Indices of 1-Forms and Newton Polyhedra

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ABSTRACT

A formula of Matsuo Oka [9] expresses the Milnor number of a germ of a complex analytic map with a generic principal part in terms of the Newton polyhedra of the components of the map. In this paper this formula is generalized to the case of the index of a 1-form on a local complete intersection singularity (Theorem 1.10, Corollaries 1.11, 4.1). In particular, the Newton polyhedron of a 1-form is defined (Definition 1.6). This also simplifies the Oka formula in some particular cases (Propositions 3.5, 3.7).

Key words: Newton polyhedra, singularities of vector fields

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1. Indices of 1-forms

In this paper we give a formula for the index of a 1-form on a local complete intersection singularity. First of all we recall the definition of this index (introduced by W. Ebeling and S. M. Gusein-Zade).

Definition 1.1 ([5,6]). Consider a germ of a map $\bar{f} = (f_1, \dots, f_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$, $k < n$ and a germ of a 1-form ω on $(\mathbb{C}^n, 0)$. Suppose that $\bar{f} = 0$ is an $(n-k)$-dimensional complete intersection with an isolated singular point at the origin, and the restriction $\omega|_{\{\bar{f}=0\}}$ has not singular points (zeroes) in a punctured neighborhood of the origin. For a small sphere S_δ^{2n-1} around the origin the set $S_\delta^{2n-1} \cap \{\bar{f} = 0\} = M^{2n-2k-1}$ is a smooth manifold. One can define the map $(\omega, df_1, \dots, df_k) : M^{2n-2k-1}$

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$\to W(n,k+1)$ to the Stiefel manifold of $(k+1)$-frames in $\mathbb{C}^n$. The image of the fundamental class of the manifold $M^{2n-2k-1}$ in the homology group $H_{2n-2k-1}(W(n,k+1)) = \mathbb{Z}$ is called the index $\mathrm{ind}_0\,\omega|_{\{\bar f=0\}}$ of the 1-form ω on the local complete intersection singularity $\{\bar f = 0\}$ (all orientations are defined by the complex structure).

Remark 1.2. One can consider this index as a generalization of the Milnor number. Indeed, let g be a complex analytic function, then $\mathrm{ind}_0\, dg|_{\{\bar f=0\}}$ is equal to the sum of the Milnor numbers of the germs $(f_1,\dots,f_k) : (\mathbb{C}^n,0) \to (\mathbb{C}^k,0)$ and $(g,f_1,\dots,f_k) : \mathbb{C}^n \to \mathbb{C}^{k+1}$ (if $k = 0$ then the first Milnor number is 0). This follows from [6, Example 2.6 and Proposition 2.8].

Now we introduce some necessary notation and recall the statement of the Oka theorem. Suppose $f_1,\dots,f_k$ are holomorphic functions on a smooth complex manifold V. Then "$f_1 = \dots = f_k = 0$ is a generic system of equations in V" means "$df_1,\dots,df_k$ are linearly independent at any point of the set $\{f_1 = \dots = f_k = 0\}$."

Definition 1.3. Suppose $f : (\mathbb{C}^n,0) \to (\mathbb{C},0)$ is a germ of a complex analytic function. Represent f as a sum over a subset of the integral lattice $f(x) = \sum_{c\in A\subset\mathbb{Z}^n_+} f_c x^c$, where $f_c \in \mathbb{C}\setminus\{0\}$, $\mathbb{Z}_+ = \{z \in \mathbb{Z}|\, z \geq 0\}$, and x^c means $x_1^{c_1},\dots,x_n^{c_n}$. The convex hull Δ_f of the set $(A+\mathbb{R}^n_+) \subset \mathbb{R}^n_+ = \{\, r\in\mathbb{R} \mid r \geq 0\,\}^n$ is called the Newton polyhedron of f.

We denote by $(\mathbb{Z}^n_+)^*$ the set of covectors $\gamma \in (\mathbb{Z}^n)^*$ such that $(\gamma, v) > 0$ for every $v \in \mathbb{Z}^n_+, v \neq 0$. Consider a polyhedron $\Delta \subset \mathbb{R}^n_+$ with integer vertices and a covector $\gamma \in (\mathbb{Z}^n_+)^*$. As a function on Δ the linear form γ achieves its minimum on a maximal compact face of Δ. Denote this face by Δ^γ. Denote by f^γ the polynomial $\sum_{c\in\Delta^\gamma_f} f_c x^c$.

Definition 1.4. A collection of germs of functions $f_1,\dots,f_k$ on $(\mathbb{C}^n,0)$ is called $\mathbb{C}$-generic, if for every $\gamma \in (\mathbb{Z}^n_+)^*$ the system $f_1^\gamma = \dots = f_k^\gamma = 0$ is a generic system of equations in $(\mathbb{C}\setminus\{0\})^n$. A collection of germs $f_1,\dots,f_k$ is called strongly $\mathbb{C}$-generic, if the collections $(f_1,\dots,f_k)$ and $(f_2,\dots,f_k)$ are $\mathbb{C}$-generic.

Theorem 1.5 ([9, Theorem (6.8), ii]). *Suppose that a collection of germs of complex analytic functions $f_1,\dots,f_k$ on $(\mathbb{C}^n,0)$ is strongly $\mathbb{C}$-generic and the polyhedra $\Delta_{f_1},\dots,\Delta_{f_k} \subset \mathbb{R}^n_+$ intersect all coordinate axes. Then the Milnor number of the map $(f_1,\dots,f_k)$ equals the number $\mu(\Delta_{f_1},\dots,\Delta_{f_k})$ which depends only on the Newton polyhedra of the components of the map.*

The explicit formula for $\mu(\Delta_1,\dots,\Delta_k)$ in terms of the integral volumes of some polyhedra associated to $\Delta_1,\dots,\Delta_k$ is given in [9], Theorem (6.8), ii. In the case $k = 1$ one has the well-known Kouchnirenko formula [8] for the Milnor number of a germ of a function.

To generalize this theorem we generalize Definitions 1.3 and 1.4 first.

Definition 1.6. One can formally represent an analytic 1-form ω on $\mathbb{C}^n$ as $\sum_{c\in A} x^c\omega_c$, where $A \subset \mathbb{Z}^n_+$, $\omega_c = \sum_{i=1}^n \omega_c^i \frac{dx_i}{x_i} \neq 0$, $\omega_c^i \in \mathbb{C}$. The convex hull Δ_ω of the set $A+\mathbb{R}^n_+ \subset \mathbb{R}^n_+$ is called the Newton polyhedron of the 1-form ω.

Remark 1.7. The Newton polyhedron of the differential of an analytic function coincides with the Newton polyhedron of the function itself.

Definition 1.8. A collection of germs of a 1-form ω and k functions $f_1, \dots, f_k$ on $(\mathbb{C}^n, 0)$ is called $\mathbb{C}$-generic, if for every $\gamma \in (\mathbb{Z}^n_+)^*$ the system $f_1^\gamma = \dots = f_k^\gamma = 0$ is a generic system of equations in $(\mathbb{C}\setminus\{0\})^n$, and the restriction $\omega^\gamma|_{\{f_1^\gamma=\dots=f_k^\gamma=0\}\cap(\mathbb{C}\setminus\{0\})^n}$ has not singular points (we define the polynomial 1-form ω^γ as $\sum_{c\in\Delta_\omega^\gamma} \omega_c x^c$ for $\omega = \sum_{c\in\Delta_\omega} \omega_c x^c$).

Remark 1.9. A collection $(dg, f_1, \dots, f_k)$ is $\mathbb{C}$-generic if and only if the collection $(g, f_1, \dots, f_k)$ is strongly $\mathbb{C}$-generic.

Non-$\mathbb{C}$-generic collections form a subset Σ in the set of germs with given Newton polyhedra $B(\Delta_0, \dots, \Delta_k) = \{\, (\omega, f_1, \dots, f_k) \mid \Delta_\omega = \Delta_0, \Delta_{f_i} = \Delta_i, i = 1, \dots, k \,\}$.

Theorem 1.10. *Suppose that the polyhedra $\Delta_0, \dots, \Delta_k$ in $\mathbb{R}^n_+$, $k < n$ intersect all coordinate axes. Then the index of a 1-form on a local complete intersection singularity as a function on $B(\Delta_0, \dots, \Delta_k) \setminus \Sigma$ is well defined and equals a constant.*

Corollary 1.11. *This constant equals $\mu(\Delta_1, \dots, \Delta_k) + \mu(\Delta_0, \dots, \Delta_k)$. (To prove it one can choose a 1-form to be the differential of a complex analytic function and use Theorem 1.5 and the remarks above.)*

Corollary 1.12. *In Theorem 1.5, one can substitute the strong $\mathbb{C}$-genericity condition by the $\mathbb{C}$-genericity condition. (To prove it one can choose a function g such that the collection $(g, f_1, \dots, f_k)$ is strongly $\mathbb{C}$-generic, and use Theorems 1.5 and 1.10 for it.)*

It is somewhat natural to express the index not in terms of the separate Newton polyhedra of the components of a 1-form, but in some sense in terms of their union. Indeed, consider a germ of a 1-form $\omega = (\omega_1, \dots, \omega_n)$ on $(\mathbb{C}^n, 0)$ and an $n\times n$ matrix C. If the entries of C are in general position, then all the components of the 1-form $C\omega$ have the same Newton polyhedron which is the convex hull of $\bigcup_{i=1}^n \Delta_{\omega_i}$. On the other hand, $\operatorname{ind}_0 C\omega = \operatorname{ind}_0 \omega$.

The definition of the Newton polyhedron of a 1-form is a bit different from the convex hull of the union of the Newton polyhedra of the components of a 1-form. This definition is more natural in the framework of toric geometry. Consider a monomial map $p : (\mathbb{C} \setminus \{0\})^m \to (\mathbb{C} \setminus \{0\})^n$, $v = p(z) = z^C$, where C is an $n \times n$ matrix with integer entries. Consider a 1-form $\omega = (\omega_1, \dots, \omega_n)$ on the torus $(\mathbb{C} \setminus \{0\})^n$. Then the lifting $p^*\omega$ satisfies the following equality: $z \cdot p^*\omega(z) = C(p(z) \cdot \omega(p(z)))$. In this equality we multiply vectors componentwise. Therefore, the Newton polyhedron in the sense of Definition 1.6 is invariant with respect to monomial mappings. Thus, multiplication by a matrix mixes the components of a 1-form, just as a monomial map mixes its "shifted" components $v_i \cdot \omega_i(v)$. This difference leads to some relations for integral volumes of polyhedra. We discuss them in section 3.

2. Proof of Theorem 1.10

The idea of the proof is the following. In fact, the set Σ is closed and its complex codimension is 1. Thus, it is enough to prove that the index is a locally constant function on $B(\Delta_0, \dots, \Delta_k) \setminus \Sigma$. The only problem is that the last set is infinite dimensional, so we substitute it by a finite dimensional "approximation."

The union of all compact faces of the Newton polyhedron of a function f is called the Newton diagram of a function f. We denote it by Δ^0_f. Suppose that $f(x) = \sum_{c \in \mathbb{Z}^n_+} f_c x^c$, then the polynomial $\sum_{c \in \Delta^0_f} f_c x^c$ is called the principal part of f. We denote it by f^0. Denote by $B(f)$ the set $\{\, g \mid \Delta_g = \Delta_f, g - g^0 = \lambda(f - f^0), \lambda \in \mathbb{C} \,\}$. Similarly, we define the Newton diagram Δ^0_ω, the principal part ω^0 and the set $B(\omega)$ for a 1-form ω.

A collection of germs of an analytic 1-form ω and k analytic functions $f_1, \dots, f_k$ on $(\mathbb{C}^n, 0)$ is $\mathbb{C}$-generic if and only if $(\omega^0, f^0_1, \dots, f^0_k)$ is $\mathbb{C}$-generic. The set of non-$\mathbb{C}$-generic collections $\Sigma \cap B(\omega) \times B(f_1) \times \dots \times B(f_k)$ is a (Zariski) closed proper subset of a finite dimensional set $B(\omega) \times B(f_1) \times \dots \times B(f_k)$. Its complex codimension is 1 (see, for instance, [1, ch. II, § 6.2, Lemma 1], for an example of the proof of such facts).

Now we can reformulate Theorem 1.10 in the following form:

Lemma 2.1. *For any $\mathbb{C}$-generic collection $(\omega, f_1, \dots, f_k)$ there exists a neighborhood $U \subset B(\omega) \times B(f_1) \times \dots \times B(f_k)$ of it and a punctured neighborhood $V \subset \mathbb{C}^n$ around the origin such that for any $(\upsilon, g_1, \dots, g_k) \in U$ the system $g_1 = \dots = g_k = 0$ is a generic system of equations in V and the restriction $\upsilon|_{\{g_1=\dots=g_k=0\} \cap V}$ has no singular points (in particular the index $\operatorname{ind}_0 \upsilon|_{\{g_1=\dots=g_k=0\}}$ is well defined and equals $\operatorname{ind}_0 \omega|_{\{f_1=\dots=f_k=0\}}$).*

Consider the toric resolution $p : (M, D) \to (\mathbb{C}^n, 0)$ related to a simplicial fan Γ compatible with $\Delta_\omega, \Delta_{f_1}, \dots, \Delta_{f_k}$ (see [1, ch. II, § 8.2, Theorem 2] or [9, § 4] for definitions). We call it a toric resolution of the collection $(\omega, f_1, \dots, f_k)$. Since the exceptional divisor D is compact, we can reformulate Lemma 2.1 as follows:

Lemma 2.2. *For any $y \in D$ there exist neighborhoods $U_y \subset B(\omega) \times B(f_1) \times \dots \times B(f_k)$ around $(\omega, f_1, \dots, f_k)$ and $V_y \subset M$ around y such that for every $(\upsilon, g_1, \dots, g_k) \in U_y$ the system $(g_1, \dots, g_k) \circ p = 0$ is a generic system of equations in $(V_y \setminus D)$ and the restriction $p^* \upsilon|_{\{(g_1,\dots,g_k) \circ p = 0\} \cap (V_y \setminus D)}$ has no critical points.*

Proof. M is a toric manifold, so we have a natural action of the complex torus $(\mathbb{C} \setminus \{0\})^n$ on M. The exceptional divisor D is invariant with respect to this action. Denote by D_y the orbit of the point y. The exceptional divisor D has the minimal decomposition into the union of disjoint smooth strata. Denote by D^0_y the stratum of D, such that $y \in D^0_y$ (if y is in the closure of the set p^{-1}(the union of coordinate planes $\setminus \{0\}$) then $D_y \subsetneq D^0_y$). If $a \in T^*_z M$ is orthogonal to the orbit of $z \in M$ under the action of the stabilizer of D^0_y, then we (formally) write $a \parallel D^0_y$.

Now we consider the three cases of location of the point y on D_y with respect to the collection $(\omega, f_1, \dots, f_k)$.

Case 1. $y \notin \overline{(\{(f_1, \dots, f_k) \circ p = 0\} \setminus D_y)} \cap D_y$.

Case 2. y doesn't satisfy the condition of the case 1, but $y \notin \overline{(\{p^*\omega \| D_y^0\} \setminus D_y)} \cap D_y$.

Case 3. y doesn't satisfy the conditions of the cases 1 and 2.

To prove Lemma 2.2 in these three cases we need a coordinate system near D_y. Let $m = n - \dim D_y$. By definition of a toric variety related to a fan the orbit D_y corresponds to some m-dimensional cone Γ_y. Denote by s the number of coordinate axes which are generatrices of Γ_y. Then $s = \dim D_y^0 - \dim D_y$. Γ_y is a face of some n-dimensional cone in the fan Γ. Coordinates of generating covectors of this cone form as row-vectors an integral square matrix B with nonnegative entries. After an appropriate reordering of variables the first m its rows correspond to the generating covectors of Γ_y, and the first s of them coincide with the first rows of the unit matrix.

This cone gives a system of coordinates $z_1, \dots, z_n$ on a (Zariski) open set containing D_y. These coordinates are given by the equation $(z_1, \dots, z_n)^B = (x_1, \dots, x_n) \circ p$ (note that $\|B\| = \pm 1$ because Γ is chosen to be simplicial). We can describe D_y, $f_1 \circ p, \dots, f_k \circ p$ and the components of

$$p^*\omega = \begin{pmatrix} (p^*\omega)^1 \\ \vdots \\ (p^*\omega)^n \end{pmatrix}$$

in this coordinate system as follows ($\bar{o}$ means a smooth function on an open neighborhood of D_y which equals zero on D_y):

(i) $D_y = \{z_1 = \cdots = z_m = 0, z_{m+1} \neq 0, \dots, z_n \neq 0\}$; $D_y^0 = \{z_{s+1} = \cdots = z_m = 0\}$; $a \parallel D_y^0 \Leftrightarrow a \perp \langle \frac{\partial}{\partial z_{s+1}}, \dots, \frac{\partial}{\partial z_m} \rangle$.

(ii) $(f_i \circ p)(z_1, \dots, z_n) = z_{s+1}^{\varphi_i^{s+1}} \cdots z_m^{\varphi_i^m} (\hat{f}_i(z_{m+1}, \dots, z_n) + \bar{o})$ where $i = 1, \dots, k$.

(iii) $(p^*\omega)^i(z_1, \dots, z_n) = z_{s+1}^{\nu^{s+1}} \cdots z_m^{\nu^m} (\widehat{(p^*\omega)_i}(z_{m+1}, \dots, z_n) + \bar{o})$ where $i = 1, \dots, s$.

(iv) $(p^*\omega)^i(z_1, \dots, z_n) = z_{s+1}^{\nu^{s+1}} \cdots z_m^{\nu^m} (\widehat{(p^*\omega)^i}(z_{m+1}, \dots, z_n) + \bar{o}) z_i^{-1}$ where $i = s+1, \dots, n$.

These descriptions are related to the functions which appear in the definition of $\mathbb{C}$-genericity. Namely, for any $\gamma \in \Gamma_y$:

(ii') $(f_i^\gamma \circ p)(z_1, \dots, z_n) = z_{s+1}^{\varphi_i^{s+1}} \cdots z_m^{\varphi_i^m} \hat{f}_i(z_{m+1}, \dots, z_n)$ where $i = 1, \dots, k$.

(iii') $(\omega^\gamma)^i = 0$ for $i = 1, \dots, s$ by definition.

(iv')

$$B\begin{pmatrix} 0\\ \vdots\\ 0\\ x_{s+1}(\omega^\gamma)^{s+1}\\ \vdots\\ x_n(\omega^\gamma)^n \end{pmatrix}\circ p = z_{s+1}^{\nu^{s+1}}\cdots z_m^{\nu^m}\begin{pmatrix} 0\\ \vdots\\ 0\\ \widehat{(p^*\omega)^{s+1}}(z_{m+1},\dots,z_n)\\ \vdots\\ \widehat{(p^*\omega)^n}(z_{m+1},\dots,z_n) \end{pmatrix}.$$

Now we can prove Lemma 2.2.

Case 1. This means that $y \notin \{\hat{f}_1 = \dots = \hat{f}_k = 0\}$. The same holds for y' close to y in M and $(g_1,\dots,g_k)$ close to $(f_1,\dots,f_k)$ in $B(f_1)\times\dots\times B(f_k)$. Thus if $y' \notin D$ is close to y then $y' \notin \{(g_1,\dots,g_k)\circ p = 0\}$.

Case 2. (Informally, in this case υ is almost orthogonal to D^0_y near y.) This means that y does not satisfy the condition of the case 1 and $y \notin \{\widehat{(p^*\omega)^{s+1}} = \dots = \widehat{(p^*\omega)^m} = 0\}$. Choose $j_0 \in \{s+1,\dots,m\}$ such that $\widehat{(p^*\omega)^{j_0}}(y) \neq 0$. Then the same holds for y' close to y in M and υ close to ω in $B(\omega)$.

From $\mathbb{C}$-genericity, (ii), and (ii') it follows that $\hat{f}_1 = \dots = \hat{f}_k = 0$ is a generic system of equations in $(\mathbb{C}\setminus\{0\})^n$. Thus we can choose $\{j_1,\dots,j_k\}\subset\{m+1,\dots,n\}$ such that $\left\|\frac{\partial \hat{f}_i}{\partial z_j}(y)\right\|^{i=1,\dots,k}_{j=j_1,\dots,j_k} \neq 0$. Then the same holds for y' close to y in M and $(g_1,\dots,g_k)$ close to $(f_1,\dots,f_k)$ in $B(f_1)\times\dots\times B(f_k)$.

The matrix $U = p^*(\upsilon, dg_1,\dots,dg_k)$ has the full rank for $y' \notin D$ close to y in M and $(\upsilon, g_1,\dots,g_k)$ close to $(\omega, f_1,\dots,f_k)$ in $B(\omega)\times B(f_1)\times\dots\times B(f_k)$. Indeed,

$$\|U_{i,j}\|^{i=1,\dots,k+1}_{j=j_0,\dots,j_k} = z_{s+1}^{\nu^{s+1}+\varphi_1^{s+1}+\dots+\varphi_k^{s+1}}\cdots z_m^{\nu^m+\varphi_1^m+\dots+\varphi_k^m} z_{j_0}^{-1}\cdots z_{j_k}^{-1}\times \\ \times\left(\widehat{(p^*\upsilon)^{j_0}}\left\|\frac{\partial \hat{g}_i}{\partial z_j}\right\|^{i=1,\dots,k}_{j=j_1,\dots,j_k} + \bar{o}\right)\neq 0.$$

Case 3. In this case $y \in \{\hat{f}_1 = \dots = \hat{f}_k = 0\}\cap\{\widehat{(p^*\omega)^{s+1}} = \dots = \widehat{(p^*\omega)^m} = 0\}$. From $\mathbb{C}$-genericity, (iii), (iii'), (iv), and (iv') it follows that the matrix $(\widehat{p^*\omega}, d\hat{f}_1,\dots, d\hat{f}_k)$ has the rank $k+1$. Thus some of its minors U_0 (suppose it consists of rows $j_0 > \dots > j_k > m$) is nonzero and the same holds for y' close to y in M and $(\upsilon, g_1,\dots,g_k)$ close to $(\omega, f_1,\dots,f_k)$ in $B(\omega)\times B(f_1)\times\dots\times B(f_k)$.

The same minor of the matrix $U = p^*(\upsilon, dg_1,\dots,dg_k)$ is equal to $z_{j_0}^{-1}\cdots z_{j_k}^{-1}\times z_{s+1}^{\nu^{s+1}+\varphi_1^{s+1}+\dots+\varphi_k^{s+1}}\cdots z_m^{\nu^m+\varphi_1^m+\dots+\varphi_k^m}(U_0+\bar{o})\neq 0$. Thus U has the full rank for $y'\notin D$ close to y in M and $(\upsilon, g_1,\dots,g_k)$ close to $(\omega, f_1,\dots,f_k)$ in $B(\omega)\times B(f_1)\times\dots\times B(f_k)$.

Lemma 2.2 and, consequently, Theorem 1.10 are proved. □

3. Interlaced polyhedra

Consider polyhedra $\Delta_1, \ldots, \Delta_n \subset \mathbb{R}^n_+$. Denote by $U_{\Delta_1,\ldots,\Delta_n}$ the convex hull of $\bigcup_{i=1}^n \Delta_i$.

Definition 3.1. Suppose that for any $\gamma \in (\mathbb{Z}^n_+)^*$ there exists $I \in \{1, \ldots, n\}$, $|I| = \dim U^\gamma_{\Delta_1,\ldots,\Delta_n} + 1$ such that $\Delta^\gamma_i \subset U^\gamma_{\Delta_1,\ldots,\Delta_n}$ for any $i \in I$. Then the polyhedra $\Delta_1, \ldots, \Delta_n$ are said to be interlaced.

The notion of interlaced polyhedra is related to the notion of $\mathbb{C}$-genericity. As a consequence, Oka formulas from [9] and Theorem 1.10 give some interrelations for the polyhedra $\Delta_1, \ldots, \Delta_n$ and $U_{\Delta_1,\ldots,\Delta_n}$ provided $\Delta_1, \ldots, \Delta_n$ are interlaced. The aim of the discussion below is to point out these facts.

Suppose that $\Delta_1, \ldots, \Delta_n \subset \mathbb{R}^n_+$ are convex polyhedra with integer vertices and the sets $\mathbb{R}^n_+ \setminus \Delta_1, \ldots, \mathbb{R}^n_+ \setminus \Delta_n$ are bounded. Suppose $\omega = \sum_{i=1}^n \omega_i \, dx_i$ is a germ of a 1-form such that the Newton polyhedra of $\omega_1, \ldots, \omega_n$ are $\Delta_1, \ldots, \Delta_n$ (with respect to a coordinate system $x_1, \ldots, x_n$ on $(\mathbb{C}^n, 0)$). We can also consider the collection $(\omega_1, \ldots, \omega_n)$ as a map $\omega_* = (\omega_1, \ldots, \omega_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Generally speaking, the $\mathbb{C}$-genericity of the map $w_* : \mathbb{C}^n \to \mathbb{C}^n$ in sense of the definition 1.4 and the $\mathbb{C}$-genericity of the 1-form w in sense of the definition 1.8 are not related. The following lemmas are obvious (they follow from the Bertini-Sard theorem, see [1, ch.II, § 6.2, Lemma 1] for an example of the proof of such facts).

Lemma 3.2. *If $\Delta_1, \ldots, \Delta_n$ are interlaced, then, for a generic complex square matrix B and generic principal parts of $\omega_1, \ldots, \omega_n$, the map $(B\omega)_* : \mathbb{C}^n \to \mathbb{C}^n$ is $\mathbb{C}$-generic.*

Denote by $e_1, \ldots, e_n$ the standard basis of $\mathbb{Z}^n$,

$$e_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, 0, \ldots, 0).$$

Lemma 3.3. *If $\Delta_1 + e_1, \ldots, \Delta_n + e_n$ are not interlaced, then the 1-form ω is not $\mathbb{C}$-generic. If they are interlaced, then the condition of $\mathbb{C}$-genericity of the 1-form ω on a local complete intersection singularity $\{f_1 = \cdots = f_k = 0\}$ is a condition of general position for the principal parts of $\omega_1, \ldots, \omega_n, f_1, \ldots, f_k$.*

Remark 3.4. This lemma implies that the Newton diagrams of $\omega_i x_i$ don't necessary belong to the Newton diagram of a $\mathbb{C}$-generic 1-form $\omega = \sum_{i=1}^n \omega_i \, dx_i$. For instance, suppose $n = 2$: the Newton diagram of Δ_i, $i = 1, 2$, consists of N edges, and the j-th edge of $\Delta_1 + e_1$ intersects the j-th edge of $\Delta_2 + e_2$ for any j. Then, by Lemma 3.3, there exists a $\mathbb{C}$-generic 1-form $\omega = \omega_1 \, dx_1 + \omega_2 \, dx_2$ such that $\Delta_{\omega_i} = \Delta_i$ for $i = 1, 2$.

Recall that $\mu(\Delta_{f_1}, \ldots, \Delta_{f_m})$ is the Milnor number $\mu(f_1, \ldots, f_m)$ of a germ of a $\mathbb{C}$-generic map $(f_1, \ldots, f_m)$. Denote by Vol the integral volume in $\mathbb{R}^n \supset \mathbb{Z}^n$ (such that $\operatorname{Vol}[0, 1]^n = 1$).

Proposition 3.5. *If $\Delta_1, \dots, \Delta_n$ are interlaced, then*

$$\mu(\Delta_1, \dots, \Delta_n) = n! \operatorname{Vol}(\mathbb{R}^n_+ \setminus U_{\Delta_1,\dots,\Delta_n}) - 1.$$

Proof. This statement is true if all the polyhedra coincide (this is a consequence of the Oka formula, see [9, Theorem (7.2)]). The following equality is obvious: $\mu(\omega_1, \dots, \omega_n) = \operatorname{ind}_0 \omega - 1 = \operatorname{ind}_0(B\omega) - 1 = \mu((B\omega)_1, \dots, (B\omega)_n)$. Now one can apply these facts to a 1-form $\omega = \sum_{i=1}^n \omega_i \, dx_i$ such that the maps ω_* and $(B\omega)_*$ are $\mathbb{C}$-generic (they exist because of Lemma 3.2), and the Newton polyhedra of all the components of $(B\omega)_*$ are equal to $U_{\Delta_1,\dots,\Delta_n}$. □

Remark 3.6. This statement gives an independent proof of Theorem 1.10 in the case $k = 0$. One can use Proposition 3.5 and the evident equation

$$\mu(x_1\omega_1, \dots, x_n\omega_n) = \sum_{\{i_1,\dots,i_m\} \subsetneq \{1,\dots,n\}} \operatorname{ind}_0 \omega|_{\{x_{i_1} = \dots = x_{i_m} = 0\}}$$

to prove this particular case by induction on n. (If the 1-form ω is $\mathbb{C}$-generic then any map

$$(x_1\omega_1, \dots, x_n\omega_n)|_{\{x_{i_1}=\dots=x_{i_m}=0\}} : \{x_{i_1} = \dots = x_{i_m} = 0\} \to \{x_{i_1} = \dots = x_{i_m} = 0\}$$

is $\mathbb{C}$-generic as well.)

Proposition 3.7. *If the polyhedra $\Delta_1 + e_1, \dots, \Delta_n + e_n$ are interlaced, then*

$$\mu(\Delta_1, \dots, \Delta_n) = \mu(U_{\Delta_1+e_1,\dots,\Delta_n+e_n}) - 1.$$

It is a consequence of Theorems 1.5 and 1.10 and the equation $\mu(\omega_1, \dots, \omega_n) = \operatorname{ind}_0 \omega - 1$ (one should choose $\omega_1, \dots, \omega_n$ such that the 1-form ω and the map ω_* are $\mathbb{C}$-generic).

Corollary 3.8. *If the polyhedra $\Delta_1, \dots, \Delta_n$ are interlaced and the polyhedra $\Delta_1 + e_1, \dots, \Delta_n + e_n$ are interlaced, then*

$$\mu(U_{\Delta_1+e_1,\dots,\Delta_n+e_n}) = n! \operatorname{Vol}(\mathbb{R}^n_+ \setminus U_{\Delta_1,\dots,\Delta_n}).$$

One can easily give a straightforward combinatorial proof of this equation (it is enough to explicitly express these volumes in terms of the coordinates of the vertices of the polyhedra).

Remark 3.9. In a similar way we can define interlaced compact polyhedra: compact polyhedra $\Delta_1, \dots, \Delta_n \subset \mathbb{R}^n$ are interlaced if for any $\gamma \in (\mathbb{R}^n)^*$ there exists $I \in \{1, \dots, n\}$, $|I| = \dim U^\gamma_{\Delta_1,\dots,\Delta_n} + 1$ such that $\Delta_i^\gamma \subset U^\gamma_{\Delta_1,\dots,\Delta_n}$ for any $i \in I$. In the same way we can prove that, for interlaced polyhedra $\Delta_1, \dots, \Delta_n \subset \mathbb{R}^n$, the mixed volume of $\Delta_1, \dots, \Delta_n$ equals $\operatorname{Vol}(U_{\Delta_1,\dots,\Delta_n})$. As a consequence, the volume $\operatorname{Vol}(U_{\Delta_1+\bar{a}_1,\dots,\Delta_n+\bar{a}_n})$ does not depend on $\bar{a}_1, \dots, \bar{a}_n \in \mathbb{R}^n$, if the polyhedra $\Delta_1 + \bar{a}_1, \dots, \Delta_n + \bar{a}_n$ are interlaced.

4. Remarks

For a 1-form on a germ of a manifold with an isolated singular point there is defined the, so called, radial index (see [6, Definition 2.1]). The radial index of a 1-form ω on a local complete intersection singularity $f_1 = \cdots = f_k = 0$ equals $\operatorname{ind}_0 \omega|_{\{\bar{f}=0\}}$ minus the Milnor number of the map $(f_1, \ldots, f_k)$.

Corollary 4.1. *Suppose a collection of germs $\omega, f_1, \ldots, f_k$ on $\mathbb{C}^n$ is $\mathbb{C}$-generic and the polyhedra $\Delta_\omega, \Delta_{f_1}, \ldots, \Delta_{f_k} \subset \mathbb{R}^n_+$ intersect all coordinate axes. Then $V^{n-k} = \{f_1 = \cdots = f_k = 0\}$ is a local complete intersection singularity and the radial index of ω on V^{n-k} equals $\mu(\Delta_\omega, \Delta_{f_1} \ldots, \Delta_{f_k})$.*

This corollary follows from Theorems 1.5 and 1.10.

This corollary and Theorem 1.10 are generalizations of the Oka theorem, which is a consequence of the A'Campo theorem (see [2]). Thus, it would be interesting to obtain this corollary and Theorem 1.10 as consequences of a generalization of the A'Campo theorem. To do it, we need the notion of a resolution of a germ of a 1-form on a germ of a manifold with an isolated singular point. Namely, we can try to generalize the notion of a toric resolution of a 1-form on a local complete intersection singularity, taking the three cases from the proof of Lemma 2.2 as a definition of a resolution.

Let $(V, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a variety. Suppose $V \setminus \{0\}$ is smooth. Let ω be a 1-form on $(\mathbb{C}^n, 0)$. Suppose $\omega|_{V\setminus\{0\}}$ has no singular points near 0.

Definition 4.2. Let $p : (M, D) \to (V, 0)$ be a proper map. Suppose M is smooth, $D = p^{-1}(0)$ is a normal crossing divisor, $D = \bigsqcup D_i$ is the minimal stratification such that D_i are smooth, and p is biholomorphic on $M \setminus D$. Suppose that, for any $y \in D_i \subset D$ and for any holomorphic vector field v near y such that $v(y) \notin T_y(D_i)$, there exists a neighborhood $U \subset M$ of y such that

(i) $\langle p^*(\omega), v\rangle = 0$ is a generic system of equations in $U \setminus D$,

(ii) $\{\langle p^*(\omega), v\rangle = 0\} \cap U$ is a normal crossing divisor.

(In coordinates, these conditions mean that $\langle p^*(\omega), v\rangle$ equals either $x_1^{a_1} \cdots x_k^{a_k}$ or $x_1^{a_1} \cdots x_k^{a_k} x_{k+1}$, where $a_i \in \mathbb{N}$, and $(x_1, \ldots, x_n)$ are coordinates near y such that $D = \{x_1 \cdots x_k = 0\}$). Then p is called a resolution of (ω, V).

The toric resolution from the proof of Lemma 2.2 is a partial case of a resolution in sense of this definition. If $w = df$, then a resolution of f in the sense of Hironaka is a resolution of w in the sense of this definition. It would be interesting to know, whether every (ω, V) is resolvable. There are some works on resolutions of singular points of vector fields and 1-forms, especially integrable and low-dimensional ones, see [3], [7], [4].

Form the sets

$$S_m = \{\, y \in D \mid \text{ the function } \langle p^*(\omega), v\rangle \text{ in a neighborhood of } y \\ \text{has the form } z^m, \text{ where } z \text{ is some local coordinate on } M \text{ near } y\,\}.$$

Consider the straightforward generalization of the A'Campo formula:

Conjecture. *The radial index of* ω *on* V *equals* $(-1)^n\bigl(-1 + \sum_{m\geq 1} m\chi(S_m)\bigr)$.

Theorem 1.10 proves this generalization in the toric case. The A'Campo formula itself proves it if ω is the differential of a function. This generalization is also obviously true in the case $n-k=1$. It would be interesting to know whether this generalization is true in the general case.

The simplest example to illustrate Theorem 1.10 is the following: $n = 2$, $k = 0$, $\omega_1 = x^a + y^b$, $\omega_2 = x^c + y^d$, $\frac{a}{b} > \frac{c}{d}$, and a, b, c, d are coprime. Then $\operatorname{ind}_0 \omega = \mu(\omega_1, \omega_2) + 1 = bc$ (the last equation illustrates the Oka formula). The Newton polyhedron Δ_ω is generated by the points $(a+1, 0), (c, 1), (1, b), (0, d+1)$. The Newton polyhedra of the components are interlaced when $c < a$, $b < d$. In accordance with Theorem 1.10, the index $\operatorname{ind}_0 \omega$ can be computed by the Kouchnirenko formula $\mu(\Delta_\omega)$ if and only if the Newton polyhedra of the components are interlaced.

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