Dotted Links, Heegaard Diagrams, and Colored Graphs for PL 4-manifolds

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ABSTRACT
The present paper is devoted to establish a connection between the 4-manifold representation method by dotted framed links (or—in the closed case—by Heegaard diagrams) and the so called crystallization theory, which visualizes general PL-manifolds by means of edge-colored graphs.

In particular, it is possible to obtain a crystallization of a closed 4-manifold $M^4$ starting from a Heegaard diagram $(\#_m(S^1 \times S^2), \omega)$, and the algorithmicity of the whole process depends on the effective possibility of recognizing $(\#_m(S^1 \times S^2), \omega)$ to be a Heegaard diagram by crystallization theory.

Key words: PL-manifold, handle-decomposition, dotted framed link, crystallization.

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1. Introduction

The classical way to understand the structure of a closed orientable PL 4-manifold $\tilde{M}^4$ is to analyze its handle-decomposition

$$\tilde{M}^4 = H^{(0)} \cup (H^{(1)}_1 \cup \cdots \cup H^{(1)}_{m_1}) \cup (H^{(2)}_1 \cup \cdots \cup H^{(2)}_{m_2}) \cup (H^{(3)}_1 \cup \cdots \cup H^{(3)}_{m_3}) \cup H^{(4)}$$

where each $p$-handle ($p \in \{0, 1, 2, 3, 4\}$) $H^{(p)} = \mathbb{D}^p \times \mathbb{D}^{4-p}$ is added to the union $W$ of the previous handles by means of an attaching map $h : \partial \mathbb{D}^p \times \mathbb{D}^{4-p} \to \partial W$. Moreover,
since the attachment of 3- and 4-handles is essentially performed in a unique way, up to PL-homeomorphisms (see [19] and [17]), the attention may be restricted to handles of index \( p \leq 2 \).

Thus, according to [19], any closed orientable PL 4-manifold may be represented by means of a Heegaard diagram \((\#_{m_1}(S^1 \times S^2), \omega)\), where \( \omega \) denotes a framed link in \((\#_{m_1}(S^1 \times S^2) = \partial(H^{(0)} \cup (H^{(1)}_1 \cup \cdots \cup H^{(1)}_{m_1}))\) corresponding to the attaching instructions for the 2-handles. Note that a pair \((\#_{m_1}(S^1 \times S^2), \omega)\) is said to be a Heegaard diagram if and only if the result of attaching 2-handles along \( \omega \) to the handlebody \( \Sigma_{m_1}^4 = H^{(0)} \cup (H^{(1)}_1 \cup \cdots \cup H^{(1)}_{m_1}) \) is a (bounded) 4-manifold whose boundary is a connected sum of \( m_3 \geq 0 \) copies of \( S^1 \times S^2 \), but no general criterion exists to test whether this happens or not.

In an analogous but less restrictive way, César de Sá introduced in [9] the notion of dotted framed link in order to identify any bounded PL 4-manifold \( M^4 = H^{(0)} \cup (H^{(1)}_1 \cup \cdots \cup H^{(1)}_{m_1}) \cup (H^{(2)}_1 \cup \cdots \cup H^{(2)}_{m_2}) \). Actually, in [9], the term “special framed link” is used, instead of “dotted framed link”; however, the original term has also a different meaning—as it happens in [3] and [4]—and we prefer to avoid confusion. In short, by a dotted framed link \((L^d, c)\), we mean a framed link consisting of \( m_1 \) unknotted and unlinked 0-framed dotted components (which correspond to hypothetic 2-handles giving rise to the same boundary as the 1-handles) and of \( m_2 \) framed components (which correspond to the actual 2-handles). Obviously, if \( \partial M^4 = \#_{m_3}(S^1 \times S^2) \), the dotted framed link uniquely determines the closed 4-manifold \( M^4 = M^4 \cup \Sigma^4_{m_3} \); hence, in this case, having a dotted framed link is perfectly equivalent to having a Heegaard diagram.

The aim of the present paper is to establish a connection between the 4-manifold representation method by dotted framed links (or equivalently—in the closed case—by Heegaard diagrams) and the so-called crystallization theory, which visualizes general PL-manifolds by means of edge-colored graphs (see [11], [1], [5], [10], [14], [16], [22], . . . ).

In particular, the following subsequent constructions are obtained in sections 3 and 4 respectively.

**Construction 1.** If \((L^d, c)\) is any dotted framed link corresponding to a bounded PL 4-manifold \( M^4 = M^4(L^d, c) \), we describe an algorithmic way to construct from \((L^d, c)\) a 5-colored graph \( \Lambda(L^d, c) \) representing \( M^4 \) (see Theorem 3.5).

Note that the boundary \( \partial \Lambda(L^d, c) = \Lambda(L^d, c) \) of the 5-colored graph \( \Gamma(L^d, c) \) turns out to be a 4-colored graph representing the closed orientable 3-manifold \( M^3(L^d, c) = \partial M^4(L^d, c) \) obtained from \( S^3 \) by Dehn surgery along the framed link underlying \((L^d, c)\).

**Construction 2.** If \( M^3 = \partial M^4 = \#_{m_3}(S^1 \times S^2) \) (i.e. if \((L^d, c)\) determines a closed 4-manifold \( M^4(L^d, c) = M^4 \cup \Sigma^4_{m_3} \)), then it is always possible to yield from \( \Lambda(L^d, c) \) a 5-colored graph \( \Lambda(L^d, c) \) representing \( M^4(L^d, c) \) (see Theorem 4.8). In particular,
if the 4-colored graph $\Lambda(L^{(d)}, c)$ does satisfy suitable combinatorial conditions (which are known to imply $M^3 = \partial M^4 = \#_{m_3}(S^1 \times S^2)$) the passage from $\Lambda(L^{(d)}, c)$ to $\Lambda(L^{(d)}, c)$ is nothing but a boundary identification (see Proposition 4.2).

Unfortunately, $\partial M^4 = \#_{m_3}(S^1 \times S^2)$ is not always sufficient to satisfy the required conditions, as proved in Proposition 4.6. This facts yields a counterexample to a conjecture stated in [16] (see Corollary 4.7).

In other words, the present paper shows how to obtain a crystallization of the closed 4-manifold $M^4$ starting from a Heegaard diagram $(\#_{m_1}(S^1 \times S^2), \omega)$, and the algorithmicity of the whole process depends on the effective possibility of recognizing $(\#_{m_1}(S^1 \times S^2), \omega)$ to be a Heegaard diagram by crystallization theory.

2. Framed links and crystallizations of simply connected 4-manifolds

Throughout the work, a framed link is intended to be a pair $(L, c)$, where $L = L_1 \cup \cdots \cup L_l$ is a link in $S^3$ with $l \geq 1$ components and $c = (c_1, \ldots, c_l)$ is an $l$-tuple of integers. According to a wide and well-established literature ([15], [18], . . . ), any framed link $(L, c)$ uniquely represents a simply-connected bounded PL 4-manifold $M^4 = M^4(L, c)$, which is obtained from the 4-disk $D^4$ by adding 2-handles along the framed link $(L, c)$:

$$M^4 = M^4(L, c) = D^4 \cup (H_1^{(2)} \cup \cdots \cup H_l^{(2)})$$

where the attaching map $f_i : S^1 \times D^2 \to \partial D^4$ of the $i$-th 2-handle $H_i^{(2)}$ ($i \in \{1, \ldots, l\}$) is such that $f_i(S^1 \times \{0\}) = L_i$ has linking number $c_i$ with $f_i(S^1 \times \{x\})$, for every $x \in D^2 - \{0\}$. Moreover, the boundary of $M^4(L, c)$ is the 3-manifold $M^3 = M^3(L, c)$ which is obtained from $S^3$ by performing a Dehn surgery on $(L, c)$.

Recently, in [7], the above representation of (3- and) 4-manifolds by framed links has been put in closed connection with “crystallization theory”: in fact, an edge-colored graph $\Lambda(L, c)$ representing $M^4(L, c)$ is easily obtained from any planar diagram of the link itself.

In order to describe the construction of $\Lambda(L, c)$, it is necessary to assume the link $L$ embedded in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, so that its projection $P$ on the plane $\pi : \mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$ consists of all regular points, and $m$ double points $p_1, \ldots, p_m$ (the crossings of $L$); thus, $\pi - P$ results to have exactly $m + 2$ connected components, which are called the regions of $L$. Actually, both the crossings and the regions ought to be referred to a planar diagram of $L$; however, the assumptions about space position allow us to identify the link $L$ and its planar diagram on $\pi$.

If an orientation is fixed on each component $L_i$ of $L$ (with $i \in \{1, 2, \ldots, l\}$), then $L_i$ is said to have writhe $w(L_i)$, where $w(L_i)$ is the algebraic sum of the signs (computed by the rule of Fig. 1) of all the (self-)crossings of $L_i$. Moreover, if $(L, c)$ is a framed link...
link, then the component $L_i$ of $L$ (with $i \in \{1, 2, \ldots, l\}$) is said to need $t_i = |c_i - w(L_i)|$ additional curls, positive or negative according to whether $c_i$ is greater or less than $w(L_i)$ (see Fig. 2).
The following rules allow us to construct a 4-colored graph $\Lambda(L,c)$ directly from $(L,c)$.

(i) For every crossing $p_j$ of $L$, construct a partial order eight graph, in the following way:

(ii) For every additional curl, construct one of the following partial order four graphs:

if the curl is a positive one

if the curl is a negative one

(iii) Finally, connect the “hanging” 0- and 1-colored edges, so that every region of $L$ (having $r$ crossings in its boundary) gives rise to a $\{1,2\}$-colored cycle of length $2r$, and every component of $L$ (having $s$ crossings and $t$ additional curls) gives rise to two $\{0,3\}$-colored cycles of length $2(s + t)$.

It is not difficult to check that (by possibly adding trivial pairs of opposite additional curls) each component $L_i$ of $L$ gives rise in $\Lambda(L,c)$ to a subgraph $Q^{(i)}$ (a quadri-color) with the following structure: $Q^{(i)}$ consists of four vertices $P_0^{(i)}, P_1^{(i)}, P_2^{(i)}, P_3^{(i)}$ and four edges $e_0^{(i)}, e_1^{(i)}, e_2^{(i)}, e_3^{(i)}, e_r^{(i)}$ being an $r$-colored edge between $P_r^{(i)}$ and $P_{r+1}^{(i)}$. 

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for every $r \in \mathbb{Z}_3$, with the condition that $P_r^{(i)}$ does not belong to the \{r + 1, r + 2\}-colored cycle containing $P_{r+1}^{(i)}, P_{r+2}^{(i)}, P_{r+3}^{(i)}$.

Now, let $\tilde{\Lambda}(L, c)$ be the 5-colored graph directly obtained from the 4-colored graph $\Lambda(L, c)$ by substituting each quadricolor $Q_i$ ($i \in \{1, \ldots, l\}$) with the order ten 5-colored subgraph depicted in Fig. 3. The following result summarizes the meaning of the above described constructions:

**Proposition 2.1 ([7]).** For every framed link $(L, c)$, the 5-colored graph $\tilde{\Lambda}(L, c)$ represents the simply connected 4-manifold $M^4(L, c)$. Moreover, $\tilde{\Lambda}(L, c)$ admits as its boundary graph (see [11] for details) the 4-colored graph $\Lambda(L, c)$, which represents the 3-manifold $M^3(L, c)$.

**Example 2.2.** If $(L, (0, 0))$ is the 0-framed Hopf link (depicted in Fig. 4(a), then the associated 4-colored graph $\Lambda(L, (0, 0))$ (resp. 5-colored graph $\tilde{\Lambda}(L, (0, 0))$) is shown in Fig. 4(b) (resp. Fig. 4(c)); by Proposition 2.1, it represents $M^3 = S^3$ (resp. $M^4 = S^2 \times S^2 - D^4$).

For the purpose of the present work, it is necessary to give a hint of the proof for Proposition 2.1. First, we have to recall some fundamental notions and terminology of crystallization theory; for a much more detailed account, we refer to [11], where a useful bibliography may also be found.
Figure 4
An \((n + 1)\)-colored graph is a pair \((\Gamma, \gamma)\), where \(\Gamma = (V(\Gamma), E(\Gamma))\) is a multigraph (i.e. multiple edges are allowed, while loops are forbidden) and \(\gamma : E(\Gamma) \to \Delta_n = \{0, 1, \ldots, n\}\) is an edge-coloration, with \(\gamma(e) \neq \gamma(f)\) for every pair \(e, f\) of adjacent edges; moreover, the vertices of \(V(\Gamma)\) may have either degree \(n + 1\) (internal vertices) or \(n\) (boundary vertices), and in this last case no incident edge can be colored by \(n + 1\).

Within crystallization theory, each \((n + 1)\)-colored graph \((\Gamma, \gamma)\) is thought of as a visualizing tool for an \(n\)-dimensional labeled pseudocomplex (see [13]) \(K(\Gamma)\), which is constructed according to the following rules:

\(\text{(i)}\) For each vertex \(v \in V(\Gamma)\), take an \(n\)-simplex \(\sigma(v)\), with its vertices labeled by \(0, 1, \ldots, n\).

\(\text{(ii)}\) For each \(j\)-colored edge between \(v\) and \(w\) (\(v, w \in V(\Gamma)\)), identify the \((n-1)\)-faces of \(\sigma(v)\) and \(\sigma(w)\) opposite to the vertex labeled by \(j\), so that equally labeled vertices coincide.

If \(K(\Gamma)\) triangulates a PL \(n\)-manifold \(M^n\), then \((\Gamma, \gamma)\) is said to represent \(M^n\); in particular, an \((n + 1)\)-colored graph representing the \(n\)-manifold \(M^n\) (with empty or connected boundary) is called a crystallization of \(M^n\), in case the subgraph \(\Gamma_j = (V(\Gamma), \gamma^{-1}(\Delta_n - \{j\}))\) is connected, for each \(j \in \Delta_n\). A basic result of the theory (known as the Pezzana Theorem) states that every PL \(n\)-manifold admits both \((n+1)\)-colored graphs and crystallizations representing it.

Now, we point out that the construction of \(K(\Gamma)\) allows us to easily prove that an \((n + 1)\)-colored graph \((\Gamma, \gamma)\) represents a bounded (resp. closed) \(n\)-manifold if and only if the \(n\)-colored subgraph \(\Gamma_j\) represents a disjoint union of copies of \(\mathbb{S}^n\) for \(j = n\), and a disjoint union of copies of either \(\mathbb{S}^n\) or \(\mathbb{D}^n\) for every \(j \in \Delta_{n-1}\) (resp. a disjoint union of copies of \(\mathbb{S}^n\), for every \(j \in \Delta_n\)).

In particular, for every framed link \((L, c)\), the subgraph \(\widehat{\lambda}_1(L, c)\) of \(\widehat{\lambda}(L, c)\) may be proved to represent a colored triangulation \(K(L, c) = K(\widehat{\lambda}_1(L, c))\) of \(\mathbb{S}^3\), whose 1-skeleton contains two copies \(L' = L_1' \cup \cdots \cup L_{l_1}'\), \(L'' = L_1'' \cup \cdots \cup L_{l_2}''\) of \(L = L_1 \cup \cdots \cup L_l \subset \mathbb{S}^3\). Further, the linking number between \(L_i'\) and \(L_i''\) in \(K(L, c)\) is equal to \(c_i\), for every \(i \in \{1, \ldots, l\}\).

More precisely, according to notations of Fig. 3, the copy \(L_i'\) (resp. \(L_i''\)) of the \(i\)-th component \(L_i\) of \(L\) (for every \(i \in \{1, \ldots, l\}\)) consists of the two \(\{0, 3\}\)-labeled edges (resp. \(\{1, 2\}\)-labeled edges) of tetrahedra \(\sigma(R^{(i)}_2\), \(\sigma(R^{(i)}_2)\) of \(K(L, c)\), having both the same \(\{0, 1\}\)-labeled edge and the same \(\{2, 3\}\)-labeled edge. Thus, \(L_i'\) and \(L_i''\) turn out to be two different longitudes of the same solid torus embedded in \(K(L, c)\), i.e. the subcomplex consisting of tetrahedra \(\sigma(R^{(i)}_2)\) and \(\sigma(R^{(i)}_2)\), for \(r \in \{1, 2, 3\}\).

At this point, it is not difficult to understand the PL-structure of the 4-dimensional pseudocomplex—\(\hat{K}(L, c)\), say—associated to \(\hat{\lambda}(L, c)\): since \(\hat{K}(L, c)\) is directly obtained from the cone over \(K(L, c)\) (i.e. a 4-disk \(\mathbb{D}^4\)) by pairwise identification of tetrahedra \(\sigma(R^{(i)}_r)\) and \(\sigma(R^{(i)}_r)\), for \(r \in \{1, 2, 3\}\) and \(i \in \{1, \ldots, l\}\), \(\hat{K}(L, c)\) admits the
handle-decomposition $\mathbb{D}^4 \cup H_1^{(2)} \cup \cdots \cup H_i^{(2)}$, with attaching maps $f_i : S^1 \times \mathbb{D}^2 \to \partial \mathbb{D}^4$ (for every $i \in \{1, \ldots, l\}$) satisfying $f_i(S^1 \times \{0\}) = L'_i$ and $f_i(S^1 \times \{x\}) = L''_i$, for some $x \in \mathbb{D}^2 - \{0\}$. This obviously implies that $\Lambda(L, c)$ represents $M^4(L, c)$, as the first part of Proposition 2.1 states. On the other hand, $\Lambda(L, c)$ exactly coincides with the boundary graph $\partial \Lambda(L, c)$ of $\Lambda(L, c)$. In fact, by construction, $\partial \Lambda(L, c)$ has a vertex for every boundary vertex of $\tilde{\Lambda}(H)$ and satisfying the following conditions:

(i) The vertices $v$ and $w$ belong to different connected components, $\Xi_1$ and $\Xi_2$ say, of the graph $\Gamma_{\Delta_n - \{j_1, \ldots, j_h\}} = (V(\Gamma), \gamma^{-1}(\Delta_n - \{j_1, \ldots, j_h\}))$.

(ii) If either $v$ or $w$ is an internal vertex, then either $\Xi_1$ or $\Xi_2$ is a regular graph of degree $n + 1 - h$.

The elimination of the $h$-dipole $\Theta$ is performed by deleting $\Theta$ from $(\Gamma, \gamma)$ and welding the “hanging” pairs of edges of the same color $j \in \Delta_n - \{j_1, \ldots, j_h\}$; if $(\Gamma', \gamma')$ is the resulting $(n + 1)$-colored graph (with $K(\Gamma') = K(\Gamma) = M^n$), then we will also say that $(\Gamma, \gamma)$ is obtained from $(\Gamma', \gamma')$ by insertion of an $h$-dipole of colors $\{j_1, j_2, \ldots, j_h\}$ and that $(\Gamma, \gamma)$ and $(\Gamma', \gamma')$ are obtained from each other by a dipole move.

3. From dotted framed links to crystallizations of bounded 4-manifolds

The starting point for the notion of dotted framed link is the fact that 1-handles in orientable 4-manifolds may be “traded for” 2-handles (see [9] and [18]).

In short, if the orientable 4-manifold $W^4_2$ is obtained from $W^4$ by adding a 1-handle $H^{(1)}$ and if $H^{(2)}$ is the complementary handle of $H^{(1)}$ in $W^4_2$, then $W^4_2 = W^4 \cup H^{(1)}$ has the same boundary as $W^4_2 = W^4 \cup \tilde{H}^{(2)}$, where $\tilde{H}^{(2)}$ is the 2-handle dual to $H^{(2)}$ in $W^4$. Moreover, the surgery instructions for the 2-handle $\tilde{H}^{(2)}$ always corresponds to an unknotted 0-framed circle in $\partial W^4$. 

Hence, if a bounded PL 4-manifold admits a handle-decomposition consisting of $m_1$ 1-handles and $m_2$ 2-handles (i.e. $M^4 = H^{(0)} \cup (H_1^{(1)} \cup \cdots \cup H_{m_1}^{(1)}) \cup (H_1^{(2)} \cup \cdots \cup H_{m_2}^{(2)})$), then it may be represented by an $(m_1 + m_2)$-component link in $S^3 = \partial H^{(0)}$, with $m_1$ unknotted and unlinked dotted 0-framed components (which correspond to traded 1-handles) and $m_2$ (possibly knotted and linked) framed components (which correspond to the surgery instructions for the actual 2-handles). If $(\mathcal{L}^{(d)}, c)$ is such a dotted framed link, the present section is devoted to describing an algorithmic way to construct a 5-colored graph representing the associated 4-manifold $M^4 = M^4(\mathcal{L}^{(d)}, c)$.

A first, minimal step is carried out using the following result.

**Proposition 3.1.** Let $(K_0^{(d)}, 0)$ be the dotted framed link consisting of a unique dotted component (i.e. $(K_0^{(d)}, 0)$ is the 0-framed dotted trivial knot). Then, the 5-colored graph $\tilde{\Lambda}(K_0^{(d)}, 0)$ depicted in Fig. 5 represents the 4-manifold $S^1 \times D^3 = M^4(K_0^{(d)}, 0)$ and admits the same boundary graph as the 5-colored graph $\tilde{\Lambda}(K_0, 0)$ associated to the underlying framed link (i.e. the 0-framed trivial knot $(K_0, 0)$).

**Proof.** It is very easy to check that the subgraph $\{H, H'\}$ of $\tilde{\Lambda}(K_0^{(d)}, 0)$ is a 2-dipole; moreover, the elimination of $\{H, H'\}$ gives rise to the standard 5-colored graph representing $S^3 \times D^3$ (see, for example, [2, Theorem 3 (iii)]). On the other hand, the last part of the statement immediately follows by direct construction of the boundary graph.

Another important step is due to the characteristic structure of graphs $\tilde{\Lambda}(\mathcal{L}, c)$.
In order to describe it, we need further definitions and results from crystallization theory.

Definition. Let \((\Gamma', \gamma')\) and \((\Gamma'', \gamma'')\) be two \((n+1)\)-colored graphs and let \(v' \in V(\Gamma')\) and \(v'' \in V(\Gamma'')\) be two internal (resp. boundary) vertices; moreover, let \(\Gamma' \#_{\{v', v''\}} \Gamma''\) be the \((n+1)\)-colored graph obtained from \(\Gamma'\) and \(\Gamma''\) by deleting \(\{v', v''\}\) and welding the “hanging” edges of the same color \(c \in \Delta_n\) (resp. \(c \in \Delta_{n-1}\)). The process leading from \(\Gamma'\), \(\Gamma''\) to \(\Gamma' \#_{\{v', v''\}} \Gamma''\) is said to be a graph connected sum, while the process leading from \(\Gamma' \#_{\{v', v''\}} \Gamma''\) to the disjoint union of \(\Gamma'\) and \(\Gamma''\) is said to be an inverse of a graph connected sum.

Proposition 3.2 ([2]). If \(\Gamma'\) and \(\Gamma''\) represent two \(n\)-manifolds \(M^n_1\) and \(M^n_2\), and if \(v'\) and \(v''\) are internal (resp. boundary) vertices, then \(\Gamma' \#_{\{v', v''\}} \Gamma''\) represents the \(n\)-manifold \(M^n_1 \# M^n_2\) (resp. \(M^{n-\theta}_1 \# M^{n-\theta}_2\)), where \(\#\) (resp. \(\#\)) is the symbol of connected sum (resp. boundary connected sum).

Let now assume \((L, c)\) is a given framed link, with \(l \geq 2\) components, and let \((L^{(l)}, c^{(l)})\) be the (possibly disconnected) framed link obtained by deleting the last component (i.e. \(L^{(l)} = L_1 \cup \cdots \cup L_{l-1}\) and \(c^{(l)} = (c_1, c_2, \ldots, c_{l-1})\)).

Proposition 3.3. Let \(\Lambda^{(l)}(L, c)\) be the \(5\)-colored graph obtained from \(\Lambda(L, c)\) by deleting the \(4\)-colored edges between \(R_r^{(l)}\) and \(R_r^{(l)}\), for \(r \in \{1, 2, 3\}\); then, \(\Lambda^{(l)}(L, c)\) represents the simply connected \(4\)-manifold associated to the framed link \((L^{(l)}, c^{(l)})\) (or the boundary connected sum of the associated \(4\)-manifolds, in case \((L^{(l)}, c^{(l)})\) has a disconnected planar projection). Moreover, a finite sequence of graph moves exists, which consists of dipole eliminations and possibly inverses of graph connected sums, that transforms \(\Lambda^{(l)}(L, c)\) into the possibly disconnected graph \(\Lambda(L^{(l)}, c^{(l)})\) (resp. \(\partial \Lambda^{(l)}(L, c)\) into the possibly disconnected graph \(\Lambda(L^{(l)}, c^{(l)})\)).

Proof. Obviously, the first part of the statement is a consequence of the last one, via Proposition 3.2. On the other hand, the \(5\)-colored graph \(\Lambda^{(l)}(L, c)\) immediately appears to contain five \(2\)-dipoles (i.e. the \(2\)-dipoles \(\theta_{(l)} = \{P_{(l)}^{(l)}, R_{(l)}^{(l)}\}, \theta_{(l)} = \{P_{(l)}^{(l)} + 1, R_{(l)}^{(l)}\}, \theta_{(l)} = \{P_{(l)}^{(l)}, R_{(l)}^{(l)}\}, \theta_{(l)} = \{P_{0}^{(l)} + 1, R_{(l)}^{(l)}\}, \theta_{(l)} = \{P_{0}^{(l)}, R_{(l)}^{(l)}\}\), whose eliminations make the quadricolor \(Q^{(l)}\) to disappear. Further, the required sequence of graph moves may be easily completed, by simply “following” the subgraph of \(\Lambda(L, c)\) (resp. of \(\Lambda(L, c)\)) corresponding to the \(l\)-th component of \(L\).

Let now \((L^{(d)}, c)\) be a dotted framed link. Without loss of generality, we may order the \(l = m_1 + m_2\) (with \(m_1, m_2 > 0\)) components of \(L\), so that \(L_i\) becomes unknotted, unlinked, dotted and \(0\)-framed, for every \(i \in \{1, \ldots, m_1\}\). If \(\Lambda(L, c)\) is the \(5\)-colored graph associated to the underlying framed link \((L, c)\), set

\[
\Lambda^{(d)}(L, c) = \Lambda^{(m_1+1) \cdots (m_1+m_2)}(L, c).
\]
This means that $\tilde{\Lambda}^{(d)}(L, c)$ is obtained from $\tilde{\Lambda}(L, c)$ by deleting the 4-colored edges corresponding to the undotted components of $(L^{(d)}, c)$.

Since $L_i = K_0$ and $c_i = 0$ hold for every $i \in \{1, \ldots, m_1\}$, Proposition 3.3 directly yields the following

**Corollary 3.4.** With the above notations, we have

(i) The 5-colored graph $\tilde{\Lambda}^{(d)}(L, c)$ represents $\partial \# m_1 (S^2 \times D^2)$.

(ii) A well-determined sequence of graph moves exists, which consists of a finite number of dipole eliminations and exactly $m_1 - 1$ inverses of graph connected sums, and transforms $\partial \tilde{\Lambda}^{(d)}(L, c)$ into $\bigsqcup_{m_1} \Lambda(K_0, 0)$ (i.e. the disjoint union of $m_1$ copies of the 4-colored graph associated to the 0-framed trivial knot).

We are now able to prove the existence of the already stated algorithmic procedure (Construction 1).

**Theorem 3.5.** Let $(L^{(d)}, c)$ be a dotted framed link and $(L, c)$ the underlying framed link. Then, there is an algorithm for constructing a 5-colored graph $\tilde{\Lambda}(L^{(d)}, c)$ such that:

(i) The graph $\tilde{\Lambda}(L^{(d)}, c)$ represents the 4-manifold $M^4(L^{(d)}, c)$, obtained from $D^4$ by adding 1-handles and 2-handles according to $(L^{(d)}, c)$.

(ii) Its boundary graph $\partial \tilde{\Lambda}(L^{(d)}, c)$ is exactly $\Lambda(L, c)$.

**Proof.** First, let us state how to construct $\tilde{\Lambda}(L^{(d)}, c)$.

**Step 1:** Consider the disjoint union $\bigsqcup_{m_1} \tilde{\Lambda}(K_0^{(d)}, 0)$ of $m_1$ copies of the 5-colored graph of Fig. 5, having $\bigsqcup_{m_1} \Lambda(K_0, 0)$ as boundary graph.

**Step 2:** By Corollary 3.4 and [8, Lemma B], a well-determined sequence of graph moves exists, which consists of a finite number of dipole insertions and exactly $m_1 - 1$ graph connected sums, and transforms $\bigsqcup_{m_1} \tilde{\Lambda}(K_0^{(d)}, 0)$ into a 5-colored graph $\Omega(L^{(d)}, c)$ of $\partial \# m_1 (S^3 \times D^3) = Y^4_{m_1}$, having the same boundary as $\tilde{\Lambda}^{(d)}(L, c)$.

**Step 3:** $\tilde{\Lambda}(L^{(d)}, c)$ is simply obtained from $\Omega(L^{(d)}, c)$ by adding a 4-colored edge between $R_r^{(i)}$ and $R_r^{(i)}$, for every $r \in \{1, 2, 3\}$ and for every $i \in \{m_1 + 1, \ldots, m_1 + m_2\}$.

Note that the aim of step 2 is to reproduce on 5-colored graphs (starting from $\bigsqcup_{m_1} \tilde{\Lambda}(K_0^{(d)}, 0)$, whose boundary graph coincides with $\bigsqcup_{m_1} \Lambda(K_0, 0)$) the inverse sequence of moves on 4-colored graphs described in Corollary 3.4. Obviously, no problem arises from graph connected sums (see Proposition 3.2). On the other hand, if $\Phi$ is an $(n + 1)$-colored graph representing an $n$-manifold $M^n$ and if $\Gamma$ is obtained from $\partial \Phi$ by inserting a dipole $\Theta$ within the colored subgraph $\Xi$, then [8, Lemma B] indicates how to obtain another graph $\tilde{\Phi}$ of $M^n$, with $\partial \tilde{\Phi} = \Gamma$. If the dipole $\Theta$ cannot be directly inserted in $\Phi$, then it may be inserted within the so called “double-layer” over $\Xi$ (which may be added to $\Phi$ by a finite sequence of dipole insertions: see [8] for details).
Let us now consider the 5-colored graph $\Omega(L^{(d)}, c)$. By construction, it really represents $\mathbb{H}_{4,1} = H^{(0)} \cup (H^{(1)}_1 \cup \cdots \cup H^{(1)}_{m_1})$ and has the same boundary as $\Lambda^{(d)}(L, c)$. Thus, for every $i \in \{m_1 + 1, \ldots, m_1 + m_2\}$, the addition of the 4-colored edges between $R^{(i)}_r$ and $R^{(i)}_r$ for $r \in \{1, 2, 3\}$, has the topological effect of adding a 2-dipole according to the surgery instructions corresponding to the $i$-th (undotted) component of $(L^{(d)}, c)$. (Recall the hint of proof for Proposition 2.1 given in the second section.)

Example 3.6. If $(L^{(d)}, c)$ is the dotted framed link depicted in Fig. 6(a), then Construction 1 allows us to algorithmically construct the 5-colored graph $\tilde{\Lambda}(L^{(d)}, c)$ of Fig. 6(b). Note that it has the same boundary graph as the 5-colored graph $\Lambda(L, (0, 0))$ shown in Fig. 4(c) (i.e. the 4-colored graph $\Lambda(L, (0, 0))$ shown in Fig. 4(b)). Moreover, the link calculus for 4-manifolds (see [9] or [19]) ensures that $\Lambda(L^{(d)}, c)$ represents the
4-disk $D^4$.

4. From Heegaard diagrams to crystallizations of closed 4-manifolds

The present section takes into account the case of a dotted framed link $(L^{(d)}, c)$ such that the 3-manifold represented by its underlying framed link is a connected sum of $m_3 \geq 0$ copies of $S^1 \times S^2$ (i.e. $(L^{(d)}, c)$ such that $\partial M^4(L^{(d)}, c) = M^3(L, c) = \# m_3(S^1 \times S^2)$, where $\# m_3(S^1 \times S^2)$ is intended to indicate the 3-sphere $S^3$, in case $m_3 = 0$). As already pointed out in the introduction, such a dotted framed link uniquely represents the closed 4-manifold $\bar{M}^4 = M^4(L^{(d)}, c) \cup Y^4_{m_3}$; in other words—according to [19]—$(L^{(d)}, c)$ turns out to be equivalent to a Heegaard diagram $(\# m_1(S^1 \times S^2), \omega)$ of $M^4$.

Unfortunately, known results about the characterization of $S^3$ and/or $\# m_3(S^1 \times S^2)$ (see [20] and [21]) are not so useful for concrete applications, both to crystallization theory and to other classical representation methods for 3-manifolds. However, the following combinatorial structures within 4-colored graphs yield interesting information about the associated 3-manifolds.

Definition. Let $(\Gamma, \gamma)$ be a 4-colored graph representing a closed orientable 3-manifold $M^3$.

(i) Two $i$-colored edges $e, f \in E(\Gamma)$ ($i \in \Delta_3$) are said to be a $\rho_2$-pair (resp. a $\rho_3$-pair) if $e$ and $f$ belong both to the same $\{i, j\}$-colored cycle and to the same $\{i, k\}$-colored cycle of $\Gamma$, with $j, k \in \Delta_3 - \{i\}$ (resp. to the same $\{i, c\}$-colored cycle of $\Gamma$, for every $c \in \Delta_3 - \{i\}$).

The switching of the $\rho_2$-pair (resp. $\rho_3$-pair) is the local process depicted in Fig. 7.

(ii) Four distinctly colored edges $e_0, e_1, e_2, e_3 \in E(\Gamma)$ are said to be a handle if they pairwise belong to the same bicolored cycle.

The breaking of the handle is the local process depicted in Fig. 8.

Remark. It is very easy to check that every $\rho_3$-pair implies the existence of a handle, too (see the captions of Fig. 7). On the contrary, if $(\Gamma, \gamma)$ contains a handle, another 4-colored graph containing a $\rho_3$-pair of color $i$ may be obtained by inserting a 1-dipole of color $i$ (see Fig. 9).

Proposition 4.1 ([16]). Let $(\Gamma, \gamma)$ be a 4-colored graph representing a closed orientable 3-manifold $M^3$.

(i) If $(\Gamma', \gamma')$ is obtained from $(\Gamma, \gamma)$ by switching a $\rho_2$-pair (resp. $\rho_3$-pair), then $|K(\Gamma')| = |K(\Gamma)| = M^3$ (resp. $|K(\Gamma)| = M^3 = |K(\Gamma')|\#(S^1 \times S^2)$).
Figure 7

Figure 8

Figure 9
(ii) If the 4-colored graph \((\Gamma', \gamma')\) obtained from \((\Gamma, \gamma)\) by breaking a handle is connected, then \(|K(\Gamma)| = M^3 = |K(\Gamma')|\#(S^1 \times S^2)|.

Remark. In case the 4-colored graph \((\Gamma', \gamma')\) obtained from \((\Gamma, \gamma)\) by breaking a handle consists of two connected components \(\Gamma_1'\) and \(\Gamma_2'\), then \(\Gamma = \Gamma_1' \# \Gamma_2'\); thus, according to Proposition 3.2, \(|K(\Gamma)| = |K(\Gamma_1')|\#|K(\Gamma_2')|\).

The following results allow us to algorithmically construct a 5-colored graph \(\Lambda(L(d), c)\) representing the closed 4-manifold \(\bar{\Lambda}(\bar{S}_4^3)\) could be recognized as being a Heegaard diagram via \(\rho_3\)-pairs and/or handles in the 4-colored graph \(\Lambda(L, c)\).

**Proposition 4.2.** Let us assume \(\Lambda(L, c)\) contains \(m_3\) \(\rho_3\)-pairs of color \(i\) \((i \in \Delta_3)\), whose switching yields a 4-colored graph \(H\) representing \(S^3\). Then, \(\Lambda(L(d), c)\) is obtained from \(\Lambda(L(d), c)\) by simply adding a 4-colored edge for every pair of \(i\)-adjacent vertices in \(H\).

**Proof.** By Proposition 4.1(i), the hypothesis implies \(\partial M^4(L(d), c) = M^3(L, c) = \#m_3(S^1 \times S^2)\).

In order to prove the statement, we have to consider the described regular 5-colored graph \(\tilde{\Lambda}\) \(\Lambda(L(d), c)\) and to check that it represents the unique closed 4-manifold \(M^4 = M^4(L(d), c) \cup Y^4_{m_3}\).

First of all, we construct a 5-colored graph \(\tilde{H}\) by applying the following procedure to the graph \(H\) (thought of as a 5-colored graph with boundary, representing \(D^4\)): for each \(\rho_3\)-pair \(\{e_r, f_r\} \quad (r \in \{1, \ldots, m_3\})\) in \(\Lambda = \Lambda(L, c)\), insert a 3-dipole \(\Theta_r = \{X_r, Y_r\}\) of colors \(\Delta_3 - \{i\}\) and add a 4-colored edge, as indicated in Figs. 10(a), 10(b). By 2. Theorem 3 (iii)], it is easy to check that the resulting 5-colored graph \(\tilde{H}\) represents a 4-dimensional handlebody \(Y^4_{m_3}\) of genus \(m_3\); moreover, the boundary graph \(\partial \tilde{H}\) exactly coincides with \(\partial \Lambda(L(d), c) = \Lambda\).

Now, let \(\tilde{H}\) be the regular 5-colored graph obtained from \(\tilde{H}\) by adding a 4-colored edge for every pair of \(i\)-adjacent vertices in \(H\) (see Fig. 10(c)). Note that, for every \(r \in \{1, \ldots, m_3\}\), \(\{X_r, Y_r\}\) are joined in \(\tilde{H}\) by three edges (colored by \(\Delta_3 - \{i\}\)), but belonging to the same \(\{i, 4\}\)-colored cycle of \(H\); hence, by 2. Theorem 14 (b’)], the 5-colored graph \(\tilde{H}'\) obtained from \(\tilde{H}\) by deleting \(\{X_r, Y_r\}\) and by welding the “hanging” edges of the same color \(c \in \{i, 4\}\), is such that \(|K(\tilde{H})| = \#m_3(S^1 \times S^2)\#|K(\tilde{H}')|\).

Moreover, since \(\tilde{H}'\) is obtained from the 4-colored graph \(H\) (representing \(S^3\)) by adding a parallel 4-colored edge for every \(i\)-colored edge, then \(|K(H')| = S^4\) easily follows (see [12, section 4], where the notion of “suspension graph” is introduced and analyzed).

Hence, the passage from \(\tilde{H}\) to \(H\) has the topological effect of transforming \(|K(\tilde{H})| = Y^4_{m_3}\) into \(|K(H)| = \#m_3(S^1 \times S^2)\). This means that the identification of tetrahedra of \(K(\tilde{H})\) associated to \(i\)-adjacent vertices in \(H\) corresponds to the unique
(see [19]) PL-homeomorphism \( \phi : \# m_3(S^1 \times S^2) \to \# m_3(S^1 \times S^2) \) giving rise to the attaching map for 3- and 4-handles.

Finally, since \( K(\bar{\Lambda}) \) is obtained from \( K(\tilde{\Lambda}) \) by means of the same identification of boundary tetrahedra, \( |K(\bar{\Lambda})| = |K(\tilde{\Lambda})| \cup \gamma_{m_3}^4 \) directly follows.

**Example 4.3.** If \((K(d)_0, 0)\) is the 0-framed dotted trivial knot, then it is very easy to check that the 5-colored graph \( \tilde{\Lambda}(K(d)_0, 0) \) depicted in Fig. 5 (and representing \( S^1 \times D^3 = \gamma_4^1 \)) satisfies the hypothesis of Proposition 4.2, with \( m_3 = 1 \) and \( i = 1 \). Hence, Construction 2 may be easily performed, by a boundary identification. The resulting regular 5-colored graph \( \bar{\Lambda}(K(d)_0, 0) \) (representing \( \bar{M}^4(K(d)_0, 0) = \gamma_4^1 \cup \gamma_4^1 = S^1 \times S^3 \)) is shown in Fig. 11.

**Example 4.4.** If \((L(d), c)\) is the dotted framed link depicted in Fig. 6(a), then the 5-colored graph \( \tilde{\Lambda}(L(d), c) \) shown in Fig. 6(b) (and representing the 4-disk \( D^4 \)) trivially satisfies the hypothesis of Proposition 4.2, with \( m_3 = 0 \); hence, Construction 2 may be easily performed, by a boundary identification. The resulting regular 5-colored graph \( \bar{\Lambda}(L(d), c) \) (representing \( \bar{M}^4(L(d), c) = D^4 \cup D^4 = S^4 \)) is shown in Fig. 12.

**Proposition 4.5.** Let us assume \( \Lambda(L, c) \) contains \( m_3 \) handles, whose breaking yields a connected 4-colored graph representing \( S^3 \). Then, a well-determined sequence of dipole moves exists, which transforms \( \tilde{\Lambda} = \tilde{\Lambda}(L(d), c) \) into a 5-colored graph \( \bar{\Lambda} \) with the following properties:

(i) The 4-colored graph \( \partial \tilde{\Lambda} \) contains \( m_3 \rho_3 \)-pairs of color \( i \) (\( i \in \Delta_3 \)).
(ii) The 5-colored graph $\bar{\Lambda}(L^{(d)}, c)$ may be obtained by suitably adding 4-colored edges to $\tilde{\Lambda}$.

Proof. As a consequence of the Remark before Proposition 4.1, $m_3$ suitable insertions of 1-dipoles of color $i$ ($i \in \Delta_3$) into $\Lambda(L, c)$ give rise to a 4-colored graph containing $m_3 \rho_3$-pairs of color $i$. By [8, Lemma B], the above sequence of dipole insertions may be reproduced on 5-colored graphs, starting from $\Lambda(L^{(d)}, c)$ (whose boundary is exactly $\Lambda(L, c)$). Now, if $\tilde{\Lambda}$ is the resulting 5-colored graph, property (i) is satisfied by construction; on the other hand, property (ii) directly follows by making use of Proposition 4.2.

Unfortunately, the following statement proves that the assumptions of Proposition 4.2 and/or of Proposition 4.5 are not always satisfied, even if $M^3(L, c) = \#_m(S^1 \times S^2)$ is assumed to hold.

Proposition 4.6. Let $(G, g)$ be the 4-colored graph depicted in Fig. 13(b). Then:

(i) $|K(G)| = S^1 \times S^2$.

(ii) No handle is contained in $(G, g)$.

Proof. As far as statement (i) is concerned, it is sufficient to note that $(G, g) = \Lambda(L, (0, 0, 0))$, where $L$ denotes the “trivial chain with three rings” depicted in Fig. 13(a) (without additional curls). Further, part (ii) follows by direct checking.

Note that in [16, page 125] a conjecture is stated, which would imply the existence of handles in every 4-colored graph representing $S^1 \times S^2$; thus, Proposition 4.6 provides a counterexample to Lins’s conjecture:

Corollary 4.7. Conjecture 5 of [16, page 125] is false.

Let us now conclude the paper with the general theorem about Construction 2.

Theorem 4.8. Let $(L^{(d)}, c)$ be any dotted framed link representing a closed 4-manifold $M^4 = M^4(L^{(d)}, c)$ (i.e. $(L^{(d)}, c)$ such that $M^3(L, c) = \#_m(S^1 \times S^2)$). Then, a finite sequence of dipole moves exists, which transforms $\Lambda = \Lambda(L^{(d)}, c)$ into a 5-colored graph $\tilde{\Lambda}$ with the following properties:

(i) The 4-colored graph $\partial \tilde{\Lambda}$ contains $m_3 \rho_3$-pairs of color $i$ ($i \in \Delta_3$).

(ii) The 5-colored graph $\bar{\Lambda}(L^{(d)}, c)$ may be obtained by suitably adding 4-colored edges to $\tilde{\Lambda}$.

Proof. By hypothesis, the 4-colored graph $\Lambda(L, c)$ represents $M^3 = M^3(L, c) = \#_m(S^1 \times S^2)$. Obviously, if $\Lambda(L, c)$ contains $m_3 \rho_3$-pairs of color $i$ ($i \in \Delta_3$), we may set $\tilde{\Lambda} = \bar{\Lambda}(L^{(d)}, c)$. On the other hand, if $\Lambda(L, c)$ contains $m_3$ handles, the required
Figure 13
5-colored graph $\tilde{\Lambda}$ is proved to exist (and well-determined) by Proposition 4.5. Otherwise, let $(G^{(m_3)}, g^{(m_3)})$ be a fixed 4-colored graph representing $M^3 = \#_{m_3}(S^1 \times S^2)$ and containing $m_3 \rho_3$-pairs of color $i$ ($i \in \Delta_3$): for example, $(G^{(m_3)}, g^{(m_3)})$ may be obtained by considering $m_3$ copies of the standard order eight 4-colored graph representing $S^1 \times S^2$ and by performing $m_3 - 1$ graph connected sums. The Main Theorem of [6] ensures the existence of a finite sequence of dipole moves which transforms $\Lambda(L, c)$ into $(G^{(m_3)}, g^{(m_3)})$: moreover, by [8, Lemma A and Lemma B], the above sequence of dipole insertions may be reproduced on 5-colored graphs, starting from $\tilde{\Lambda}(L^{(d)}, c)$ (whose boundary is exactly $\Lambda(L, c)$). Now, if $\tilde{\Lambda}$ is the resulting 5-colored graph, property (i) is satisfied by construction, while property (ii) directly follows by making use of Proposition 4.2.

Example 4.9. If $(L^{(d)}, c)$ is the dotted framed link depicted in Fig. 14, then the associated 5-colored graph $\tilde{\Lambda}(L^{(d)}, c)$ has the 4-colored graph $\Lambda(L, c) = (G, g)$ depicted in Fig. 13(b)) as boundary graph. Since $(G, g)$ does not contain $\rho_3$-pairs, Proposition 4.2 can not be applied. Notwithstanding this, it is easy to check that a finite sequence of dipole eliminations (more precisely, the subsequent eliminations of 1-dipole $\{v_1, v_2\}$ and 2-dipoles $\{v_3, v_4\}, \{v_5, v_6\}, \{v_7, v_8\}, \{v_9, v_{10}\}$, according to the captions of Fig. 13(b)) transforms $(G, g)$ into a 4-colored graph containing a $\rho_3$-pair of color 2 (which corresponds to the pair of edges $\{e, f\}$ of $(G, g)$, according to the captions of Fig. 13(b)). Hence, by Theorem 4.8, a regular 5-colored graph $\Lambda(L^{(d)}, c)$ of the associated closed 4-manifold $M^4$ may be constructed by reproducing on $\tilde{\Lambda}(L^{(d)}, c)$ the above sequence of moves, and finally by applying Proposition 4.2. It is not difficult to check—by making use of [8, Lemma A]—that the resulting 5-colored graph $\Lambda(L^{(d)}, c)$ is simply obtained from $\tilde{\Lambda}(L^{(d)}, c)$ by adding a 4-colored edge for every pair of boundary vertices corresponding to vertices of type $\{v_i, v_{i+1}\}$ in $(G, g)$, for any odd index $i$.

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