Study of the Limit of Transmission Problems in a Thin Layer by the Sum Theory of Linear Operators

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Recibido: 19 de Abril de 2004
Aceptado: 27 de Junio de 2004

ABSTRACT

We consider a family (P_δ), where δ is a small positive parameter, of singular elliptic transmission problems in the juxtaposition $\Omega^\delta =]-1,\delta[\times G$ of two bodies, the cylindric medium $\Omega_- =]-1,0[\times G$ and the thin layer $\Omega_+^\delta =]0,\delta[\times G$. It is assumed that the coefficient in Ω_+^δ is $1/\delta$. Such problems model for instance heat propagation between the body Ω_-, the layer Ω_+^δ (when supposed with infinite conductivity), and the ambient space. After performing a rescaling in the thin layer to transform the problem in the fixed domain $]-1,1[\times G$, it is shown that the sum of operators' method by Da Prato and Grisvard works and gives an existence and uniqueness result in the framework L^p spaces, $p>1$. We deduce that the family of solutions u^δ converges in L^p to a function u in the case of second member in L^p and converges in $W^{1+2\theta,p}$ for a second member in $W^{2\theta,p}$, ($\theta \in]0,1/2[$). We then prove that the restriction of the limit u to $]-1,0[\times G$ is in fact the solution to an elliptic problem on $]-1,0[\times G$ with a boundary condition of Ventcel's type and it has an optimal regularity.

Key words: sums of linear operators, elliptic problems, interpolation spaces, Ventcel's problem.

2000 Mathematics Subject Classification: 34K06, 34K10, 34K30.

1. Introduction and main results

Consider the boundary value problems

$$(P_\delta) \quad \begin{cases} -\frac{1}{b_\delta}\operatorname{div}(b_\delta \nabla v^\delta) = g^\delta & \text{in } \Omega^\delta, \\ v^\delta = 0 & \text{on } \partial\Omega^\delta \setminus \Gamma^\delta, \\ \partial_\xi v^\delta = 0 & \text{on } \Gamma^\delta, \end{cases} \tag{1}$$

where δ is a small parameter given in $]0,1[$, Ω^δ is the cylinder $]-1,\delta[\times G$ of $\mathbb{R}^n$ in variables (ξ,η), G is a regular open domain of $\mathbb{R}^{n-1}$, $\Gamma^\delta = \{\delta\} \times G$ and b_δ is the function defined by

$$b_\delta(\xi) = \begin{cases} 1 & \text{if } \xi \in]-1,0[, \\ 1/\delta & \text{if } \xi \in]0,\delta[. \end{cases}$$

g^δ is given in $L^p(\Omega^\delta)$, $1 < p < \infty$.

This problem models, for instance, the heat propagation between the fixed body

$$\Omega_- =]-1,0[\times G,$$

the thin layer

$$\Omega_+^\delta =]0,\delta[\times G,$$

(when supposed with infinite conductivity) and the ambient space.

The main purpose of this paper will consist in solving (P_δ) for some fixed $\delta > 0$ and then in studying the convergence of the family (P_δ) when $\delta \to 0$.

If we denote by v_-^δ and g_-^δ the respective restrictions of v^δ and g^δ to Ω_-, and by v_+^δ and g_+^δ the restrictions of v^δ and g^δ to Ω_+^δ, problem (1) is equivalent to the following singular transmission problem

$$\begin{cases} \text{(eq)} \quad -\Delta v_-^\delta = g_-^\delta \text{ in } \Omega_- \quad \text{and} \quad -\Delta v_+^\delta = g_+^\delta \text{ in } \Omega_+^\delta \\[2ex] \text{(bc)} \quad v_-^\delta = 0 \text{ on } \partial\Omega_- \setminus \Gamma^0, \qquad v_+^\delta = 0 \text{ on } \partial\Omega_+^\delta \setminus \left(\Gamma^0 \cup \Gamma^\delta\right) \\ \qquad\quad \text{and} \quad \partial_\xi v_+^\delta = 0 \text{ on } \Gamma^\delta \\[2ex] \text{(tc)} \quad v_-^\delta = v_+^\delta \text{ on } \Gamma^0 \quad \text{and} \quad \partial_\xi v_-^\delta = \frac{1}{\delta}\partial_\xi v_+^\delta \text{ on } \Gamma^0, \end{cases} \tag{2}$$

where $\Gamma^0 = \{0\} \times G$ and $\partial\Omega_+^\delta$ is the boundary of Ω_+^δ. The two last transmission conditions mean that the jumps of v^δ and $b_\delta(\cdot)\partial_\xi v^\delta$ through Γ^0 are equal to 0.

Recall that $h \in W^{2\theta,p}(U)$ if and only if $h \in L^p(U)$ and

$$\begin{aligned} \|h\|^p_{W^{2\theta,p}(U)} &= \|h\|^p_{L^p(U)} + \iint_{U\times U} \frac{|h(\sigma)-h(\tau)|^p}{\|\sigma-\tau\|^{2\theta p+n}}\, d\sigma\, d\tau \\ &= \|h\|^p_{L^p(U)} + [h]_{2\theta,p,U} < \infty, \end{aligned}$$

where U is an open set of $\mathbb{R}^n$, $1 < p < \infty$ and $0 < \theta < 1/2$ (see [7]).

The main results in this work are given by the following Theorems:

Theorem 1.1. *Let $1 < p < \infty$.*

(i) *For any $g^\delta \in L^p(\Omega^\delta)$, there exists a strong solution*

$$v^\delta = \begin{cases} v_-^\delta & \text{on } \Omega_- \\ v_+^\delta & \text{on } \Omega_+^\delta \end{cases}$$

in $L^p(\Omega^\delta)$ of Problem (2).

(ii) *Moreover if $g_-^\delta \in W^{2\theta,p}(\Omega_-)$ and $g_+^\delta \in W^{2\theta,p}(\Omega_+^\delta)$ with $0 < \theta < 1/2$, then v^δ is a strict solution satisfying*

(a) $v^\delta \in L^p(-1, \delta; W^{2,p}(G) \cap W_0^{1,p}(G))$,

(b) $v_-^\delta \in W^{2,p}(-1, 0; L^p(G))$,

(c) $v_+^\delta \in W^{2,p}(0, \delta; L^p(G))$,

(d) $\Delta_\eta v^\delta \in W^{2\theta,p}(\Omega_+^\delta)$, $\partial_\xi^2 v_-^\delta \in W^{2\theta,p}(\Omega_-)$ *and* $\partial_\xi^2 v_+^\delta \in W^{2\theta,p}(\Omega_+^\delta)$.

Theorem 1.2. *Let $1 < p < \infty$. For any $g^\delta \in L^p(\Omega^\delta)$ such that*

(a) *g_-^δ tends to g_- in $L^p(\Omega_-)$ when $\delta \to 0$,*

(b) *there exists a constant $M > 0$ such that for any $\delta > 0$*

$$\frac{1}{\delta} \int_{\Omega_+^\delta} |g_+^\delta(\xi, \eta)|^p \, d\xi \, d\eta \leq M,$$

(c) *$\frac{1}{\delta} \int_0^\delta g_+^\delta(\xi, \cdot) \, d\xi = m_+^\delta$ tends to m in $L^p(G)$ when $\delta \to 0$,*

there exists a unique $(v_-, v_+) \in L^p(\Omega_-) \times L^p(G)$ and a constant C independent of δ such that

(i) $\lim_{\delta \to 0} \|v_-^\delta - v_-\|_{L^p(\Omega_-)} = 0$, *and*

$$\begin{aligned} &\|v_-^\delta - v_-\|_{L^p(\Omega_-)} \\ &\qquad \leq C \cdot \delta(\|g_-^\delta\|_{L^p(\Omega_-)} + \delta^{-1/p} \|g_+^\delta\|_{L^p(\Omega_+^\delta)}) \\ &\qquad\quad + C(\|g_-^\delta - g_-\|_{L^p(\Omega_-)} + \|m_+^\delta - m\|_{L^p(G)}), \end{aligned}$$

(ii) $\lim_{\delta \to 0} \delta^{-1/p} \|v_+^\delta - v_+\|_{L^p(\Omega_+^\delta)} = 0$, *and*

$$\begin{aligned} &\delta^{-1/p} \|v_+^\delta - v_+\|_{L^p(\Omega_+^\delta)} \\ &\qquad \leq C \cdot \delta(\|g_-^\delta\|_{L^p(\Omega_-)} + \delta^{-1/p} \|g_+^\delta\|_{L^p(\Omega_+^\delta)}) \\ &\qquad\quad + C(\|g_-^\delta - g_-\|_{L^p(\Omega_-)} + \|m_+^\delta - m\|_{L^p(G)}) \end{aligned}$$

(here, v_+ stands for the function $(\xi, \eta) \mapsto v_+(\eta)$),

(iii) v_- *is the unique strong solution of the following non homogeneous Ventcel's Problem*

$$\begin{cases} -\Delta v_- = g_- & \text{in } \Omega_-, \\ v_- = 0 & \text{on } \partial\Omega_- \setminus \Gamma^0, \\ \partial_\xi v_-(0,\cdot) - \Delta_\eta v_-(0,\cdot) = m & \text{on } G. \end{cases}$$

Theorem 1.3. *Let* $1 < p < \infty$ *and* $0 < \theta < 1/2$. *Set* $\Omega_+ =]0,1[\times G$. *For any* $g^\delta \in W^{2\theta,p}(\Omega^\delta)$, *such that*

(a) g_-^δ *tends to* g_- *in* $W^{2\theta,p}(\Omega_-)$ *when* $\delta \to 0$,

(b) *there exists a constant* $M > 0$ *such that for any* $\delta > 0$

$$\|g_+^\delta(\delta\cdot,\cdot)\|_{W^{2\theta,p}(\Omega_+)} \leq M,$$

(c) $\frac{1}{\delta}\int_0^\delta g_+^\delta(\xi,\cdot)d\xi = m_+^\delta$ *tends to* m *in* $W^{2\theta,p}(G)$ *when* $\delta \to 0$,

then $(v_-, v_+) \in W^{2,p}(\Omega_-) \times W^{2,p}(G)$ *and*

(i) *for any* $\eta \in G$

$$v_+(\eta) = v_-(0,\eta),$$

(ii) $\lim_{\delta\to 0}\|v_-^\delta - v_-\|_{W^{1+2\theta,p}(\Omega_-)} = 0$ *and*

$$\begin{aligned} \|v_-^\delta - &v_-\|_{W^{2\theta+1,p}(\Omega_-)} \\ &\leq C\cdot\delta\cdot(\|g_-^\delta\|_{W^{2\theta,p}(\Omega_-)} + \|g_+^\delta(\delta\cdot,\cdot)\|_{W^{2\theta,p}(\Omega_+)}) \\ &\quad + C(\|g_-^\delta - g_-\|_{W^{2\theta,p}(\Omega_-)} + \|m_+^\delta - m\|_{W^{2\theta,p}(G)}), \end{aligned}$$

where C *is a constant independent of* δ,

(iii) $\lim_{\delta\to 0}\|v_+^\delta(\delta\cdot,\cdot) - v_+\|_{W^{1+2\theta,p}(\Omega_+)} = 0$ *(here,* v_+ *stands for the function* $(\xi,\eta) \mapsto v_+(\eta)$*),*

(iv) v_- *is the unique strict solution of the following non homogeneous Ventcel's Problem*

$$\begin{cases} -\Delta v_- = g_- & \text{in } \Omega_-, \\ v_- = 0 & \text{on } \partial\Omega_- \setminus \Gamma^0, \\ \partial_\xi v_-(0,\cdot) - \Delta_\eta v_-(0,\cdot) = m & \text{on } G. \end{cases}$$

The paper is organized as follows.

In section 2, we perform a rescaling in the thin layer to transform Problem (2) in a problem set in the fixed domain

$$\Omega =]-1,1[\times G.$$

Section 3 contains the main results on the solution u^δ of the transmission problem in Ω and gives its regularity. In a first step we write this problem in the form of a sum of linear operators and we show that the sum theory developed by Da Prato-Grisvard [1], gives an explicit writing of the strong solution u^δ as a Dunford Integral in the L^p spaces. In the second step, we study the behavior of u^δ as $\delta \to 0$, we prove that u^δ converges in $L^p(\Omega_-)$ to a function u_- in the spaces L^p and $W^{1,p}$, generalizing the results obtained, in the Hilbertian case, by Lemrabet [4]. Moreover, in virtue of the techniques used in the study of abstract differential equations (as in [3]), we prove that u_- is solution of a boundary value problem of Ventcel's type and we precise its regularity.

Finally, in section 4 we go back to our first problem (2) in order to translate all the above results on the initial function v^δ.

2. New writing of Problem (2)

It is difficult to study directly the convergence of Problem (2) since it is given in an open set which depends on the parameter δ (the coefficients of the operator depend also on δ). In order to work in a fixed domain, we perform the following rescaling in the thin layer by setting

$$\begin{array}{rcl}\Psi^\delta : \Omega =]-1,1[\times G & \longrightarrow & \Omega^\delta =]-1,\delta[\times G \\ (x,y) & \mapsto & (\xi,\eta) = \begin{cases}(x,y) & \text{if } x \leq 0, \\ (\delta x, y) & \text{if } x \geq 0.\end{cases}\end{array}$$

Put

$$\begin{aligned} u^\delta = v^\delta \circ \Psi^\delta, \quad & f^\delta = g^\delta \circ \Psi^\delta, \\ u_-^\delta = (v^\delta \circ \Psi^\delta)_{|\Omega_-} &= v_-^\delta, \\ f_-^\delta = (g^\delta \circ \Psi^\delta)_{|\Omega_-} &= g_-^\delta, \end{aligned}$$

and

$$\begin{array}{rlcl} u_+^\delta = (v^\delta \circ \Psi^\delta)_{|\Omega_+} : & \Omega_+ =]0,1[\times G & \longrightarrow & \mathbb{R} \\ & (x,y) & \mapsto & v_+^\delta(\delta x, y) \\[2mm] f_+^\delta = (g^\delta \circ \Psi^\delta)_{|\Omega_+} : & \Omega_+ =]0,1[\times G & \longrightarrow & \mathbb{R} \\ & (x,y) & \mapsto & g_+^\delta(\delta x, y). \end{array}$$

Then (2) transforms into the problem

$$\begin{cases} \text{(eq)} & \begin{cases} -\Delta u_-^\delta = f_-^\delta & \text{in } \Omega_-, \\ -\left(\frac{1}{\delta^2}\partial_x^2 u_+^\delta + \Delta_y u_+^\delta\right) = f_+^\delta & \text{in } \Omega_+, \end{cases} \\ \text{(bc)} & \begin{cases} u_-^\delta = 0 & \text{on } \partial\Omega_- \setminus \Gamma^0, \\ u_+^\delta = 0 & \text{on } \partial\Omega_+ \setminus \left(\Gamma^0 \cup \Gamma^1\right), \\ \partial_x u_+^\delta = 0 & \text{on } \Gamma^1, \end{cases} \\ \text{(tc)} & \begin{cases} u_-^\delta = u_+^\delta & \text{on } \Gamma^0, \\ \partial_x u_-^\delta = \frac{1}{\delta^2}\partial_x u_+^\delta & \text{on } \Gamma^0, \end{cases} \end{cases} \tag{3}$$

where $\Gamma^1 = \{1\} \times G$.

3. Study of the problem in the fixed domain

The main results in this section are the following

Theorem 3.1. *Let $1 < p < \infty$.*

(i) *For any $f^\delta \in L^p(\Omega)$, there exists a strong solution*

$$u^\delta = \begin{cases} u_-^\delta & \text{on } \Omega_- \\ u_+^\delta & \text{on } \Omega_+, \end{cases}$$

in $L^p(\Omega)$ of Problem (3).

(ii) *Moreover if $f^\delta \in W^{2\theta,p}(\Omega)$ with $0 < \theta < 1/2$, then u^δ is a strict solution satisfying*

(a) $u^\delta \in L^p(-1, 1; W^{2,p}(G) \cap W_0^{1,p}(G))$,

(b) $u_-^\delta \in W^{2,p}(-1, 0; L^p(G))$,

(c) $u_+^\delta \in W^{2,p}(0, 1; L^p(G))$,

(d) $\Delta_y u^\delta \in W^{2\theta,p}(\Omega)$, $\partial_x^2 u_-^\delta \in W^{2\theta,p}(\Omega_-)$ *and* $(\frac{1}{\delta^2}\partial_x^2 u_+^\delta) \in W^{2\theta,p}(\Omega_+)$.

Theorem 3.2. *Let $1 < p < \infty$. For any $f^\delta \in L^p(\Omega)$ satisfying*

(a) *f_-^δ tends to f_- in $L^p(\Omega_-)$ when $\delta \to 0$,*

(b) *f_+^δ is bounded in $L^p(\Omega_+)$,*

(c) *$\int_0^1 f_+^\delta(x, \cdot)\, dx = m_+^\delta$ tends to m in $L^p(G)$ when $\delta \to 0$,*

then there exists a unique $(u_-, u_+) \in L^p(\Omega_-) \times L^p(G)$ such that

(i) $\lim_{\delta\to 0}(u_\pm^\delta - u_\pm) = 0$ *in* $L^p(\Omega_\pm)$ *and we have*

$$\begin{aligned}\|u_\pm^\delta - &u_\pm\|_{L^p(\Omega_\pm)}\\ &\le C\cdot\delta(\|f_-^\delta\|_{L^p(\Omega_-)} + \|f_+^\delta\|_{L^p(\Omega_+)})\\ &\quad + C(\|f_-^\delta - f_-\|_{L^p(\Omega_-)} + \|m_+^\delta - m\|_{L^p(\Omega_+)}),\end{aligned}$$

where C *is independent of* δ *(here,* u_+ *stands for the function* $(x,y)\mapsto u_+(y)$*),*

(ii) u_- *is the unique strong solution of the following non homogeneous Ventcel's Problem*

$$\begin{cases}-\Delta u_- = f_- & \text{in } \Omega_-\\ u_- = 0 & \text{on } \partial\Omega_-\setminus\Gamma^0\\ \partial_x u_-(0,\cdot) - \Delta_y u_-(0,\cdot) = m & \text{on } G.\end{cases}$$

Theorem 3.3. *Let* $1 < p < \infty$ *and* $0 < \theta < 1/2$. *For any* $f^\delta \in W^{2\theta,p}(\Omega)$ *satisfying*

(a) f_-^δ *tends to* f_- *in* $W^{2\theta,p}(\Omega_-)$ *when* $\delta\to 0$,

(b) f_+^δ *is bounded in* $W^{2\theta,p}(\Omega_+)$,

(c) $\int_0^1 f_+^\delta(x,\cdot)dx = m_+^\delta$ *tends to* m *in* $W^{2\theta,p}(G)$ *when* $\delta\to 0$,

then $(u_-,u_+)\in W^{2,p}(\Omega_-)\times W^{2,p}(G)$ *and*

(i) *for any* $y\in G$

$$u_+(y) = u_-(0,y),$$

(ii) $\lim_{\delta\to 0}(u_\pm^\delta - u_\pm) = 0$ *in* $W^{1+2\theta,p}(\Omega_\pm)$ *and*

$$\begin{aligned}\|u_-^\delta - &u_-\|_{W^{2\theta+1,p}(\Omega_-)}\\ &\le C\cdot\delta\cdot(\|f_-^\delta\|_{W^{2\theta,p}(\Omega_-)} + \|f_+^\delta\|_{W^{2\theta,p}(\Omega_+)})\\ &\quad + C(\|f_-^\delta - f_-\|_{W^{2\theta,p}(\Omega_-)} + \|m_+^\delta - m\|_{W^{2\theta,p}(G)})\end{aligned}$$

(here, u_+ *stands for the function* $(x,y)\mapsto u_+(y)$*),*

(iii) u_- *is the unique strict solution of the following non homogeneous Ventcel's Problem*

$$\begin{cases}-\Delta u_- = f_- & \text{in } \Omega_-\\ u_- = 0 & \text{on } \partial\Omega_-\setminus\Gamma^0\\ \partial_x u_-(0,\cdot) - \Delta_y u_-(0,\cdot) = m & \text{on } G.\end{cases}$$

We begin by recalling the sum theory in a commutative framework developed by Da Prato-Grisvard [1].

3.1. On the commutative sums

Let A and B be two closed linear operators in a complex Banach E, with domains $D(A)$ and $D(B)$. We assume that there exist positive constants C_A, C_B, R, ϵ_A, ϵ_B such that

$$(DG.0)\quad \begin{cases} \overline{D(A)+D(B)} = E, \\ A \text{ or } B \text{ is boundedly invertible,} \end{cases}$$

$$(DG.1)\quad \begin{cases} \rho(-A) \supset \sum_{\epsilon_A} = \{\, z \ \in \mathbb{C} : |z| \geq R \text{ and } |\arg(z)| < \epsilon_A \,\} \\ \qquad\qquad\qquad \forall\, z \in \sum_{\epsilon_A}, \quad \|(A+zI)^{-1}\|_{L(E)} \leq C_A/|z|, \\ \rho(-B) \supset \sum_{\epsilon_B} = \{\, z \ \in \mathbb{C} : |z| \geq R \text{ and } |\arg(z)| < \epsilon_B \,\} \\ \qquad\qquad\qquad \forall\, z \in \sum_{\epsilon_B}, \quad \|(B+zI)^{-1}\|_{L(E)} \leq C_B/|z|, \\ \epsilon_A + \ \epsilon_B > \pi, \\ \sigma(-A) \cap \sigma(B) = \emptyset, \end{cases}$$

$$(DG.2)\quad \begin{cases} \forall \xi \in \rho(-A), \forall \eta \in \rho(-B), \\ (A+\xi I)^{-1}(B+\eta I)^{-1} = (B+\eta I)^{-1}(A+\xi I)^{-1}, \end{cases}$$

where $\rho(-A)$ and $\rho(-B)$ are the resolvent sets of $-A$ and $-B$, $\sigma(-A)$ and $\sigma(B)$ are the spectra of $-A$ and B.

Due to Da Prato-Grisvard [1] we have

Theorem 3.4. *Under assumptions* (DG.0), (DG.1) *and* (DG.2)*, the sum* $A+B$ *is closable and* $0 \in \rho\big(\overline{A+B}\big)$.

The inverse of $\overline{A+B}$ is given by a Dunford integral

$$u = \big(\overline{A+B}\big)^{-1} f = \frac{-1}{2i\pi} \int_\Gamma (B-zI)^{-1}(A+zI)^{-1} f \, dz,$$

where Γ is a sectorial curve separating $\sigma(-A)$ and $\sigma(B)$ and lying in $\rho(-A) \cap \rho(B)$. The function u is called the strong solution of the equation $Au + Bu = f$.

Theorem 3.5. *Assume* (DG.0), (DG.1), (DG.2)*, and* $f \in D_B(\theta,q)$ *(resp.* $f \in D_A(\theta,q)$*),* $0 < \theta < 1$*,* $1 \leq q \leq +\infty$*. Then* $u \in D(A) \cap D(B)$ *and* $Au, Bu \in D_B(\theta,q)$ *(resp.* $Au, Bu \in D_A(\theta,q)$*).*

For $\theta \in]0,1[$ and $q \in [1,\infty]$, the real interpolation space $D_B(\theta,q)$ is characterized by

$$D_B(\theta,q) = \left\{ u \in E \, : \, \int_0^\infty t^{\theta q} \|B(B+tI)^{-1}u\|_E^q \frac{dt}{t} < \infty \right\}$$

with the usual modification for $q = \infty$.

3.2. Proof of Theorem 3.1

It is essentially a direct application of the sum theory.

Set $E = L^p(\Omega) = L^p(-1, 1; L^p(G))$ and define A and B_δ by

$$D(A) = L^p(-1,\ 1; D(Q)), \qquad (Aw)(x) = Q(w(x)),\ w \in D(A),$$

where Q is the realization in $X = L^p(G)$ characterized by

$$D(Q) = W^{2,p}(G) \cap W_0^{1,p}(G), \quad (Q\psi)(y) = -\Delta_y \psi(y), \quad \psi \in D(Q), \tag{4}$$

and

$$\begin{aligned} D(B_\delta) = \{\, & w^\delta \in L^p(-1,1;X) \,:\, w_-^\delta \in W^{2,p}(-1,0;X), \quad w_+^\delta \in W^{2,p}(0,1;X), \\ & w_-^\delta(-1) = 0, \quad w_-^\delta(0) = w_+^\delta(0), \quad \partial_x w_-^\delta(0) = (1/\delta^2)\partial_x w_+^\delta(0) \text{ and } \partial_x w_+^\delta(1) = 0 \,\} \end{aligned}$$

$$(B_\delta w^\delta)(x) = -\partial_x(a_\delta(x)\partial_x w^\delta)(x), \quad x \in (-1,1), \qquad w^\delta \in D(B_\delta),$$

with

$$a_\delta(x) = \begin{cases} 1 & \text{if } x \in]-1,0[, \\ 1/\delta^2 & \text{if } x \in]0,1[. \end{cases}$$

So, problem (3) writes

$$Au^\delta + B_\delta u^\delta = f^\delta.$$

Proposition 3.6. *The operators A and B_δ are linear closed with dense domains in E and satisfy assumptions* (DG.0), (DG.1), *and* (DG.2).

Proof. Step 1. For $\rho = \sqrt{\lambda}$ with $\operatorname{Re}\rho > 0$, equation $Aw + \lambda w = g \in E$ is equivalent, for a.e. $x \in (-1,1)$, to

$$\begin{cases} -\Delta_y \psi_x(y) + \lambda\psi_x(y) = g_x(y) & \text{in } G, \\ \psi_x = 0 & \text{on } \partial G, \end{cases}$$

where $\psi_x = w(x,\cdot)$ and $g_x = g(x,\cdot) \in L^p(G)$. We know that this equation has a unique solution $\psi_x(\cdot) \in W^{2,p}(G) \cap W_0^{1,p}(G)$ and that there exist $\epsilon_0 \in]0,\pi/2[$ and $M_Q > 0$ such that

$$\|(Q+\lambda)^{-1}\|_{L(X)} \le \frac{M_Q}{1+|\lambda|},$$

for any λ in the set

$$\{\, \lambda \in \mathbb{C}^* \,:\, |Arg(\lambda)| < (\pi - \epsilon_0) = \epsilon_Q \,\} \cup b(0, r_Q),$$

where $b(0, r_Q)$ is some open ball with sufficiently small radius $r_Q = r_A > 0$. We deduce the same spectral properties for the operator A.

Step 2. For B_δ, we solve the equation $B_\delta w^\delta + \lambda w^\delta = h \in E$ with $\sqrt{\lambda} = \rho > 0$. It can be written

$$\begin{cases} -\partial_x^2 w_-^\delta(x) + \lambda\, w_-^\delta(x) = h_-(x), \quad x \in (-1,0), \\ -\frac{1}{\delta^2}\partial_x^2 w_+^\delta(x) + \lambda\, w_+^\delta(x) = h_+(x), \quad x \in (0,1), \\ \\ w_-^\delta(-1) = 0, \quad \partial_x w_+^\delta(1) = 0, \\ w_-^\delta(0) = w_+^\delta(0), \quad \partial_x w_-^\delta(0) = \frac{1}{\delta^2}\partial_x w_+^\delta(0). \end{cases}$$

Then we obtain

$$\begin{aligned} w_-^\delta(x) &= \int_{-1}^0 H_{\rho,\delta}^-(x,s) h_-(s)\,ds + \int_0^1 \frac{\delta \sinh \rho(x+1)}{\rho \Delta_0(\rho,\delta)} \cosh \rho\delta(1-s) \cdot h_+(s)\,ds \\ &= \int_{-1}^0 H_{\rho,\delta}^-(x,s) h_-(s)\,ds + \int_0^1 N_{\rho,\delta}^-(x,s) h_+(s)\,ds \\ &= (K_{\rho,\delta}^- h_-)(x) + (T_{\rho,\delta}^- h_+)(x), \end{aligned}$$

$$\begin{aligned} w_+^\delta(x) &= \int_0^1 N_{\rho,\delta}^+(x,s) h_+(s)\,ds + \int_{-1}^0 \frac{\delta \cosh \rho\delta(1-x)}{\rho \Delta_0(\rho,\delta)} \sinh \rho(s+1) \cdot h_-(s)\,ds \\ &= \int_0^1 N_{\rho,\delta}^+(x,s) h_+(s)\,ds + \int_{-1}^0 H_{\rho,\delta}^+(x,s) h_-(s)\,ds \\ &= (T_{\rho,\delta}^+ h_+)(x) + (K_{\rho,\delta}^+ h_-)(x), \end{aligned}$$

where

$$H_{\rho,\delta}^-(x,s) = \begin{cases} \frac{\Delta_-(\rho,\delta,x)}{\rho\Delta_0(\rho,\delta)} \sinh \rho(s+1) & \text{if} \quad -1 < s < x, \\ \frac{\Delta_-(\rho,\delta,s)}{\rho\Delta_0(\rho,\delta)} \sinh \rho(x+1) & \text{if} \quad x \le s < 0, \end{cases}$$

$$N_{\rho,\delta}^+(x,s) = \begin{cases} \frac{\Delta_+(\rho,\delta,s)}{\rho\Delta_0(\rho,\delta)} \cosh \rho\delta(1-x) & \text{if} \quad 0 < s < x, \\ \frac{\Delta_+(\rho,\delta,x)}{\rho\Delta_0(\rho,\delta)} \cosh \rho\delta(1-s) & \text{if} \quad x \le s < 1, \end{cases}$$

and

$$\begin{aligned} \Delta_0(\rho,\delta) &= \sinh \rho \cdot \sinh \rho\delta + \delta \cosh \rho \cdot \cosh \rho\delta \\ \Delta_-(\rho,\delta,\xi) &= -\sinh \rho\xi \cdot \sinh \rho\delta + \delta \cosh \rho\delta \cdot \cosh \rho\xi \\ \Delta_+(\rho,\delta,\xi) &= \delta(\sinh \rho \cdot \cosh \rho\delta\xi + \delta \cosh \rho \cdot \sinh \rho\delta\xi). \end{aligned}$$

We verify that

$$\inf_{\rho \ge 0} \Delta_0(\rho,\delta) = \delta > 0.$$

Step 3. The function

$$w^\delta(x) = \begin{cases} w_-^\delta(x) & x \in [-1,0], \\ w_+^\delta(x) & x \in [0,1] \end{cases}$$

belongs to $D(B_\delta)$ and

$$\begin{aligned}\|w^\delta\|_{L^p(-1,1;X)} &= \|(B_\delta+\lambda)^{-1}h\|_E \\ &\le \|w_-^\delta\|_{L^p(-1,0;X)} + \|w_+^\delta\|_{L^p(0,1;X)} \\ &\le \|K_{\rho,\delta}^- h_-\|_{L^p(-1,0;X)} + \|T_{\rho,\delta}^- h_+\|_{L^p(-1,0;X)} \\ &\quad + \|K_{\rho,\delta}^+ h_-\|_{L^p(0,1;X)} + \|T_{\rho,\delta}^+ h_+\|_{L^p(0,1;X)}.\end{aligned}$$

We use the Schur interpolation lemma to estimate this norm. The symmetry of the kernel $H_{\rho,\delta}^-$ allows us to write

$$\begin{aligned}&\|K_{\rho,\delta}^-\|_{L(L^p(-1,0;X))} \\ &\quad\le \sup_{-1\le x\le 0}\left(\int_{-1}^0 |H_{\rho,\delta}^-(x,s)|\,ds\right) \\ &\quad= \sup_{-1\le x\le 0}\Bigg(\frac{\delta\cosh\rho\delta\cdot\cosh\rho x - \sinh\rho x\cdot\sinh\rho\delta}{\rho\Delta_0(\rho,\delta)}\int_{-1}^x \sinh\rho(s+1)\,ds \\ &\qquad - \sinh\rho(x+1)\int_0^x \frac{\delta\cosh\rho\delta\cdot\cosh\rho s - \sinh\rho s\cdot\sinh\rho\delta}{\rho\Delta_0(\rho,\delta)}\,ds\Bigg) \\ &\quad= \sup_{-1\le x\le 0}\Bigg(\frac{[\delta\cosh\rho\delta\cdot\cosh\rho x - \sinh\rho x\cdot\sinh\rho\delta][\cosh\rho(x+1)-1]}{\lambda\Delta_0(\rho,\delta)} \\ &\qquad + \frac{[\sinh\rho\delta\cdot\cosh\rho x - \sinh\rho\delta - \delta\cosh\rho\delta\cdot\sinh\rho x]\cdot\sinh\rho(x+1)}{\lambda\Delta_0(\rho,\delta)}\Bigg) \\ &\quad= \sup_{-1\le x\le 0}\left(\frac{\Delta_0(\rho,\delta) - [(\sinh\rho(1+x)-\sinh\rho x)\sinh\rho\delta + \delta\cosh\rho\delta\cosh\rho x]}{\lambda\Delta_0(\rho,\delta)}\right) \\ &\quad\le \frac{\Delta_0(\rho,\delta)}{\lambda\Delta_0(\rho,\delta)} \le \frac{1}{\lambda},\end{aligned}$$

and since $N_{\rho,\delta}^+$ is symmetric, we have

$$\begin{aligned}&\|T_{\rho,\delta}^+\|_{L(L^p(0,1;X))} \\ &\quad\le \sup_{0\le x\le 1}\left(\int_0^1 |N_{\rho,\delta}^+(s,x)|\,ds\right) \\ &\quad\le \sup_{0\le x\le 1}\Bigg(\frac{\delta}{\rho\Delta_0(\rho,\delta)}\cosh\rho\delta(1-x)\int_0^x [\sinh\rho\cosh\rho\delta s + \delta\cosh\rho\sinh\rho\delta s]\,ds \\ &\qquad + \delta\frac{[\sinh\rho\cosh\rho\delta x + \delta\cosh\rho\sinh\rho\delta x]}{\rho\Delta_0(\rho,\delta)}\int_x^1 \cosh\rho\delta(1-s)ds\Bigg)\end{aligned}$$

$$
\begin{aligned}
&= \sup_{0\le x\le 1}\Bigg(\frac{\cosh\rho\delta(1-x)[\sinh\rho\sinh\rho\delta x+\delta\cosh\rho\cosh\rho\delta x+\delta\cosh\rho]}{\lambda\Delta_0(\rho,\delta)}\\
&\quad+\frac{\sinh\rho\delta(1-x)[\sinh\rho\cosh\rho\delta x+\delta\cosh\rho\sinh\rho\delta x]}{\lambda\Delta_0(\rho,\delta)}\Bigg)\\
&= \sup_{0\le x\le 1}\Bigg(\frac{\sinh\rho\sinh\rho\delta-\delta\cosh\rho[\cosh\rho\delta(1-x)-\cosh\rho\delta]}{\lambda\Delta_0(\rho,\delta)}\Bigg)\\
&= \frac{\Delta_0(\rho,\delta)}{\lambda\Delta_0(\rho,\delta)}\le\frac{1}{\lambda}.
\end{aligned}
$$

For $T^-_{\rho,\delta}$ and $K^+_{\rho,\delta}$ which are not symmetric, we have

$$
\begin{aligned}
&\|T^-_{\rho,\delta}\|_{L(L^p(0,1;X),L^p(-1,0;X))}\\
&\qquad\le \sup_{-1\le x\le 0}\left(\int_0^1|N^-_{\rho,\delta}(x,s)|\,ds\right)+\sup_{0\le s\le 1}\left(\int_{-1}^0|N^-_{\rho,\delta}(x,s)|\,dx\right)\\
&\qquad\le\frac{\sinh\rho\sinh\rho\delta}{\rho^2\Delta_0(\rho,\delta)}+\frac{\delta\cosh\rho\cosh\rho\delta}{\rho^2\Delta_0(\rho,\delta)}\le\frac{1}{\lambda},
\end{aligned}
$$

and also

$$
\begin{aligned}
&\|K^+_{\rho,\delta}\|_{L(L^p(-1,0;X),L^p(0,1;X))}\\
&\qquad\le \sup_{0\le x\le 1}\left(\int_{-1}^0|H^+_{\rho,\delta}(x,s)|\,ds\right)+\sup_{-1\le s\le 0}\left(\int_0^1|H^+_{\rho,\delta}(x,s)|\,dx\right)\le\frac{1}{\lambda}.
\end{aligned}
$$

Summarizing these results we obtain

$$
\forall\lambda>0\quad\|(B_\delta+\lambda)^{-1}h\|_E\le\frac{C_1}{\lambda}\|h\|_E, \tag{5}
$$

where C_1 is a positive constant which does not depend on the parameter δ.

Step 4. An explicit calculus shows that

$$
[(B_\delta)^{-1}h](x)=\begin{cases}\phi^\delta_-(x) & x\in(-1,0),\\ \phi^\delta_+(x) & x\in(0,1),\end{cases}
$$

where

$$
\begin{aligned}
\phi^\delta_-(x)&=\int_{-1}^x(1+s)h_-(s)\,ds+(1+x)\int_x^0h_-(s)\,ds\\
&\quad+(1+x)\int_0^1h_+(s)\,ds,
\end{aligned}
$$

and

$$\phi_+^\delta(x) = \int_0^x (\delta^2 s + 1) h_+(s)\, ds + (\delta^2 x + 1) \int_x^1 h_+(s)\, ds + \int_{-1}^0 (1+s) h_-(s)\, ds.$$

We verify that there exists a positive constant C_2, which does not depend on the parameter δ, such that

$$\|(B_\delta)^{-1}\|_{L(E)} \le C_2. \tag{6}$$

So there exist positive constants $M_{B_\delta} = M_1$, $\epsilon_{B_\delta} = \epsilon_1 \in]0, \pi/2[$ and $r_{B_\delta} = r_1$ (independent of δ) such that for any λ in the set

$$\{\, \lambda \in \mathbb{C}^* \,:\, |Arg(\lambda)| < \epsilon_1 \,\} \cup b(0, r_1),$$

we have

$$\|(B_\delta + \lambda)^{-1}\|_{L(E)} \le \frac{M_1}{1 + |\lambda|}, \tag{7}$$

($b(0, r_1)$ is the open ball of central point 0 and radius r_1).

Choosing $\epsilon_0 < \epsilon_1$, we have

$$\epsilon_A + \epsilon_{B_\delta} = \pi - \epsilon_0 + \epsilon_1 > \pi.$$

Step 5. Moreover the resolvents of operators A and B_δ commute.

Hypotheses (DG.0), (DG.1) and (DG.2) are, thus, satisfied. □

So we can apply Theorems 3.4 and 3.5. It follows that:

(i) For any $f^\delta \in L^p(\Omega)$ there exists a unique $u^\delta \in D(\overline{A + B_\delta})$ strong solution of $Au + B_\delta u = f^\delta$. This means that there exists a sequence $u_n^\delta \in D(A) \cap D(B_\delta)$ such that for $n \to \infty$ we get

$$\begin{cases} u_n \to u^\delta & \text{in } L^p(\Omega), \\ -(\frac{1}{\delta^2}\partial_x^2 u_{n,+}^\delta + D_y^2 u_{n,+}^\delta) \to f_+^\delta & \text{in } L^p(\Omega_+), \\ -\Delta u_{n,-}^\delta \to f_-^\delta & \text{in } L^p(\Omega_-). \end{cases}$$

(ii) If $f_-^\delta \in W^{\theta,p}(\Omega_-)$ and $f_+^\delta \in W^{\theta,p}(\Omega_+)$ $(0 < \theta < 1/2)$, the solution u^δ belongs to $D(A) \cap D(B_\delta)$ and satisfies

$$\begin{cases} \text{(eq)} & -\Delta u_-^\delta = f_-^\delta \text{ in } \Omega_- \text{ and } -(\frac{1}{\delta^2}\partial_x^2 u_+^\delta + \Delta_y u_+^\delta) = f_+^\delta \text{ in } \Omega_+, \\ \text{(bc)} & u_-^\delta = 0 \text{ on } \partial\Omega_- \setminus \Gamma^0, u_+^\delta = 0 \text{ on } \partial\Omega_+ \setminus (\Gamma^0 \cup \Gamma^1) \text{ and } \partial_x u_+^\delta = 0 \text{ on } \Gamma^1, \\ \text{(tc)} & u_-^\delta = u_+^\delta \text{ on } \Gamma^0 \text{ and } \partial_x u_-^\delta = \frac{1}{\delta^2}\partial_x u_+^\delta \text{ on } \Gamma^0. \end{cases}$$

Note that, here, $u^\delta \in D(A) \cap D(B_\delta)$ does not imply $u^\delta \in W^{2,p}(\Omega)$.

3.3. Representation of the solution

We have seen that $\rho(B_\delta)$ contains the sector (independent of δ)

$$\{\, z \in \mathbb{C}^* \,:\, |\arg(z)| \geq \pi - \epsilon_1 \,\} \cup b(0, r_0),$$

where $r_0 = \min(r_A, r_1)$. If Γ denotes the boundary curve of this sector, oriented positively, the solution of problem

$$Au^\delta + B_\delta u^\delta = f^\delta,$$

is given by

$$u^\delta = \frac{1}{2i\pi} \int_\Gamma (B_\delta - zI)^{-1}(A + zI)^{-1} f^\delta \, dz.$$

Note that for $z \in \partial b(0, r_0)$, the resolvent $(B_\delta - zI)^{-1}$ is given in the sense of the analytic extension of B_δ^{-1} by the classical Neumann series, and, for $z \in \Gamma \setminus \partial b(0, r_0)$, $\sqrt{-z}$ is the analytic determination defined by $\operatorname{Re}\sqrt{-z} > 0$.

So we have, for $-1 \leq x \leq 0$,

$$\begin{aligned} u_-^\delta(x) = &\frac{1}{2i\pi} \int_\Gamma (Q + zI)^{-1} \big(K^-_{\sqrt{-z},\delta}(f_-^\delta)(x)\big) \, dz \\ &+ \frac{1}{2i\pi} \int_\Gamma (Q + zI)^{-1} \big(T^-_{\sqrt{-z},\delta}(f_+^\delta)(x)\big) \, dz, \end{aligned}$$

or

$$u_-^\delta = \frac{1}{2i\pi} \int_\Gamma (A + zI)^{-1} K^-_{\sqrt{-z},\delta}(f_-^\delta) \, dz + \frac{1}{2i\pi} \int_\Gamma (A + zI)^{-1} T^-_{\sqrt{-z},\delta}(f_+^\delta) \, dz,$$

and, for $0 \leq x \leq 1$,

$$\begin{aligned} u_+^\delta(x) = &\frac{1}{2i\pi} \int_\Gamma (Q + zI)^{-1} \big(K^+_{\sqrt{-z},\delta}(f_-^\delta)(x)\big) \, dz \\ &+ \frac{1}{2i\pi} \int_\Gamma (Q + zI)^{-1} \big(T^+_{\sqrt{-z},\delta}(f_+^\delta)(x)\big) \, dz, \end{aligned}$$

or

$$u_+^\delta = \frac{1}{2i\pi} \int_\Gamma (A + zI)^{-1} K^+_{\sqrt{-z},\delta}(f_-^\delta) \, dz + \frac{1}{2i\pi} \int_\Gamma (A + zI)^{-1} T^+_{\sqrt{-z},\delta}(f_+^\delta) \, dz.$$

We have used the classical notations $u_\pm^\delta(x) = u_\pm^\delta(x, \cdot)$ and $f_\pm^\delta(x) = f_\pm^\delta(x, \cdot)$.

3.4. Proof of Theorem 3.2

Recall that $X = L^p(G)$ and denote $E_- = L^p(-1, 0; X)$ and $E_+ = L^p(0, 1; X)$.

We assume in this subsection that $f^\delta \in L^p(\Omega)$ with $1 < p < \infty$ and

(F1) f_-^δ has a limit f_- in $L^p(\Omega_-)$ when $\delta \to 0$,

(F2) f_+^δ is bounded in $L^p(\Omega_+)$,

(F3) $\int_0^1 f_+^\delta(x,\cdot)\,dx = m_+^\delta$ has a limit m in X when $\delta \to 0$.

Hypothesis (F2) implies that $m_+^\delta \in X$ and from (F3) we deduce that

$$\|m_+^\delta - m\|_{E_+} \to 0, \quad \delta \to 0.$$

3.4.1. Limit of the kernels when $\delta \to 0$

The functions $\Delta_0(\rho,\delta)$ and $\Delta_-(\rho,\delta,\xi)$ are differentiable with respect to δ, so it is easy to check that, for any $x \in [-1,0]$ and any $z \in \Gamma$, $H^-_{\sqrt{-z},\delta}(x,s)$ tends to

$$H^-_{\sqrt{-z}}(x,s) = \begin{cases} \frac{\partial_\delta \Delta_-(\sqrt{-z},0,x)}{\partial_\delta \Delta_0(\sqrt{-z},0)} \cdot \frac{\sinh\sqrt{-z}(s+1)}{\sqrt{-z}} & \text{if } 1 < s < x \\ \frac{\partial_\delta \Delta_-(\sqrt{-z},0,s)}{\partial_\delta \Delta_0(\sqrt{-z},0)} \cdot \frac{\sinh\sqrt{-z}(x+1)}{\sqrt{-z}} & \text{if } x \le s < 0 \end{cases}$$

$$= \begin{cases} \frac{-\sqrt{-z}\sinh\sqrt{-z}x+\cosh\sqrt{-z}x}{\sqrt{-z}\sinh\sqrt{-z}+\cosh\sqrt{-z}} \cdot \frac{\sinh\sqrt{-z}(s+1)}{\sqrt{-z}} & \text{if } 1 < s < x \\ \frac{-\sqrt{-z}\sinh\sqrt{-z}s+\cosh\sqrt{-z}s}{\sqrt{-z}\sinh\sqrt{-z}+\cosh\sqrt{-z}} \cdot \frac{\sinh\sqrt{-z}(x+1)}{\sqrt{-z}} & \text{if } x \le s < 0. \end{cases}$$

Set

$$K^-_{\sqrt{-z}}(f_-)(x) = \int_{-1}^0 H^-_{\sqrt{-z}}(x,s) f_-(s)\,ds,$$

then

$$\begin{aligned} &K^-_{\sqrt{-z},\delta}(f_-^\delta)(x) - K^-_{\sqrt{-z}}(f_-)(x) \\ &= \int_{-1}^0 (H^-_{\sqrt{-z},\delta}(x,s) - H^-_{\sqrt{-z}}(x,s)) f_-^\delta(s)\,ds + \int_{-1}^0 H^-_{\sqrt{-z}}(x,s)(f_-^\delta(s) - f_-(s))\,ds, \end{aligned}$$

and

$$\begin{aligned} &\|K^-_{\sqrt{-z},\delta}(f_-^\delta) - K^-_{\sqrt{-z}}(f_-)\|_{E_-} \\ &\qquad \le \|K^-_{\sqrt{-z},\delta} - K^-_{\sqrt{-z}}\|_{L(E_-)} \cdot \|f_-^\delta\|_{E_-} + \frac{C}{|z|}\|f_-^\delta - f_-\|_{E_-}. \end{aligned}$$

We estimate $\|K^-_{\sqrt{-z},\delta} - K^-_{\sqrt{-z}}\|_{L(E_-)}$ using the Schur interpolation Lemma. It is enough to do it for $\sqrt{-z} = \rho > 0$. We have

$$\|K^-_{\sqrt{-z},\delta} - K^-_{\sqrt{-z}}\|_{L(E_-)} \le \sup_{-1 \le x \le 0} \left(\int_{-1}^0 |(H^-_{\rho,\delta} - H^-_\rho)(x,s)|\,ds \right),$$

and by the mean value Theorem, there exists $\delta^* \in]0,\delta[$ such that

$$(H^-_{\rho,\delta} - H^-_\rho)(x,s) = \delta \cdot \frac{\sinh \rho(1+x)}{\rho} \cdot \frac{\partial_\delta \Delta_-(\rho,\delta^*,s)\Delta_0(\rho,\delta^*) - \partial_\delta \Delta_0(\rho,\delta^*)\Delta_-(\rho,\delta^*,s)}{\Delta_0^2(\rho,\delta^*)},$$

if $-1 < s \le x$, and

$$(H^-_{\rho,\delta} - H^-_\rho)(x,s) = \delta \cdot \frac{\sinh \rho(1+s)}{\rho} \cdot \frac{\partial_\delta \Delta_-(\rho,\delta^*,x)\Delta_0(\rho,\delta^*) - \partial_\delta \Delta_0(\rho,\delta^*)\Delta_-(\rho,\delta^*,x)}{\Delta_0^2(\rho,\delta^*)},$$

if $x \le s \le 0$. But

$$\frac{\partial \Delta_-}{\partial \delta}(\rho,\delta^*,\xi)\Delta_0(\rho,\delta^*) - \frac{\partial \Delta_0}{\partial \delta}(\rho,\delta^*)\Delta_-(\rho,\delta^*,\xi) = \sinh \rho(1+\xi)[\sinh \rho\delta^* \cosh \rho\delta^* - \rho\delta^*],$$

so, integration gives

$$\begin{aligned}
\sup_{-1\le x\le 0}\left(\int_{-1}^0 |(H^-_{\rho,\delta} - H^-_\rho)(x,s)|\,ds\right)
&\le 2\frac{\delta \sinh \rho}{\rho} \cdot \frac{(\cosh \rho - 1)}{\rho} \cdot \frac{|\rho\delta^* - \cosh \rho\delta^* \sinh \rho\delta^*|}{\Delta_0^2(\rho,\delta^*)} \\
&\le 2\frac{\delta}{\rho} \cdot \frac{\sinh \rho \cdot \cosh \rho \cdot (\rho\delta^*) \cdot \cosh \rho\delta^* \cdot \sinh \rho\delta^*}{(\sinh \rho \cdot \sinh \rho\delta^*) \cdot \rho \cdot (\delta^* \cosh \rho\delta^* \cdot \cosh \rho)} \cdot \left|\frac{1}{\sinh \rho\delta^* \cdot \cosh \rho\delta^*} - \frac{1}{\rho\delta^*}\right| \\
&\le 2\frac{\delta}{\rho} \cdot \sup_{t\ge 0}\left|\frac{1}{t} - \frac{2}{\sinh 2t}\right| \le C\frac{\delta}{\rho},
\end{aligned}$$

because the function $t \mapsto 1/t - 2/\sinh 2t$ is continuous and bounded on $\mathbb{R}$. Thus

$$\|K^-_{\sqrt{-z},\delta}(f^\delta_-) - K^-_{\sqrt{-z}}(f_-)\|_{E_-} \le C\Big(\frac{\delta}{|z|^{1/2}}\|f^\delta_-\|_{E_-} + \frac{1}{|z|}\|f^\delta_- - f_-\|_{E_-}\Big).$$

Similarly, for $x \in [-1,0]$ and $z \in \Gamma$, $N^-_{\sqrt{-z},\delta}(x,s)$ tends to

$$N^-_{\sqrt{-z}}(x,s) = N^-_{\sqrt{-z}}(x) = \frac{\sinh \sqrt{-z}(x+1)}{\sqrt{-z}(\sqrt{-z}\sinh \sqrt{-z} + \cosh \sqrt{-z})} = H^-_{\sqrt{-z}}(x,0).$$

Setting

$$\begin{cases} T^-_{\sqrt{-z}}(m)(x) = N^-_{\sqrt{-z}}(x)m, \\ M^-_{\sqrt{-z}}(f^\delta_+)(x) = N^-_{\sqrt{-z}}(x)m^\delta_+ = N^-_{\sqrt{-z}}(x)\int_0^1 f^\delta_+(s)\,ds, \end{cases}$$

we get

$$
\begin{aligned}
&T^-_{\sqrt{-z},\delta}(f^\delta_+)(x) - T^-_{\sqrt{-z}}(m)(x)\\
&\qquad = \int_0^1 \big(N^-_{\sqrt{-z},\delta}(x,s) - N^-_{\sqrt{-z}}(x)\big) f^\delta_+(s)\,ds + N^-_{\sqrt{-z}}(x)m^\delta_+ - N^-_{\sqrt{-z}}(x)m\\
&\qquad = T^-_{\sqrt{-z},\delta}(f^\delta_+)(x) - M^-_{\sqrt{-z}}(f^\delta_+)(x) + N^-_{\sqrt{-z}}(x)(m^\delta_+ - m)\\
&\qquad = (T^-_{\sqrt{-z},\delta} - M^-_{\sqrt{-z}}(f^\delta_+)(x) + T^-_{\sqrt{-z}}(m^\delta_+ - m)(x),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|T^-_{\sqrt{-z},\delta}(f^\delta_+) - T^-_{\sqrt{-z}}(m)\|_{E_-}\\
&\qquad \le \|(T^-_{\sqrt{-z},\delta} - M^-_{\sqrt{-z}})(f^\delta_+)\|_{E_-} + \|N^-_{\sqrt{-z}}(\cdot)(m^\delta_+ - m)\|_{E_-}\\
&\qquad \le \|(T^-_{\sqrt{-z},\delta} - M^-_{\sqrt{-z}})\|_{L(E_+,E_-)}\|f^\delta_+\|_{E_+} + \|N^-_{\sqrt{-z}}(\cdot)\|_{L^p(-1,0)}\|m^\delta_+ - m\|_{E_-}.
\end{aligned}
$$

As previously, there exists another $\delta^* \in]0,\delta[$ such that, for $-1 \le x \le 0$ and $0 \le s \le 1$, we have

$$
\begin{aligned}
(N^-_{\rho,\delta} - N^-_\rho)(x,s) = \delta \frac{\sinh\rho(1+x)}{\rho\Delta_0^2(\rho,\delta^*)} \times \{&\cosh\rho\delta^*(1-s)\times\\
&[\sinh\rho\sinh\rho\delta^* - \delta^*\rho(\sinh\rho\cosh\rho\delta^* + \delta^*\cosh\rho\sinh\rho\delta^*)]\\
&+ \sinh\rho\delta^*(1-s)\cdot[\delta^*\rho(1-s)(\sinh\rho\sinh\rho\delta^* + \delta^*\cosh\rho\cosh\rho\delta^*)]\},
\end{aligned}
$$

so

$$
\begin{aligned}
&\sup_{-1\le x\le 0}\left(\int_0^1 |(N^-_{\rho,\delta} - N^-_\rho)(x,s)|\,ds\right)\\
&\qquad \le \delta\frac{\sinh\rho}{\rho}\cdot\frac{\sinh\rho\delta^*}{\rho\delta^*}\cdot\frac{1}{\Delta_0^2(\rho,\delta^*)}\times\\
&\qquad\qquad (|\sinh\rho\sinh\rho\delta^* - \rho\delta^*(\sinh\rho\cosh\rho\delta^* + \delta^*\cosh\rho\sinh\rho\delta^*)|)\\
&\qquad\quad + \delta\frac{\sinh\rho}{\rho}\cdot\frac{\cosh\rho\delta^*}{\rho\delta^*}\cdot\frac{\rho\delta^*}{\Delta_0(\rho,\delta^*)}\\
&\qquad \le \frac{\delta}{\rho}\sinh\rho\sinh\rho\delta^*\frac{(\sinh\rho\cosh\rho\delta^* + \delta^*\cosh\rho\sinh\rho\delta^*)}{\sinh\rho\sinh\rho\delta^*\Delta_0(\rho,\delta^*)}\times\\
&\qquad\qquad \left|\frac{\sinh\rho\sinh\rho\delta^*}{\rho\delta^*(\sinh\rho\cosh\rho\delta^* + \delta^*\cosh\rho\sinh\rho\delta^*)} - 1\right| + \frac{\delta}{\rho}\\
&\qquad \le \frac{\delta}{\rho}\left|\frac{\sinh\rho\delta^*}{\rho\delta^*(\cosh\rho\delta^* + \delta^*\sinh\rho\delta^*)} - 1\right| + \frac{\delta}{\rho} \le C\frac{\delta}{\rho},
\end{aligned}
$$

since $t \to \sinh t/t\cosh t$ is bounded on $[0,+\infty[$.

Moreover we can show that for $x \in]-1,0[$, we have

$$|N^-_{\sqrt{-z}}(x)| = O\Big(\frac{e^{\mathrm{Re}\sqrt{-z}x}}{|z|}\Big) \qquad \text{for } |z| \to \infty,$$

so there exists a constant $C > 0$ such that

$$\|N^-_{\sqrt{-z}}(\cdot)\|_{L^p(-1,0)} \le \frac{C}{|z|^{1+1/2p}}.$$

We deduce that

$$\|T^-_{\sqrt{-z},\delta}(f^\delta_+) - T^-_{\sqrt{-z}}(m)\|_{E_-} \le \frac{C\cdot\delta}{|z|^{1/2}}\|f^\delta_+\|_{E_+} + \frac{C}{|z|^{1+1/2p}}\|m^\delta_+ - m\|_{E_+}.$$

The situation for $x \in [0,1]$ is dual, in some sense. In fact the kernels $H^+_{\sqrt{-z},\delta}(x,s)$ and $N^+_{\sqrt{-z},\delta}(x,s)$ tend to

$$\begin{cases} H^+_{\sqrt{-z}}(s) = \dfrac{\sinh\sqrt{-z}(s+1)}{\sqrt{-z}(\sqrt{-z}\sinh\sqrt{-z} + \cosh\sqrt{-z})} = H^-_{\sqrt{-z}}(0,s), \\ N^+_{\sqrt{-z}} = \dfrac{\sinh\sqrt{-z}}{\sqrt{-z}(\sqrt{-z}\sinh\sqrt{-z} + \cosh\sqrt{-z})} = N^-_{\sqrt{-z}}(0). \end{cases} \tag{8}$$

Setting, for $x \in [0,1]$,

$$\begin{cases} K^+_{\sqrt{-z}}(f_-)(x) = \displaystyle\int_{-1}^0 H^+_{\sqrt{-z}}(s) f_-(s)\,ds, \\ T^+_{\sqrt{-z}}(m)(x) = N^+_{\sqrt{-z}}\cdot m, \\ M^+_{\sqrt{-z}}(f^\delta_+)(x) = N^+_{\sqrt{-z}}\cdot m^\delta_+, \end{cases} \tag{9}$$

we then obtain (thanks to Hölder inequality)

$$\|K^+_{\sqrt{-z}}\|_{L(E_-,E_+)} \le \frac{C}{|z|^{1+1/2q}}.$$

(where $1/p + 1/q = 1$). From

$$\begin{aligned} &K^+_{\sqrt{-z},\delta}(f^\delta_-)(x) - K^+_{\sqrt{-z}}(f_-)(x) \\ &\quad = \int_{-1}^0 (H^+_{\sqrt{-z},\delta}(x,s) - H^+_{\sqrt{-z}}(s)) f^\delta_-(s)\,ds + \int_{-1}^0 H^+_{\sqrt{-z}}(s)(f^\delta_-(s) - f_-(s))\,ds, \end{aligned}$$

we get

$$\begin{aligned}
&\|K^+_{\sqrt{-z},\delta}(f^\delta_-) - K^+_{\sqrt{-z}}(f_-)\|_{E_+} \\
&\qquad\qquad \leq \|K^+_{\sqrt{-z},\delta} - K^+_{\sqrt{-z}}\|_{L(E_-,E_+)}\|f^\delta_-\|_{E_-} + \frac{C}{|z|^{1+1/2q}}\|f^\delta_- - f_-\|_{E_-} \\
&\qquad\qquad \leq \frac{C\cdot\delta}{|z|^{1/2}}\|f^\delta_-\|_{E_-} + \frac{C}{|z|^{1+1/2q}}\|f^\delta_- - f_-\|_{E_-},
\end{aligned}$$

and from

$$\begin{aligned}
&T^+_{\sqrt{-z},\delta}(f^\delta_+)(x) - T^+_{\sqrt{-z}}(m)(x) \\
&\qquad\qquad = \int_0^1 (N^+_{\sqrt{-z},\delta}(x,s) - N^+_{\sqrt{-z}})f^\delta_+(s)\,ds + N^+_{\sqrt{-z}}m^\delta_+ - N^+_{\sqrt{-z}}m \\
&\qquad\qquad = (T^+_{\sqrt{-z},\delta} - M^+_{\sqrt{-z}})(f^\delta_+)(x) + T^+_{\sqrt{-z}}(m^\delta_+ - m)(x),
\end{aligned}$$

one has

$$\|T^+_{\sqrt{-z},\delta}(f^\delta_+) - T^+_{\sqrt{-z}}(m)\|_{E_+} \leq \frac{C\cdot\delta}{|z|^{1/2}}\|f^\delta_+\|_{E_+} + \frac{C}{|z|}\|m^\delta_+ - m\|_{E_+}.$$

Note that m^δ_+ and m are constants in E_-, E_+, and E.

3.4.2. Convergence in $E = L^p(\Omega)$

The previous study suggests to set

$$\begin{aligned}
u_- &= \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}K^-_{\sqrt{-z}}(f_-)\,dz + \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}T^-_{\sqrt{-z}}(m)\,dz \\
u_+ &= \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}K^+_{\sqrt{-z}}(f_-)\,dz + \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}T^+_{\sqrt{-z}}(m)\,dz.
\end{aligned}$$

On the other hand, in virtue of (9), we get

$$\begin{aligned}
u_+(x) &= \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1}(K^+_{\sqrt{-z}}(f_-)(x))\,dz \\
&\quad + \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1}(T^+_{\sqrt{-z}}(m)(x))\,dz \\
&= \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1}\left(\int_{-1}^0 H^+_{\sqrt{-z}}(s)f_-(s)\,ds\right)dz \\
&\quad + \frac{1}{2i\pi}\int_\Gamma N^+_{\sqrt{-z}}(Q+zI)^{-1}(m)\,dz.
\end{aligned}$$

where $N^+_{\sqrt{-z}}$ is the complex number

$$N^+_{\sqrt{-z}} = \frac{\sinh\sqrt{-z}}{\sqrt{-z}(\sqrt{-z}\sinh\sqrt{-z} + \cosh\sqrt{-z})}.$$

Thus u_+ is independent of $x \in [0,1]$. We set

$$u_+(x) = u_+ \in L^p(G).$$

Now, we have

$$\begin{aligned} u^\delta_- - u_- &= \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}\big(K^-_{\sqrt{-z},\delta}(f^\delta_-) - K^-_{\sqrt{-z}}(f_-)\big)\,dz \\ &\quad + \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}\big(T^-_{\sqrt{-z},\delta}(f^\delta_+) - T^-_{\sqrt{-z}}(m)\big)\,dz \\ &= \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}\big(K^-_{\sqrt{-z},\delta} - K^-_{\sqrt{-z}}\big)(f^\delta_-)\,dz \\ &\quad + \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}K^-_{\sqrt{-z}}(f^\delta_- - f_-)\,dz \\ &\quad + \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}\big(T^-_{\sqrt{-z},\delta} - M^-_{\sqrt{-z}}\big)(f^\delta_+)\,dz \\ &\quad + \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}T^-_{\sqrt{-z}}(m^\delta_+ - m)\,dz \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

and

$$\begin{aligned} \|I_1\|_{E_-} &= \Big\|\frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}(K^-_{\sqrt{-z},\delta} - K^-_{\sqrt{-z}})(f^\delta_-)\,dz\Big\|_{E_-} \\ &\leq C\cdot\delta\int_\Gamma \frac{1}{|z|}\frac{1}{|z|^{1/2}}\,|dz|\,\|f^\delta_-\|_E \leq C\cdot\delta\|f^\delta_-\|_{E_-}, \end{aligned}$$

$$\begin{aligned} \|I_2\|_{E_-} &= \Big\|\frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1}K^-_{\sqrt{-z}}(f^\delta_- - f_-)\,dz\Big\|_{E_-} \\ &\leq C\int_\Gamma \frac{1}{|z|^2}\,|dz|\,\|f^\delta_- - f_-\|_{E_-} \leq C\|f^\delta_- - f_-\|_{E_-}, \end{aligned}$$

We do the same for the other integrals and for $u^\delta_+ - u_+$. So we get the estimate

$$\|u^\delta_\pm - u_\pm\|_{E_-} \leq C\cdot\delta(\|f^\delta_-\|_{E_-} + \|f^\delta_+\|_{E_+}) + C(\|f^\delta_- - f_-\|_{E_-} + \|m^\delta_+ - m\|_{E_+}),$$

where C is independent of δ.

3.4.3. The formal Ventcel's Problem

The Dunford operational calculus shows that u_- satisfies formally

$$\begin{cases} -u_-''(x) + Qu_-(x) = f_-(x), & x \in]-1, 0[, \\ u_-(-1) = 0. \end{cases}$$

Operator Q has been defined in (4).

To find the boundary condition on u_- at 0, we integrate between 0 and 1, with respect to x, the equation

$$-\Big(\frac{1}{\delta^2} D_x^2 u_+^\delta + \Delta_y u_+^\delta\Big) = f_+^\delta.$$

We get

$$D_x u_-^\delta(0, y) = \frac{1}{\delta^2} D_x u_+^\delta(0, y) = \int_0^1 f_+^\delta(\tau, y) d\tau + \int_0^1 \Delta_y u_+^\delta(\tau, y) d\tau.$$

Let $\delta \to 0$. We formally obtain the boundary condition of Ventcel's type

$$D_x u_-(0, y) = \lim_{\delta \to 0} \int_0^1 f_+^\delta(\tau, y) d\tau + \lim_{\delta \to 0} \int_0^1 \Delta_y u_+^\delta(\tau, y) d\tau = m(y) + \Delta_y u_-(0, y).$$

So u_- satisfies the abstract boundary value problem

$$\begin{cases} -u_-''(x) + Qu_-(x) = f_-(x), & x \in]-1, 0[, \\ u_-(-1) = 0, \\ u_-'(0) - Qu_-(0) = m, \end{cases}$$

that is

$$\begin{cases} -\Delta u_- = f_- & \text{in } \Omega_- \\ u_- = 0 & \text{on } \partial\Omega_- \setminus \Gamma^0 \\ \partial_x u_-(0, \cdot) - \Delta_y u_-(0, \cdot) = m & \text{on } G. \end{cases}$$

3.4.4. Study of the Ventcel's Problem

The solution of the Ventcel Problem

$$\begin{cases} -u''(x) + Qu(x) = f_-(x), & x \in]-1, 0[, \\ u(-1) = 0, \\ u'(0) - Qu(0) = m, \end{cases} \tag{10}$$

can be written

$$u_- = u_-(f, m) = u_-^1(f) + u_-^2(m),$$

where u_-^1 is the solution of the following problem with homogeneous condition of Ventcel's type

$$\begin{cases} -(u_-^1)''(x) + Qu_-^1(x) = f_-(x), & x \in]-1,0[, \\ u_-^1(-1) = 0, \\ (u_-^1)'(0) - Qu_-^1(0) = 0, \end{cases} \tag{11}$$

and u_-^2 is the solution of the following problem with non homogeneous condition of Ventcel's type

$$\begin{cases} -\left(u_-^2\right)''(x) + Qu_-^2(x) = 0, & x \in]-1,0[, \\ u_-^2(-1) = 0, \\ (u_-^2)'(0) - Qu_-^2(0) = m. \end{cases} \tag{12}$$

Set

$$\begin{aligned} D(A_-) &= L^p(-1,0;D(Q)), \qquad (A_-\phi)(x) = Q(\phi(x)), \\ D(A_+) &= L^p(0,1;D(Q)), \qquad (A_+\phi)(x) = Q(\phi(x)), \end{aligned}$$

and

$$\begin{cases} D(B_-) = \{\, \psi \in W^{2,p}(-1,0;X), \psi(-1) = 0, \psi(0) \in D(Q), \psi'(0) = Q(\psi(0)) \,\} \\ (B_-\psi)(x) = -\psi''(x) \qquad \text{a.e. } x \in (-1,0), \quad \psi \in D(B_-). \end{cases}$$

The sum theory of operators applies since we can show, as previously, that A_- and B_- satisfy (DG.0), (DG.1), and (DG.2). So, the strong solution of the homogeneous Ventcel's problem in E_- is

$$\begin{aligned} u_-^1 &= \frac{1}{2i\pi} \int_\Gamma (A_- + zI)^{-1} K^-_{\sqrt{-z}}(f_-)\, dz \\ &= \frac{1}{2i\pi} \int_\Gamma (A_- + zI)^{-1} (B_- - zI)^{-1} f_-\, dz. \end{aligned}$$

For the non homogeneous problem, we set in E_-

$$u_-^2 = \frac{1}{2i\pi} \int_\Gamma (A_- + zI)^{-1} T^-_{\sqrt{-z}}(m)\, dz,$$

and verify that

$$\begin{cases} -(u_-^2)''(x) + Qu_-^2(x) = 0, & \forall x \in]-1,0[, \\ u_-^2(-1) = 0. \end{cases}$$

Moreover, for any $x \in [-1,0[$

$$\begin{aligned} &(u_-^2)'(x) - Qu_-^2(x) \\ &\qquad = \frac{1}{2i\pi} \int_\Gamma \frac{\sqrt{-z}\sinh\sqrt{-z}(x+1) + \cosh\sqrt{-z}(x+1)}{\sqrt{-z}\sinh\sqrt{-z} + \cosh\sqrt{-z}} (Q + zI)^{-1} m\, dz, \end{aligned}$$

the absolute convergence of the integral being assured by the following estimate, given for $z \in \Gamma$ and $x \in [-1,0[$

$$\left|\frac{\sqrt{-z}\sinh\sqrt{-z}(x+1)+\cosh\sqrt{-z}(x+1)}{(\sqrt{-z}\sinh\sqrt{-z}+\cosh\sqrt{-z})}\right| = O\big(e^{-\operatorname{Re}(\sqrt{-z})|x|}\big) \qquad \text{for } |z|\to\infty.$$

We show, due to [3], that there exists a constant C, which depends only on Γ, such that, for any $x \in [-1,0[$, we have

$$\|(u_-^2)'(x) - Qu_-^2(x)\|_X \le C\|m\|_X.$$

This estimate yields

$$u_2'(0) - Qu_2(0) = m,$$

since $(u_-^2)'(0) - Qu_2(0) = m$ when $m \in D(Q)$ and $\overline{D(Q)} = X$. So u_-^2 is the strong solution of the non homogeneous Ventcel's problem.

Theorem 3.2 is completely proved. □

3.5. Proof of Theorem 3.3

Let $f^\delta \in W^{2\theta,p}(\Omega)$, where $1 < p < \infty$ and $0 < \theta < 1/2$. We assume in this subsection that

(R1) f_-^δ has a limit f_- in $W^{2\theta,p}(\Omega_-)$ when $\delta \to 0$,

(R2) f_+^δ is bounded in $W^{2\theta,p}(\Omega_+)$,

(R3) $\int_0^1 f_+^\delta(x,\cdot)\,dx = m_+^\delta$ tends to m in $W^{2\theta,p}(G)$ when $\delta \to 0$.

To simplify we suppose that $\theta < 1/2p$. The case $2\theta > 1/p$ can be treated in the same manner when necessary compatibility hypotheses at the boundary of Ω are assumed.

3.5.1. Regularity of u_-

Lemma 3.7. *Let $\theta \in]0, 1/2p[$, $p \in]1,\infty[$ and $X = L^p(G)$. Then*

$$\begin{aligned}
&D_{B_\delta}(\theta,p) = W^{2\theta,p}(-1,1;X)\\
&W^{2\theta,p}(\Omega) = L^p(-1,1;W^{2\theta,p}(G)) \cap W^{2\theta,p}(-1,1;X)\\
&D_{A_-}(\theta,p) = L^p(-1,0;W^{2\theta,p}(G)) = L^p(-1,0;D_Q(\theta,p))\\
&D_{A_+}(\theta,p) = L^p(0,1;W^{2\theta,p}(G)) = L^p(0,1;D_Q(\theta,p))\\
&D_{B_-}(\theta,p) = W^{2\theta,p}(-1,0;X).
\end{aligned}$$

Proof. Let us show, for instance, the first equality. Set

$$W_0^{2,p}(-1,0;X) = \{ f \in W^{2,p}(-1,0;X) \,:\, f(-1) = f(0) = f'(-1) = f'(0) = 0 \},$$
$$W_0^{2,p}(0,1;X) = \{ f \in W^{2,p}(0,1;X) \,:\, f(0) = f(1) = f'(0) = f'(1) = 0 \}.$$

Then

$$\{ u \in L^p(-1,1;X) \,:\, u_- \in W_0^{2,p}(-1,0;X) \text{ and } u_+ \in W_0^{2,p}(0,1;X) \} \subset D(B_\delta),$$

and

$$D(B_\delta) \subset \{ u \in L^p(-1,1;X) \,:\, u_- \in W^{2,p}(-1,0;X) \text{ and } u_+ \in W^{2,p}(0,1;X) \},$$

so by interpolation

$$\Big\{ u \in L^p(-1,1;X) \,:\, u_- \in \big(W_0^{2,p}(-1,0;X); L^p(-1,0;X)\big)_{1-\theta,p} \text{ and } u_+ \in \big(W_0^{2,p}(0,1;X); L^p(0,1;X)\big)_{1-\theta,p} \Big\} \subset D_{B_\delta}(\theta,p),$$

and

$$D_{B_\delta}(\theta,p) \subset \Big\{ u \in L^p(-1,1;X) \,:\, u_- \in \big(W^{2,p}(-1,0;X); L^p(-1,0;X)\big)_{1-\theta,p} \text{ and } u_+ \in \big(W^{2,p}(0,1;X); L^p(0,1;X)\big)_{1-\theta,p} \Big\},$$

or

$$\{ u \in L^p(-1,1;X) \,:\, u_- \in W^{2\theta,p}(-1,0;X) \text{ and } u_+ \in W^{2\theta,p}(0,1;X) \} \subset D_{B_\delta}(\theta,p),$$

and

$$D_{B_\delta}(\theta,p) \subset \{ u \in L^p(-1,1;X) \,:\, u_- \in W^{2\theta,p}(-1,0;X) \text{ and } u_+ \in W^{2\theta,p}(0,1;X) \},$$

thus $D_{B_\delta}(\theta,p) = W^{2\theta,p}(-1,1;X)$. □

We deduce the following equivalent abstract form of our previous hypotheses

(D1) f_-^δ tends to f_- in $D_{A_-}(\theta,p) \cap D_{B_-}(\theta,p)$ when $\delta \to 0$,

(D2) f_+^δ is bounded in $L^p(0,1;W^{2\theta,p}(G)) = D_{A_+}(\theta,p)$,

(D3) $\int_0^1 f_+^\delta(x)\,dx = m_+^\delta$ tends to $m \in D_Q(\theta,p)$ when $\delta \to 0$,

and $f^\delta \in D_A(\theta,p) \cap D_{B_\delta}(\theta,p) = D_{(A+B_\delta)}(\theta,p)$ (see Grisvard [5]).

Now using Theorem 3.5, the solution u_-^1 of the homogeneous Ventcel's problem is in $D(A_-)\cap D(B_-)$, so $u_-^1 \in W^{2,p}(\Omega_-)$, $u^1_{-|\Gamma^0} \in W^{2,p}(\Gamma^0)\cap W_0^{1,p}(\Gamma^0)$ and

$$\begin{cases} -\Delta u_-^1 = f_- & \text{in } \Omega_- \\ u_-^1 = 0 & \text{on } \partial\Omega_- \setminus \Gamma^0 \\ \partial_x u_-^1(0,\cdot) - \Delta_y u_-^1(0,\cdot) = 0 & \text{on } G, \end{cases}$$

and moreover $\partial_x^2 u_-^1 \in W^{2\theta,p}(\Omega_-)$ and $\Delta_y u_-^1 \in W^{2\theta,p}(\Omega_-)$.

Similarly, m being in the space $W^{2\theta,p}(G) = D_Q(\theta,p)$, we use the results of Labbas [2, 3] to obtain $u_-^2 \in W^{2,p}(-1,0;X)\cap L^p(-1,0;D(A_-))$ and moreover

$$\begin{cases} (u_-^2)'' \in W^{2\theta,p}(-1,0;L^p(G)) \\ A_- u_-^2 \in W^{2\theta,p}(-1,0;L^p(G)) \\ (u_-^2)'' \in L^p(-1,0;D_Q(\theta,p)) = D_{A_-}(\theta,p). \end{cases}$$

So $u_-^2 \in W^{2,p}(\Omega_-)$.The global regularity is obtained due to Mihlin's Theorem. We also have

$$\partial_x^2 u_-^2 \in W^{2\theta,p}(\Omega_-) \quad \text{and} \quad \Delta_y u_-^2 \in W^{2\theta,p}(\Omega_-).$$

3.5.2. Relation between u_- and u_+

Let us verify that for all $y \in G$,

$$u_+(y) = u_-(0,y).$$

We recall that

$$u_- = \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1} K^-_{\sqrt{-z}}(f_-)\,dz + \frac{1}{2i\pi}\int_\Gamma (A+zI)^{-1} T^-_{\sqrt{-z}}(m)\,dz,$$

and

$$\begin{aligned} u_+ = {} & \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1}\Big(\int_{-1}^0 H^+_{\sqrt{-z}}(s) f_-(s)\,ds\Big)\,dz \\ & + \frac{1}{2i\pi}\int_\Gamma N^+_{\sqrt{-z}}(Q+zI)^{-1}(m)\,dz. \end{aligned}$$

Using the regularity of u_-, we have

$$\begin{aligned} u_-(x)(\cdot) = u_-(x,\cdot) & = \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1}(K^-_{\sqrt{-z}}(f_-)(x))\,dz \\ & \quad + \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1}(T^-_{\sqrt{-z}}(m)(x))\,dz \\ & = \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1}\left(\int_{-1}^0 H^-_{\sqrt{-z}}(x,s) f_-(s)\,ds\right)dz \\ & \quad + \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1} N^-_{\sqrt{-z}}(x) m\,dz, \end{aligned}$$

which implies that

$$\begin{aligned} u_-(0,\cdot) &= \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1}\Big(\int_{-1}^0 H^-_{\sqrt{-z}}(0,s)f_-(s)\,ds\Big)\,dz \\ &\quad + \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1}N^-_{\sqrt{-z}}(0)m\,dz \\ &= \frac{1}{2i\pi}\int_\Gamma (Q+zI)^{-1}\Big(\int_{-1}^0 H^+_{\sqrt{-z}}(s)f_-(s)\,ds\Big)\,dz \\ &\quad + \frac{1}{2i\pi}\int_\Gamma N^+_{\sqrt{-z}}(Q+zI)^{-1}(m)\,dz = u_+, \end{aligned}$$

in virtue of (8).

3.5.3. Convergence in $W^{1+2\theta,p}(-1,0;X)$

We use here the representation

$$\begin{aligned} u_-^\delta &= \frac{1}{2i\pi}\int_\Gamma K^-_{\sqrt{-z},\delta}((A_-+zI)^{-1}f_-^\delta)\,dz \\ &\quad + \frac{1}{2i\pi}\int_\Gamma T^-_{\sqrt{-z},\delta}((A_++zI)^{-1}f_+^\delta)\,dz \\ &= \frac{1}{2i\pi}\int_\Gamma K^-_{\sqrt{-z},\delta}(A_-(A_-+zI)^{-1}f_-^\delta)\,\frac{dz}{z} \\ &\quad + \frac{1}{2i\pi}\int_\Gamma T^-_{\sqrt{-z},\delta}(A_+(A_++zI)^{-1}f_+^\delta)\,\frac{dz}{z}, \end{aligned}$$

The first equality follows from the commutativity of the two resolvents of B_δ and A, and the second is a consequence of the classical Dunford calculus. Similarly we get

$$\begin{aligned} u_- &= \frac{1}{2i\pi}\int_\Gamma K^-_{\sqrt{-z}}((A_-+zI)^{-1}f_-)\,dz \\ &\quad + \frac{1}{2i\pi}\int_\Gamma T^-_{\sqrt{-z}}((A+zI)^{-1}m)\,dz \\ &= \frac{1}{2i\pi}\int_\Gamma K^-_{\sqrt{-z}}(A_-(A_-+zI)^{-1}f_-)\,\frac{dz}{z} \\ &\quad + \frac{1}{2i\pi}\int_\Gamma T^-_{\sqrt{-z}}(A(A+zI)^{-1}m)\,\frac{dz}{z}. \end{aligned}$$

Thus

$$\begin{aligned}\partial_x u_-^\delta(x) - \partial_x u_-(x) &= \frac{1}{2i\pi}\int_\Gamma \partial_x K^-_{\sqrt{-z},\delta}(A_-(A_- + zI)^{-1} f_-^\delta)(x)\,\frac{dz}{z}\\ &\quad - \frac{1}{2i\pi}\int_\Gamma \partial_x K^-_{\sqrt{-z}}(A_-(A_- + zI)^{-1} f_-)(x)\,\frac{dz}{z}\\ &\quad + \frac{1}{2i\pi}\int_\Gamma \partial_x T^-_{\sqrt{-z},\delta}(A_+(A_+ + zI)^{-1} f_+^\delta)(x)\,\frac{dz}{z}\\ &\quad - \frac{1}{2i\pi}\int_\Gamma \partial_x T^-_{\sqrt{-z}}(A(A + zI)^{-1} m)(x)\,\frac{dz}{z},\end{aligned}$$

with

$$\begin{aligned}\partial_x K^-_{\sqrt{-z},\delta}(h_-)(x) &= \int_{-1}^0 \partial_x H^-_{\rho,\delta}(x,s) h_-(s)\,ds,\\ \partial_x K^-_{\sqrt{-z}}(h_-)(x) &= \int_{-1}^0 \partial_x H^-_{\rho}(x,s) h_-(s)\,ds,\end{aligned}$$

$$\partial_x H^-_{\rho,\delta}(x,s) = \begin{cases} \frac{\partial_x \Delta_-(\rho,\delta,x)}{\rho\Delta_0(\rho,\delta)} \sinh\rho(s+1) & \text{if } -1 < s < x,\\ \frac{\Delta_-(\rho,\delta,s)}{\Delta_0(\rho,\delta)} \cosh\rho(x+1) & \text{if } x \le s < 0,\end{cases}$$

$$\partial_x H^-_{\rho}(x,s) = \begin{cases} \frac{-\sqrt{-z}\cosh\sqrt{-z}x + \sinh\sqrt{-z}x}{\sqrt{-z}\sinh\sqrt{-z} + \cosh\sqrt{-z}} \sinh\sqrt{-z}(s+1) & \text{if } -1 < s < x,\\ \frac{-\sqrt{-z}\sinh\sqrt{-z}s + \cosh\sqrt{-z}s}{\sqrt{-z}\sinh\sqrt{-z} + \cosh\sqrt{-z}} \cosh\sqrt{-z}(x+1) & \text{if } x \le s < 0,\end{cases}$$

and

$$\partial_x \Delta_-(\rho,\delta,x) = \sqrt{-z}(-\cosh\sqrt{-z}x \cdot \sinh\sqrt{-z}\delta + \delta\cosh\sqrt{-z}\delta \cdot \sinh\sqrt{-z}x).$$

Thus, in E_-, we get

$$\begin{aligned}\partial_x u_-^\delta - \partial_x u_- &= \frac{1}{2i\pi}\int_\Gamma \partial_x K^-_{\sqrt{-z},\delta}\big(A_-(A_- + zI)^{-1}(f_-^\delta - f_-)\big)\,\frac{dz}{z}\\ &\quad + \frac{1}{2i\pi}\int_\Gamma (\partial_x K^-_{\sqrt{-z},\delta} - \partial_x K^-_{\sqrt{-z}})\big(A_-(A_- + zI)^{-1} f_-\big)\,\frac{dz}{z}\\ &\quad + \frac{1}{2i\pi}\int_\Gamma (\partial_x T^-_{\sqrt{-z},\delta} - \partial_x M^-_{\sqrt{-z}})\big(A_+(A_+ + zI)^{-1} f_+^\delta\big)\,\frac{dz}{z}\\ &\quad + \frac{1}{2i\pi}\int_\Gamma \partial_x T^-_{\sqrt{-z}}\big(A(A + zI)^{-1}(m_+^\delta - m)\big)\,\frac{dz}{z}\\ &= \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4.\end{aligned}$$

Now, using again the Schur interpolation lemma, we prove that there exists a constant C, independent of δ and z, such that

$$\begin{cases} \|\partial_x K^-_{\sqrt{-z},\delta}\|_{L(E_-)} \leq C/|z|^{1/2}, \\ \|\partial_x K^-_{\sqrt{-z},\delta} - \partial_x K^-_{\sqrt{-z}}\|_{L(E_-)} \leq C\delta, \\ \|\partial_x T^-_{\sqrt{-z},\delta} - \partial_x M^-_{\sqrt{-z}}\|_{L(E_-)} \leq C\delta, \\ \|\partial_x T^-_{\sqrt{-z}}\|_{L(E_-)} \leq C/|z|^{1/2+1/2p}, \end{cases}$$

and, due to these estimates, we deduce that

$$\begin{aligned} \|u^\delta_- - u_-\|_{W^{1,p}(-1,0;X)} \leq & C \cdot \delta \cdot (\|f^\delta_-\|_{D_{A_-}(\theta,\infty)} + \|f^\delta_+\|_{D_{A_+}(\theta,\infty)}) \\ & + C(\|f^\delta_- - f_-\|_{D_{A_-}(\theta,\infty)} + \|m^\delta_+ - m\|_{D_{A_+}(\theta,\infty)}). \end{aligned}$$

If we show analogous estimates in the space

$$W^{1+2\theta,\infty}(-1,0;X) = C^{1+2\theta,\infty}([-1,0];X),$$

we then deduce the result in $W^{1+2\theta,p}(-1,0;X)$ by interpolation.

Let x_1 and x_2 such that $-1 \leq x_1 < x_2 \leq 0$. One has

$$\|(\partial_x u^\delta_- - \partial_x u_-)(x_2) - (\partial_x u^\delta_- - \partial_x u_-)(x_1)\|_X \leq \sum_{i=1}^{4} \|\Psi_i(x_2) - \Psi_i(x_1)\|.$$

Let us estimate the second term, for example. The other terms can be treated similarly.

$$\begin{aligned} \Psi_2(x_2) - \Psi_2(x_1) &= \frac{1}{2i\pi} \int_\Gamma (\partial_x K^-_{\sqrt{-z},\delta} - \partial_x K^-_{\sqrt{-z}})(A_-(A_- + zI)^{-1} f_-)(x_2) \frac{dz}{z} \\ &\quad - \frac{1}{2i\pi} \int_\Gamma (\partial_x K^-_{\sqrt{-z},\delta} - \partial_x K^-_{\sqrt{-z}})(A_-(A_- + zI)^{-1} f_-)(x_1) \frac{dz}{z} \\ &= (I(x_2) - I(x_1)) + (J(x_2) - J(x_1)), \end{aligned}$$

where

$$
\begin{aligned}
I(x_2)-I(x_1)& \\
&= \frac{1}{2i\pi}\int_{-1}^{x_2}\int_\Gamma \partial_x\big(H^-_{\sqrt{-z},\delta}-H^-_{\sqrt{-z}}\big)(x_2,s)\,Q\,(Q+zI)^{-1}f_-(s)\,\frac{dz}{z}\,ds\\
&\quad-\frac{1}{2i\pi}\int_{-1}^{x_1}\int_\Gamma \partial_x\big(H^-_{\sqrt{-z},\delta}-H^-_{\sqrt{-z}}\big)(x_1,s)\,Q\,(Q+zI)^{-1}f_-(s)\,\frac{dz}{z}\,ds\\
&= \frac{1}{2i\pi}\int_{x_1}^{x_2}\int_\Gamma \partial_x\big(H^-_{\sqrt{-z},\delta}-H^-_{\sqrt{-z}}\big)(x_2,s)\,Q\,(Q+zI)^{-1}f_-(s)\,\frac{dz}{z}\,ds\\
&\quad+\frac{1}{2i\pi}\int_{-1}^{x_1}\int_\Gamma \partial_x\big(H^-_{\sqrt{-z},\delta}-H^-_{\sqrt{-z}}\big)(x_2,s)-\partial_x\big(H^-_{\sqrt{-z},\delta}-H^-_{\sqrt{-z}}\big)(x_1,s)\\
&\qquad\qquad\cdot\Big\{Q(Q+zI)^{-1}f_-(s)\frac{dz}{z}ds\Big\}\\
&= J_1+J_2.
\end{aligned}
$$

Now, for $-1\le x_1\le s<x_2$ and $\rho>0$, we have

$$
\begin{aligned}
\partial_x(H^-_{\rho,\delta}-H^-_\rho)(x,s)&\\
&=\delta\sinh\rho(1+x)\frac{\partial_\delta\Delta_-(\rho,\delta^*,s)\Delta_0(\rho,\delta^*)-\partial_\delta\Delta_0(\rho,\delta^*)\Delta_-(\rho,\delta^*,s)}{\Delta_0^2(\rho,\delta^*)}\\
&=\delta\sinh\rho(1+x)\frac{\sinh\rho(1+s)[\sinh\rho\delta^*\cosh\rho\delta^*-\rho\delta^*]}{\Delta_0^2(\rho,\delta^*)},
\end{aligned}
$$

and

$$
\begin{aligned}
|\partial_x(H^-_{\rho,\delta}-H^-_\rho)(x_2,s)|&\\
&\le\delta\frac{\sinh\rho(x_2+1)\cdot\cosh\rho(s+1)\cdot(\rho\delta^*)\cdot\cosh\rho\delta^*\cdot\sinh\rho\delta^*}{(\sinh\rho\cdot\sinh\rho\delta^*)(\delta^*\cosh\rho\delta^*\cdot\cosh\rho)}\\
&\qquad\times\left|\frac{1}{\sinh\rho\delta^*\cdot\cosh\rho\delta^*}-\frac{1}{\rho\delta^*}\right|\\
&\le\delta\rho\frac{\sinh\rho(x_2+1)\cdot\cosh\rho(s+1)}{\sinh\rho\cdot\cosh\rho}\cdot\sup_{t\ge0}\left|\frac{1}{t}-\frac{2}{\sinh 2t}\right|\\
&\le\delta\rho\frac{\sinh\rho(x_2+1)\cdot\cosh\rho(s+1)}{\sinh\rho\cdot\cosh\rho}\le C\delta\rho e^{-\rho(x_2-s)},
\end{aligned}
$$

where $\rho=\operatorname{Re}\sqrt{-z}=c\,|z|^{1/2}$ on Γ. It follows that

$$
\begin{aligned}
\|J_1\|_X&=\Big\|\frac{1}{2i\pi}\int_{x_1}^{x_2}\int_\Gamma \partial_x\big(H^-_{\sqrt{-z},\delta}-H^-_{\sqrt{-z}}\big)(x_2,s)\,Q\,(Q+zI)^{-1}f_-(s)\,\frac{dz}{z}\,ds\Big\|\\
&\le C\cdot\delta\cdot\int_{x_1}^{x_2}\int_\Gamma\frac{|z|^{1/2}e^{-c|z|^{1/2}(x_2-s)}}{|z|^{\theta+1}}\,|dz|\,ds\,\|f_-\|_{D_{A_-}(\theta,\infty)}
\end{aligned}
$$

$$\begin{aligned}&\le C\cdot\delta\cdot\int_{x_1}^{x_2}\int_0^\infty \frac{e^{-c\mu}}{(\frac{\mu^2}{(x_2-s)^2})^{\theta+1/2}}\cdot\frac{\mu\,d\mu}{(x_2-s)^2}\,ds\,\|f_-\|_{D_{A_-}(\theta,\infty)}\\&\le C\cdot\delta\cdot\int_{x_1}^{x_2}(x_2-s)^{2\theta-1}\Big(\int_0^\infty\frac{e^{-c\mu}d\mu}{\mu^{2\theta}}\Big)\,ds\,\|f_-\|_{D_{A_-}(\theta,\infty)}\\&\le C\cdot\delta|x_2-x_1|^{2\theta}\|f_-\|_{D_{A_-}(\theta,\infty)}.\end{aligned}$$

Finally

$$\begin{aligned}&\|\Psi_2(x_2)-\Psi_2(x_1)\|_X+\|\Psi_3(x_2)-\Psi_3(x_1)\|_X\\&\qquad\le C\cdot\delta|x_2-x_1|^{2\theta}\|f_-\|_{D_{A_-}(\theta,\infty)}+C\cdot\delta|x_2-x_1|^{2\theta}\|f_+^\delta\|_{D_{A_+}(\theta,\infty)},\end{aligned}$$

and similarly

$$\begin{aligned}&\|\Psi_1(x_2)-\Psi_1(x_1)\|_X+\|\Psi_4(x_2)-\Psi_4(x_1)\|_X\\&\qquad\le C\cdot|x_2-x_1|^{2\theta}\|f_-^\delta-f_-\|_{D_{A_-}(\theta,\infty)}+C\cdot|x_2-x_1|^{2\theta}\|m_+^\delta-m\|_{D_{A_+}(\theta,\infty)}.\end{aligned}$$

Thus

$$\begin{aligned}&\|u_-^\delta-u_-\|_{W^{1+2\theta,\infty}(-1,0;X)}\\&\qquad\le C\cdot\delta\cdot(\|f_-^\delta\|_{D_{A_-}(\theta,\infty)}+\|f_+^\delta\|_{D_{A_+}(\theta,\infty)})\\&\qquad\quad+C(\|f_-^\delta-f_-\|_{D_{A_-}(\theta,\infty)}+\|m_+^\delta-m\|_{D_{A_+}(\theta,\infty)}),\end{aligned}$$

and Theorem 3.3 is then obtained.

In addition, we study now the convergence, in $D_{A_\pm}(\theta+1/2,p)$, of $u_\pm^\delta$ when $\delta\to 0$.

3.5.4. Convergence in $D_{A_\pm}(\theta+1/2,p)$

We use the following lemma whose proof is not difficult.

Lemma 3.8. (i) *There exists $C>0$ such that for any $z\in\Gamma$ and $r>0$, we have*

$$|z\pm r|\ge Cr,\qquad |z\pm r|\ge C|z|,$$

(ii) *Let $\upsilon\in]0,1[$. Then there exists $C>0$ such that for any $r>0$,*

$$\int_\Gamma\frac{1}{|r\pm z||z|^\upsilon}\,|dz|\le C/r^\upsilon.$$

We give the proof in the case of interpolation spaces corresponding to $p=\infty$. The case $p\in]1,\infty[$ can be deduced by interpolation. We must estimate

$$\|(u_-^\delta-u_-)\|_{D_{A_-}(\theta+1/2,\infty)}=\sup_{r>0}\|r^{\theta+1/2}A_-(A_-+rI)^{-1}(u_-^\delta-u_-)\|_{E_-}.$$

Let $r > 0$. Then

$$\begin{aligned}
A_-(A_- + rI)^{-1}(u_-^\delta - u_-)
&= \frac{1}{2i\pi}\int_\Gamma \frac{1}{z-r}\big(K^-_{\sqrt{-z},\delta} - K^-_{\sqrt{-z}}\big)(A_-(A_- + zI)^{-1}f_-^\delta)\,dz\\
&\quad + \frac{1}{2i\pi}\int_\Gamma \frac{1}{z-r}K^-_{\sqrt{-z}}\big(A_-(A_- + zI)^{-1}(f_-^\delta - f_-)\big)\,dz\\
&\quad + \frac{1}{2i\pi}\int_\Gamma \frac{1}{z-r}\big(T^-_{\sqrt{-z},\delta} - M^-_{\sqrt{-z}}\big)(A(A + zI)^{-1}f_+^\delta)\,dz\\
&\quad + \frac{1}{2i\pi}\int_\Gamma \frac{1}{z-r}T^-_{\sqrt{-z}}\big(A(A + zI)^{-1}(m_+^\delta - m)\big)\,dz\\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

in virtue of Dunford's calculus. We also have used the fact that $D(A) \subset D(A_-)$ and $(A + zI)^{-1} = (A_- + zI)^{-1}$ on E_-.

We get

$$\begin{aligned}
\|I_1\|_E &\le C\int_\Gamma \frac{1}{|z-r|}\frac{\delta}{|z|^{1/2}}\frac{1}{|z|^\theta}\,|dz|\,\|f_-^\delta\|_{D_{A_-}(\theta,+\infty)}\\
&\le \frac{C}{r^{\theta+1/2}}\cdot\delta\cdot\|f_-^\delta\|_{D_{A_-}(\theta,+\infty)},
\end{aligned}$$

by the previous Lemma. Similarly

$$\begin{aligned}
\|I_2\|_E &\le C\int_\Gamma \frac{1}{|z-r|}\cdot\frac{1}{|z|^{\theta+1}}\,|dz|\,\|f_-^\delta - f_-\|_{D_{A_-}(\theta,+\infty)}\\
&\le C\int_\Gamma \frac{1}{|z-r|^{\theta+1/2}}\cdot\frac{1}{|z-r|^{-\theta+1/2}}\cdot\frac{1}{|z|^{\theta+1}}\,|dz|\,\|f_-^\delta - f_-\|_{D_{A_-}(\theta,+\infty)}\\
&\le \frac{C}{r^{\theta+1/2}}\int_\Gamma \frac{1}{|z|^{1+1/2}}\,|dz|\,\|f_-^\delta - f_-\|_{D_{A_-}(\theta,+\infty)}\\
&\le \frac{C}{r^{\theta+1/2}}\|f_-^\delta - f_-\|_{D_{A_-}(\theta,+\infty)}.
\end{aligned}$$

The other integrals can be treated in the same manner. So we obtain

$$\begin{aligned}
&\|u_-^\delta - u_-\|_{D_{A_-}(\theta+1/2,\infty)}\\
&\qquad\le C\cdot\delta\cdot(\|f_-^\delta\|_{D_{A_-}(\theta,\infty)} + \|f_+^\delta\|_{D_{A_+}(\theta,\infty)})\\
&\qquad\quad + C(\|f_-^\delta - f_-\|_{D_{A_-}(\theta,\infty)} + \|m_+^\delta - m\|_{D_{A_+}(\theta,\infty)}),
\end{aligned}$$

and the same estimate for $\|u_+^\delta - u_+\|_{D_{A_+}(\theta+1/2,\infty)}$.

In our case we have precisely

$$\begin{cases}
D_{A_-}(\theta + 1/2, p) = \{\, u \in L^p(-1,0; W^{2\theta+1,p}(G)) \,:\, u = 0 \text{ on }]-1,0[\times\partial G \,\}\\
D_{A_+}(\theta + 1/2, p) = \{\, u \in L^p(0,1; W^{2\theta+1,p}(G)) \,:\, u = 0 \text{ on }]0,1[\times\partial G \,\}
\end{cases}$$

(see Grisvard [6]).

4. Go back to Problem (2)

Let us recall that $v_-^\delta = u_-^\delta$, $g_-^\delta = f_-^\delta$ on Ω_- and

$$\begin{aligned} v_+^\delta(\xi,\eta) &= u_+^\delta(\xi/\delta,\eta) \\ g_+^\delta(\xi,\eta) &= f_+^\delta(\xi/\delta,\eta), \end{aligned}$$

for any $(\xi,\eta) \in \Omega_+^\delta$. Setting

$$v^\delta = \begin{cases} v_-^\delta & \text{on } \Omega_-, \\ v_+^\delta & \text{on } \Omega_+^\delta, \end{cases}$$

we will translate below all properties of u^δ in properties on v^δ.

4.1. Proof of Theorem 1.1

Fix $\delta > 0$. Since $g^\delta \in L^p(\Omega^\delta)$ then $f^\delta \in L^p(\Omega)$. Due to Theorem 3.1 (statement (i)), there exists a strong solution

$$u^\delta = \begin{cases} u_-^\delta & \text{on } \Omega_-, \\ u_+^\delta & \text{on } \Omega_+, \end{cases}$$

in $L^p(\Omega)$ of Problem (3), so v^δ, defined previously, is a strong solution in $L^p(\Omega^\delta)$ of Problem (2).

Assume that $g_-^\delta \in W^{2\theta,p}(\Omega_-)$ and $g_+^\delta \in W^{2\theta,p}(\Omega_+^\delta)$ then

$$f_-^\delta = g_-^\delta \in W^{2\theta,p}(\Omega_-).$$

Moreover, for any $\delta > 0$

$$\begin{aligned} \|f_+^\delta\|_{L^p(\Omega_+)}^p &= \int_{\Omega_+} |f_+^\delta(x,y)|^p\,dx\,dy = \int_0^1 \int_G |f_+^\delta(x,y)|^p\,dx\,dy \\ &= \frac{1}{\delta}\int_0^\delta \int_G |f_+^\delta(\xi/\delta,\eta)|^p\,d\xi\,d\eta \\ &= \frac{1}{\delta}\int_{\Omega_+^\delta} |g_+^\delta(\xi,\eta)|^p\,d\xi\,d\eta < \infty, \end{aligned}$$

and

$$\begin{aligned}
[f_+^\delta]_{2\theta,p,\Omega_+} &= \iint_{\Omega_+\times\Omega_+} \frac{|f_+^\delta(x,y)-f_+^\delta(x',y')|^p}{\|(x,y)-(x',y')\|^{2\theta p+n}}\,dx\,dx'\,dy\,dy' \\
&= \left(\int_0^1\int_0^1\left(\int_{G\times G}\frac{|g_+^\delta(\delta x,y)-g_+^\delta(\delta x',y')|^p}{\|(x,y)-(x',y')\|^{2\theta p+n}}\,dy\,dy'\right)dx\,dx'\right) \\
&= \frac{1}{\delta^2}\int_0^\delta\int_0^\delta\left(\int_{G\times G}\frac{|g_+^\delta(\xi,\eta)-g_+^\delta(\xi',\eta')|^p}{\|(\xi/\delta,\eta)-(\xi'/\delta,\eta')\|^{2\theta p+n}}\,d\eta\,d\eta'\right)d\xi\,d\xi' \\
&\le C(\delta)\iint_{\Omega_+^\delta\times\Omega_+^\delta}\frac{|g_+^\delta(\xi,\eta)-g_+^\delta(\xi',\eta')|^p}{\|(\xi,\eta)-(\xi',\eta')\|^{2\theta p+n}}\,d\eta\,d\eta'\,d\xi\,d\xi' \\
&= C(\delta)[g_+^\delta]_{2\theta,p,\Omega_+^\delta} < \infty.
\end{aligned}$$

Thus $f_+^\delta \in W^{2\theta,p}(\Omega_+)$. Using the decomposition $\Omega = \Omega_- \cup \Gamma^0 \cup \Omega_+$ and in virtue of [7, page 323, Remark 3] one deduce

$$f^\delta \in W^{2\theta,p}(\Omega).$$

Then Theorem 3.1 (statement (ii)) applies and gives statement (ii) of Theorem 1.1.

4.2. Proofs of Theorems 1.2 and 1.3

Statements (a)–(c) of Theorem 1.2 lead us to set $f_- = g_-$ on Ω_- and imply that

$$\begin{gathered}
\lim_{\delta\to 0}\|f_-^\delta - f_-\|_{L^p(\Omega_-)} = \lim_{\delta\to 0}\|g_-^\delta - g_-\|_{L^p(\Omega_-)} = 0, \\
\int_{\Omega_+}|f_+^\delta(x,y)|^p\,dx\,dy = \frac{1}{\delta}\int_{\Omega_+^\delta}|g_+^\delta(\xi,\eta)|^p\,d\xi\,d\eta \le M^p,
\end{gathered}$$

and

$$\int_0^1 f_+^\delta(x,\cdot)\,dx = \frac{1}{\delta}\int_0^\delta g_+^\delta(\xi,\cdot)\,d\xi = m_+^\delta \to m \quad \text{in } L^p(G) \text{ as } \delta\to 0.$$

Thus, Theorem 3.2 can be applied. So there exists a unique $(u_-,u_+)\in L^p(\Omega_-)\times L^p(G)$ such that

$$\lim_{\delta\to 0}\left\|u_\pm^\delta - u_\pm\right\|_{L^p(\Omega_\pm)} = 0.$$

Setting $v_- = u_-$ on Ω_-, $v_+ = u_+$ on G, we obtain

$$\lim_{\delta\to 0}\|v_-^\delta - v_-\|_{L^p(\Omega_-)} = 0,$$

and

$$\begin{aligned}\lim_{\delta\to 0}\delta^{-1/p}\|v_+^\delta - v_+\|_{L^p(\Omega_+^\delta)} &= \lim_{\delta\to 0}\left(\frac{1}{\delta}\int_0^\delta\int_G |u_+^\delta(\xi/\delta,\eta) - u_+(\eta)|^p\,d\xi\,d\eta\right)^{1/p}\\ &= \lim_{\delta\to 0}\left(\int_0^1\int_G |u_+^\delta(x,y) - u_+(y)|^p\,dx\,dy\right)^{1/p}\\ &= \lim_{\delta\to 0}\|u_+^\delta - u_+\|_{L^p(\Omega_+)} = 0.\end{aligned}$$

Using the estimates in Theorem 3.2, we get

$$\begin{aligned}\|v_-^\delta - v_-\|_{L^p(\Omega_-)} \le{}& C\cdot\delta\big(\|g_-^\delta\|_{L^p(\Omega_-)} + C\delta^{-1/p}\|g_+^\delta\|_{L^p(\Omega_+^\delta)}\big)\\ &+ C\big(\big\|g_-^\delta - g_-\big\|_{L^p(\Omega_-)} + \|m_+^\delta - m\|_{L^p(G)}\big),\end{aligned}$$

and

$$\begin{aligned}\delta^{-1/p}\|v_+^\delta - v_+\|_{L^p(\Omega_+^\delta)} \le{}& C\cdot\delta\big(\|g_-^\delta\|_{L^p(\Omega_-)} + \delta^{-1/p}\|g_+^\delta\|_{L^p(\Omega_+^\delta)}\big)\\ &+ C\big(\|g_-^\delta - g_-\|_{L^p(\Omega_-)} + \|m_+^\delta - m\|_{L^p(\Omega_+)}\big).\end{aligned}$$

Finally $v_- = u_-$ is the unique strong solution of the non homogeneous Ventcel's cited Problem.

By an analogous way, we obtain Theorem 1.3.

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