions: there exists a constant $C > 0$ and a polynomial P of degree $d^o P \leq [s]$ ($P = 0$ if $s < 0$) such that:

$$\left(\int_{B(x_0,r)} |f - P|^q \, d\mu \right)^{1/q} \leq C r^s, \qquad \text{for all } 0 < r \leq 1$$

where the Lebesgue measure μ on the ball is $dx/|B(x_0, r)|$ (with the usual generalization if $q = \infty$ Then $T_s^\infty(x_0)$ is the Hölder space $\dot{\mathbf{C}}^s(x_0)$).

These conditions contrarily to the classical Hölder-type conditions, are stable under various operators (fractional integration, singular integral transformations) and were first introduced by Calderón and Zygmund, in their study of solutions of elliptic partial differential equations, in order to obtain pointwise estimates of solutions and their derivatives (see [3]). Their new results come after they notice and show that a differential operator is in some sense a composition of a fractional differentiation with a singular integral transformation (see [2]).

The L^2 based criterion for regularity is given by $\|f - P\|_{L^2(B(x_0,r))} \le Cr^s$, i.e. $f \in T_s^2(x_0)$.

Remark. $T_s^q(x_0) \subset T_s^{q\prime}(x_0)$ if $q\prime \le q$. Therefore the condition for $q = 2$ is weaker than the condition for $q = \infty$ (the usual Hölder condition).

We can now state our main result:

Theorem 1.2. *If a function f satisfies* (1)*, then there exists a set $E^* \subset \mathbf{R}^n$ with Hausdorff dimension $\dim_{\mathrm{H}}(E^*) = 0$ such that, for all $x_0 \notin E^*$, $f \in T_s^2(x_0)$ for all $s \ge 0$.*

The proof of this Theorem will be given in the following section.

2. Proof of the main result

Theorem 1.2 is an immediate consequence of the following result, which gives in addition some information on the pointwise regularity of elements in Besov spaces.

For the sake of simplicity, we write $\mathbf{B}_p$ instead of $\dot{\mathbf{B}}_p^{n(1/p-1/2),p}$.

Theorem 2.1. *Let $0 < p \le 1$ and $-n/2 \le s$. Then for all $f \in \mathbf{B}_p$, there exists an exceptional set E_f, with Hausdorff dimension $\dim_H(E_f) \le p(s + n/2)$, such that $\forall x_0 \notin E_f$, $f \in T_s^2(x_0)$.*

Remark first that $\mathbf{B}_p$ is included in $\mathbf{B}_2 = \mathbf{L}^2(\mathbf{R}^n)$ (for $p \le 2$) and that $\mathbf{L}^2(\mathbf{R}^n)$ is included in $T_s^2(x_0)$ for $s \le -n/2$.

This proof is a straightforward adaptation of [4].

Proof of Theorem 2.1. The conclusion for $s = -n/2$ follows directly from the embeddings we just mention, thus we can take $s > -n/2$. To prove this result, we fix an integer $N > \max\{s, n(1/p - 1/2)\}$ and we use an orthonormal basis $2^{nj/2}\psi(2^j x - k)$ of compactly supported wavelets with regularity N. The support of ψ is included in $[-M, M]^n$.

Then $f \in \mathbf{B}_p$ if and only if $f(x) = \sum_j \sum_k c_{j,k}\psi_{j,k}(x)$ where $\psi_{j,k}(x) = 2^{nj/2}\psi(2^j x - k)$ and the wavelet coefficients of f satisfy

$$\|f\|^p \equiv \sum_j \sum_k |c_{j,k}|^p < \infty.$$

Here $\|f\|$ denotes the quasi-norm of $\mathbf{B}_p$. Observe that $f(x)$ and $\lambda^{n/2} f(\lambda x)$ have the same quasi-norm for all $\lambda > 0$.

The exceptional set E_f will be the union of two sets V^σ and W^σ which we now define. The norm of $x \in \mathbf{R}^n$ will be $|x| = \sup(|x_1|, \dots, |x_n|)$. Let $\beta = \sigma^{-1}$, $\sigma = s+n/2$ and

$$V_j^\sigma = \bigcup_k B(k2^{-j}, |c_{j,k}|^\beta).$$

We set

$$V^\sigma = \limsup_{j\to+\infty} V_j^\sigma.$$

If $x_0 \notin V^\sigma$, there exists j_0 such that $x_0 \notin V_j^\sigma$, for all $j \geq j_0$. This means that

$$|x_0 - k2^{-j}| > |c_{j,k}|^\beta \qquad (j \geq j_0, \quad k \in \mathbf{Z}^n),$$

i.e.,

$$|c_{j,k}| < |x_0 - k2^{-j}|^\sigma, \qquad \forall j \geq j_0, \qquad \forall k \in \mathbf{Z}^n. \tag{2}$$

Lemma 2.2. *We have* $\dim_H V^\sigma \leq p\sigma$*.*

Indeed, V_j^σ is covered by the balls $B(k2^{-j}, |c_{j,k}|^\beta)$, and we have

$$\sum_j \sum_k |c_{j,k}|^{\beta p \sigma} = \sum_j \sum_k |c_{j,k}|^p < \infty.$$

The conclusion directly follows from the definition of the Hausdorff dimension.

The second set W^σ is constructed as follows. Let us denote by Λ_j the collection of dyadic cubes λ of side length 2^{-j}, and let $\Lambda = \bigcup_j (\Lambda_j)$. For every $\lambda \in \Lambda_j$, 4λ is the cube with same center, and side length $4 \cdot 2^{-j}$.

Let

$$\sigma(\lambda) = \sum_{\lambda' \subset 4\lambda} |c(\lambda')|^p$$

and M_j be the collection of $\lambda \in \Lambda_j$ such that

$$\sigma(\lambda) \geq 2^{-p\sigma j}.$$

Finally,

$$W_j^\sigma = \bigcup_{\lambda \in M_j} (\lambda) \qquad \text{and} \qquad W^\sigma = \limsup_{j\to\infty} W_j^\sigma.$$

Lemma 2.3. *With the setting above we have the majoration:* $\dim_H W^\sigma \leq p\sigma$*.*

Indeed,

$$\sum_{\lambda \in \Lambda_j} \sigma(\lambda) = \sum_{\lambda' \in \Lambda} \sum_{\substack{\lambda \in \Lambda_j \\ 4\lambda \supset \lambda'}} |c(\lambda')|^p.$$

But for every λ' there exist at most 4^n cubes $\lambda \in \Lambda_j$ such that $\lambda' \subset 4\lambda$. So,

$$\sum_{\lambda\in\Lambda_j} \sigma(\lambda) = \sum_{\lambda'\in\Lambda} \sum_{\substack{\lambda\in\Lambda_j \\ 4\lambda\supset\lambda'}} |c(\lambda')|^p \leq 4^n \sum_{\lambda\in\Lambda} |c(\lambda)|^p.$$

Finally, for any given j we have

$$\sum_{\lambda\in\Lambda_j} \sigma(\lambda) \leq C'.$$

This obviously gives, for a given j,

$$\sharp M_j \leq 2^{p\sigma j} C'.$$

Since the side length of every $\lambda \in M_j$ is 2^{-j}, it follows that $\dim_H W^\sigma \leq p\sigma$ as announced.

We finally set

$$E_f = V^\sigma \cup W^\sigma$$

and we show that for every $x_0 \notin E_f$, we have

$$f \in T_s^2(x_0). \tag{3}$$

To prove (3), we split the wavelet expansion of f into three components, $f = f_0 + f_1 + f_2$. Here f_0 is the approximation $E_{j_0}(f)$ wich is evidently of class C^N, where N is the regularity of the used wavelet. Obviously this term satisfies (3) if $s \leq N$.

The second term f_1 is the part of the wavelet expansion which is inside the "cone of influence":

$$f_1(x) = \sum_{j\geq j_0} \sum_{|x_0 - k2^{-j}| \leq 4M2^{-j}} c_{j,k} 2^{nj/2} \psi(2^j x - k).$$

The following Lemma obviously implies that $f_1 \in T_s^2(x_0)$.

Lemma 2.4. *We have $f_1 \in \dot{\mathbf{C}}^s(\mathbf{R}^n)$ for every $x_0 \notin V^\sigma$.*

Proof. We use (2) and $|x_0 - k2^{-j}| \leq 4M2^{-j}$. It implies $|c_{j,k}|2^{nj/2} \leq C''2^{-js}$ and Lemma 2.4 follows. □

Then we treat

$$f_2(x) = \sum_{j\geq j_0} \sum_{|x_0 - k2^{-j}| > 4M2^{-j}} c_{j,k} 2^{nj/2} \psi(2^j x - k).$$

We now define the integer J by

$$2^{-J-1} \leq r < 2^{-J}$$

and the integer a by

$$2^a \leq M < 2^{a+1}.$$

Then we have the following:

Lemma 2.5. *If the support of $\psi(2^j \cdot -k)$ has a non-empty intersection with $B(x_0, r)$ and if $|x_0 - k2^{-j}| > 4M2^{-j}$, then $j \geq J + a + 2$ and $\lambda(j,k) \subset 4Q_J$, where Q_J is the dyadic cube of side 2^{-J} containing x_0.*

Proof. The assumptions are $|x - k2^{-j}| \leq 2^{-j}M$, $|x - x_0| \leq 2^{-J}$ and $|x_0 - k2^{-j}| > 4M2^{-j}$. But $|x_0 - k2^{-j}| \leq |x_0 - x| + |x - k2^{-j}| \leq 2^{-j}M + 2^{-J}$. It yields $3M2^{-j} < 2^{-J}$ and $j \geq J + a + 2$. Then

$$|x_0 - k2^{-j}| \leq 2^{-j}M + 2^{-J} \leq 2^{-j+a} + 2^{-J} \leq \frac{5}{4}2^{-J} \tag{4}$$

and the conclusion of Lemma 2.5 follows. □

Our last Lemma is immediate:

Lemma 2.6. *If $x_0 \notin W^\sigma$, then there exists an integer j_0 such that if $j \geq j_0$ and $x_0 \in \lambda(j,k)$ then $\sigma(\lambda(j,k)) \leq 2^{-pj\sigma}$.*

We can now finish the proof of Theorem 2.1.
We have

$$\begin{aligned}
\|f_2\|_{L^2(B(x_0,r))} &\leq \sum_{j \geq J+a+2} \sum_{|x_0 - k2^{-j}| > 4M2^{-j}} |c_{j,k}| \, \|\psi_{j,k}\|_{L^2(B(x_0,r))} \\
&\leq \sum_{j \geq J+a+2} \sum_{4M2^{-j} < |x_0 - k2^{-j}| \leq 5\, 2^{-J}/4} |c_{j,k}| \\
&\leq \sup_{|x_0 - k2^{-j}| \leq \frac{5}{4}2^{-J}} |c_{j,k}|^{1-p} \sum_{\lambda(j,k) \subset 4Q_J} |c_{j,k}|^p \\
&\leq C2^{-\sigma J(1-p)} 2^{-pJ\sigma} = C2^{-\sigma J}.
\end{aligned}$$

In the second and third inequalities we used Lemma 2.5 and (4). In the last inequality we used that $x_0 \notin V^\sigma$, $x_0 \notin W^\sigma$, (2) and Lemma 2.6.
Since $\sigma = \frac{n}{2} + s$, Theorem 2.1 is established. □

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