

# Boundedness for Threefolds in $\mathbb{P}^6$ Containing a Smooth Ruled Surface as Hyperplane Section

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## ABSTRACT

Let  $X \subset \mathbb{P}^6$  be a smooth irreducible projective threefold, and  $d$  its degree. In this paper we prove that there exists a constant  $\beta$  such that for all  $X$  containing a smooth ruled surface as hyperplane section and not contained in a fourfold of degree less than or equal to 15,  $d \leq \beta$ . Under some more restrictive hypothesis we prove an analogous result for threefolds containing a smooth ruled surface as hyperplane section and contained in a fourfold of degree less than or equal to 15.

*Key words:* threefolds, low codimensional subvarieties, ruled surface as hyperplane section, bounded degree.

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## Introduction

Let  $\mathcal{S}(\mathbb{P}^N)$  be the set of all irreducible smooth projective  $n$  dimensional varieties of the complex projective space  $\mathbb{P}^N$ . As in [5] we say that a subset  $\mathcal{S} \subset \mathcal{S}(\mathbb{P}^N)$  is bounded, or equivalently that varieties in  $\mathcal{S}$  are bounded, if

$$\sup\{\deg(X) : X \in \mathcal{S}\} < +\infty.$$

A famous conjecture of Hartshorne and Lichtenbaum states that  $n$  dimensional non general type varieties of  $\mathbb{P}^N$ , with  $N \geq 2n$ , are bounded. In [7] the conjecture is proved for  $n = 2, N = 4$ , in [2] for  $n = 3, N = 5$ , and in [13] it is proved in the range  $n \geq \frac{N+2}{2}$ . As far as we know, nothing is known for  $n = 3, N = 6$ .

In this paper we will discuss boundedness for smooth threefolds in  $\mathbb{P}^6$  containing a smooth ruled surface as hyperplane section, or *threefolds with ruled surface section* for short. The reason for restricting to this class of varieties is that in this case we have a, at least coarse, classification (see [1, p. 205, table 7.3]. By analogy with [7] (see also [5]), we first bound the degree of threefolds with ruled surface section not contained in a fourfold of fixed degree less than or equal to 15. This is the content of our main result, Theorem 2.1. Section 2 is devoted to the proof of this Theorem. Next we face boundedness for threefolds contained in a fourfold of fixed degree. We are able to prove only a partial result in this direction, this is discussed in section 3.

### 1. Preliminaries

We work over the complex numbers.

**Notation 1.1.** Denote by  $X \subset \mathbb{P}^6$  a smooth irreducible projective variety,  $\dim(X) = 3$ , and  $H$  the class of an hyperplane section. By  $d = H^3$  we will denote the degree of  $X$ , by  $g$  its sectional genus and by  $K_X$  the class of its canonical bundle. If  $T_X$  denotes the tangent bundle of  $X$  we will adopt the following notation for its Chern classes  $c_i(X) := c_i(T_X)$ .

In this section we will collect, for the reader's convenience, a number of results and definitions that we will use in the following sections. Proofs are omitted or only sketched.

**Proposition 1.2.** *If  $N_{X/\mathbb{P}^6}$  denotes the normal bundle of  $X$  in  $\mathbb{P}^6$  then the following identities hold:*

$$c_3(N_{X/\mathbb{P}^6}) = d^2 \tag{1}$$

$$c_1(N_{X/\mathbb{P}^6}) = 7H - c_1(X) \tag{2}$$

$$c_2(N_{X/\mathbb{P}^6}) = 21H^2 - c_2(X) - c_1(X) \cdot (7H - c_1(X)) \tag{3}$$

$$c_3(N_{X/\mathbb{P}^6}) = 35H^3 + 7H \cdot c_1(X)^2 - 21H^2 \cdot c_1(X) - c_1(X)^3 - 7H \cdot c_2(X) + 2c_1(X) \cdot c_2(X) - c_3(X) \tag{4}$$

$$d^2 - 35d = 7H \cdot c_1(X)^2 - 21H^2 \cdot c_1(X) - c_1(X)^3 - 7H \cdot c_2(X) + 2c_1(X) \cdot c_2(X) - c_3(X). \tag{5}$$

*Proof (Sketch).* (1) is a particular case of [10, p. 431]. (2)–(4) are consequence of the standard exact sequence

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}^6}|_X \rightarrow N_{X/\mathbb{P}^6} \rightarrow 0. \quad \square$$

**Proposition 1.3.** *If  $E$  is a rank  $r$  vector bundle on an irreducible smooth projective variety  $B$  of dimension  $b$ ,  $\mathbb{P}(E)$  the corresponding projectivized bundle and  $p : \mathbb{P}(E) \rightarrow B$  the canonical projection then*

(i)

$$\sum_{i=0}^r (-1)^i p^*(c_i(E)) \cdot h^{r-i} = 0.$$

(ii) *If  $r = 2$  and  $b = 2$*

$$\begin{aligned} c_1(\mathbb{P}(E)) &= 2h - p^*c_1(E) + p^*c_1(B), \\ c_2(\mathbb{P}(E)) &= 2h \cdot p^*c_1(B) - p^*c_1(E) \cdot p^*c_1(B) + p^*c_2(B), \\ c_3(\mathbb{P}(E)) &= 2h \cdot p^*c_2(B). \end{aligned}$$

(iii) *If  $r = 4$  and  $b = 1$*

$$\begin{aligned} c_1(\mathbb{P}(E)) &= 4h - p^*c_1(E) + p^*c_1(B), \\ c_2(\mathbb{P}(E)) &= 6h^2 + 4h \cdot p^*c_1(B) - 3h \cdot p^*c_1(E), \\ c_3(\mathbb{P}(E)) &= 4h^3 - 3h^2 \cdot p^*c_1(E) + 6h^2 \cdot p^*c_1(B), \\ c_4(\mathbb{P}(E)) &= 4h^3 \cdot p^*c_1(B). \end{aligned}$$

(iv) *If  $r = 3$  and  $b = 1$*

$$\begin{aligned} c_1(\mathbb{P}(E)) &= 3h - p^*c_1(E) + p^*c_1(B) \\ c_2(\mathbb{P}(E)) &= 3h^2 + 3h \cdot p^*c_1(B) - 2h \cdot p^*c_1(E) \\ c_3(\mathbb{P}(E)) &= 3h^2 \cdot p^*c_1(B) \end{aligned}$$

where  $h \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$ .

*Proof (Sketch).* (i) is [10, p. 429]. ii)–iv) follow from the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(E)} \rightarrow (p^*E^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(1)) \rightarrow T_{\mathbb{P}(E)} \rightarrow p^*T_B \rightarrow 0$$

([12, Proposition (17.12), pp. 80–82]). □

**Definition 1.4.** We call an effective divisor  $E \subset X$ , an *exceptional plane* if  $E \cong \mathbb{P}^2$ ,  $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$  and  $\mathcal{O}_X(H) \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^2}(1)$ .

**Definition 1.5.** Consider the pairs  $(X', H')$  where  $X'$  is a smooth irreducible projective variety and  $H'$  is an ample divisor on  $X'$ , we call  $(X', H')$  a *reduction* of  $(X, H)$ , if there exists a morphism  $\sigma : X \rightarrow X'$  such that  $\sigma$  is the contraction of all the exceptional planes  $E_1, \dots, E_k$  contained in  $X$ , and  $H'$  is such that  $H = \sigma^*(H') - E_1 - \dots - E_k$ .

A standard reference for the notion of reduction is [1].

**Proposition 1.6.** (i) *The set of exceptional planes of  $X$  is finite.*

(ii) *Two exceptional planes of  $X$  are disjoint.*

(iii) *If  $(X', H')$  is a reduction of  $(X, H)$ , and  $\sigma : X \rightarrow X'$  is the corresponding contraction morphism, then  $X$  is the blow-up of  $X'$  in a finite number of distinct points and  $\sigma$  is the corresponding morphism.*

*Proof.* [8, p. 102 (11.11)]. □

**Notation 1.7.** If  $\mathcal{S}$  is a set and  $h_1 : \mathcal{S} \rightarrow \mathbb{N}$ ,  $h_2 : \mathcal{S} \rightarrow \mathbb{N}$  two functions we write  $h_1 = O(h_2)$  if exists a constant  $c$  such that  $|h_1(s)| \leq ch_2(s)$ ,  $\forall s \in \mathcal{S}$ .

## 2. Statement and proof of the theorem

**Theorem 2.1.** *Smooth threefolds in  $\mathbb{P}^6$  with ruled surface section, not contained in a fourfold of degree less than or equal to 15, are bounded.*

*Remark 2.2.* Threefolds with ruled surface section are all listed in [1, p. 205, table 7.3] or [11, p. 339, Theorem II]. Since Del Pezzo varieties are classified in [8, p. 71, (8.11)], we have to prove Theorem 2.1 for scrolls in lines, quadric fibrations, Veronese fibrations and scrolls in planes (see [1, 11] for the definitions).

First of all we find a lower bound for the sectional genus of  $X$ . This is made by a case by case analysis.

*Remark 2.3.* If  $(X, H)$  is a scroll on a smooth surface then  $X \cong \mathbb{P}(E)$ , where  $E$  is a rank two vector bundle on a smooth projective surface  $B$  and  $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$ .

**Notation 2.4.** For a scroll in lines  $X \subset \mathbb{P}^6$  we will denote by  $k$  the number of lines of  $X$  that are contained in a general hyperplane section.

**Proposition 2.5.** *For a scroll in lines in  $\mathbb{P}^6$  with ruled surface section*

$$\frac{d^2}{28} + O(d) \leq g.$$

*Proof.* By Proposition 1.3 (ii) we have

$$\begin{aligned} c_1(X)^2 &= 4H^2 + p^*c_1(E)^2 + p^*c_1(B)^2 - 4H \cdot p^*c_1(E) \\ &\quad + 4H \cdot p^*c_1(B) - 2p^*c_1(E) \cdot p^*c_1(B), \\ c_1(X)^3 &= 8H^3 + 6H \cdot p^*c_1(E)^2 + 6H \cdot p^*c_1(B)^2 - 12H^2 p^*c_1(E) \\ &\quad + 12H^2 \cdot p^*c_1(B) - 12H \cdot p^*c_1(E) \cdot p^*c_1(B), \\ c_1(X) \cdot c_2(X) &= 4H^2 \cdot p^*c_1(B) - 4H \cdot p^*c_1(E) \cdot p^*c_1(B) \\ &\quad + 2H \cdot p^*c_2(B) + 2H \cdot p^*c_1(B)^2, \end{aligned}$$

that substituted in (5) yields

$$d^2 - 13d = H \cdot p^*c_1(E)^2 + 5H \cdot p^*c_1(B)^2 + 5H^2 \cdot p^*c_1(E) - 11H^2 \cdot p^*c_1(B) - 3Hp^*c_1(E) \cdot p^*c_1(B) - 5H \cdot p^*c_2(B). \quad (6)$$

Now intersecting the equation in Proposition 1.3 (i) respectively with the cycles  $p^*c_1(E)$  and  $p^*c_1(B)$  we get

$$\begin{aligned} H \cdot p^*c_1(E)^2 &= H^2 \cdot p^*c_1(E), \\ H \cdot p^*c_1(E) \cdot p^*c_1(B) &= H^2 \cdot p^*c_1(B), \end{aligned} \quad (7)$$

and substituting in (6)

$$d^2 - 13d = 6H^2 \cdot p^*c_1(E) - 14H^2 \cdot p^*c_1(B) + 5K_B^2 - 5H \cdot p^*c_2(B). \quad (8)$$

By the adjunction formula

$$2g - 2 = H^2 \cdot p^*c_1(E) - H^2 \cdot p^*c_1(B)$$

moreover, by [1, p. 282, Theorem (11.1.2)],  $d = c_1(E)^2 - c_2(E)$  and  $c_2(E) = k$ , and then

$$\begin{aligned} H^2 \cdot p^*c_1(E) &= d + k, \\ H^2 \cdot p^*c_1(B) &= d + k - 2g - 2. \end{aligned}$$

Substituting in (8) we get

$$d^2 - 5d + 8k + 28 = 28g + 5K_B^2 - 5c_2(B).$$

To conclude the proof observe that  $k \geq 0$  and since  $B$  is ruled  $K_B^2 - c_2(B) \leq 6$ .  $\square$

*Remark 2.6.* Let  $X$  be a quadric fibration,  $X$  is a smooth divisor in  $\mathbb{P}(E)$ , projectivized of a vector bundle  $E$ ,  $\text{rk}(E) = 4$ , on  $B$  a smooth curve. If  $h$  denotes the class of  $\mathcal{O}_{\mathbb{P}(E)}(1)$  in  $\text{Pic}(X)$  then  $X \in |2h - \pi^*L|$ , where  $L$  is a divisor on  $B$ , and  $\pi : \mathbb{P}(E) \rightarrow B$  is the canonical projection associated  $\mathbb{P}(E)$ .

$$N_{X/\mathbb{P}(E)} = (2h - \pi^*L)|_X = 2h|_X - (\pi^*L)|_X = 2H - p^*L$$

where  $H = h|_X$ , and  $p$  denotes the restriction of  $\pi$  to  $X$ .  $H$  embeds  $X$  in  $\mathbb{P}^6$ .

**Proposition 2.7.** *For a quadric fibration in  $\mathbb{P}^6$*

$$\frac{3d^2}{70} + O(d) \leq g.$$

*Proof.* From the exact sequence

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}(E)}|_X \rightarrow N_{X/\mathbb{P}(E)} \rightarrow 0,$$

we have

$$\begin{aligned} c_1(X) &= c_1(T_{\mathbb{P}(E)}|_X) - 2H + p^*L, \\ c_2(X) &= c_2(T_{\mathbb{P}(E)}|_X) - 2H \cdot c_1(X) + c_1(X) \cdot p^*L, \\ c_3(X) &= c_3(T_{\mathbb{P}(E)}|_X) - 2H \cdot c_2(X) + c_2(X) \cdot p^*L. \end{aligned}$$

But by Proposition 1.3 (iii)

$$\begin{aligned} c_1(T_{\mathbb{P}(E)}|_X) &= 4H - p^*c_1(E) + p^*c_1(B), \\ c_2(T_{\mathbb{P}(E)}|_X) &= 6H^2 + 4H \cdot p^*c_1(B) - 3H \cdot p^*c_1(E), \\ c_3(T_{\mathbb{P}(E)}|_X) &= 4H^3 - 3H^2 \cdot p^*c_1(E) + 6H^2 \cdot p^*c_1(B), \end{aligned}$$

hence

$$\begin{aligned} c_1(X) &= 2H - p^*c_1(E) + p^*c_1(B) + p^*L, \\ c_2(X) &= 2H^2 - H \cdot p^*c_1(E) + 2H \cdot p^*c_1(B), \\ c_3(X) &= -H^2 \cdot p^*c_1(E) + 2H^2 \cdot p^*c_1(B) + 2H^2 \cdot p^*L. \end{aligned}$$

Substituting in (5), and noting that  $-H^2 \cdot p^*c_1(B) = 4g(B) - 4$  (where  $g(B)$  denote the geometric genus of  $B$ ), we get

$$d^2 - 7d = 5H^2 \cdot p^*c_1(E) + 36g(B) - 36 - 3H^2 \cdot p^*L. \tag{9}$$

Denote by  $C$  the general curve section of  $X$ , then we have a 2:1 map  $p|_C : C \rightarrow B$ . By adjunction formula

$$K_C = H^2 \cdot p^*c_1(E) - H^2 \cdot p^*c_1(B) - H^2 \cdot p^*L,$$

substituting in

$$K_C = (p|_C)^*(K_B) + R,$$

(where as usual  $R$  denotes the ramification divisor of  $p|_C$ ) and using Hurwitz's formula for the degree of  $R$  we have

$$2(g + 1) - 4g(B) = H^2 \cdot p^*c_1(E) - H^2 \cdot p^*L.$$

In  $\mathbb{C}(E)$  we have  $X \in |2h - \pi^*L|$ , and, by Proposition 1.3 (i),  $h^4 = h^3 \cdot \pi^*c_1(E)$  and

$$2d = 2H^2 \cdot p^*c_1(E) - H^2 \cdot p^*L. \tag{10}$$

The two equalities above imply

$$\begin{aligned} c_1(E) &= d - (g + 1) + 2g(B), \\ \text{deg}(L) &= d - 2(g + 1) + 4g(B), \end{aligned}$$

and substituting in (9) we get

$$d^2 - 11d + 36 = 2(g + 1) + 32g(B). \tag{11}$$

$N_{X/\mathbb{P}^6}(-1)$  is generated by its sections. Then, by the positivity results of Fulton and Lazarsfeld ([9, chapter 12]), we have that the second Segre class of  $N_{X/\mathbb{P}^6}(-1)$  satisfies the following inequality

$$H \cdot s_2(N_{X/\mathbb{P}^6}(-1)) \geq 0.$$

By the same type of computations as before, writing this inequality for a quadric fibration we obtain

$$12g(B) \leq 8g + 4 \tag{12}$$

that with (11) proves the Proposition.  $\square$

*Remark 2.8.* Let  $X$  be a Veronese fibration, then exists a reduction  $\sigma$  of  $(X, H)$  to  $(X', H')$ , where  $\sigma : X \rightarrow X'$  is the blow-up in  $k$  distinct points (Proposition 1.6),  $X'$  is the projectivized bundle of a rank 3 vector bundle on  $B$ , a smooth curve.  $H' = 2h - p^*L$ ,  $H = \sigma^*(H') - \sum_{i=1}^k E_i$ , where  $p$  is the canonical projection on the base,  $h \in |\mathcal{O}_{X'}(1)|$ ,  $E_i$  are the exceptional divisors of  $\sigma$  and  $L$  is a divisor on  $B$ .

**Proposition 2.9.** *For a Veronese fibration in  $\mathbb{P}^6$*

$$\frac{d^2}{24} + O(d) \leq g.$$

*Proof.* Denote by  $c(X)$  and  $c(X')$  the total Chern class respectively of  $X$  and  $X'$ . By [9, pp. 300–301, Theorem (15.4) and Example (15.4.2)],

$$c(X) = \sigma^* c(X') + \sum_{i=1}^k \{-2E_i + 2E_i^3\}. \tag{13}$$

Since  $X$  and  $X'$  are smooth,  $\sigma^* : A^* X' \rightarrow A^* X$  is a morphism of graded rings ([9, p. 140]), and since  $\mathcal{O}_{E_i}(E_i) \cong \mathcal{O}(-1)$  then  $E_i^3 = 1$ . From (13), recalling that  $E_i$

are disjoint by Proposition 1.6 we obtain

$$\begin{aligned} H &= \sigma^*(H') - \sum_{i=1}^k E_i, \\ H^2 &= \sigma^*(H')^2 + \sum_{i=1}^k E_i^2, \\ c_1(X) &= \sigma^*c_1(X') - 2 \sum_{i=1}^k E_i, \\ c_1(X)^2 &= \sigma^*c_1(X')^2 + 4 \sum_{i=1}^k E_i^2, \\ c_1(X)^3 &= \sigma^*c_1(X')^3 - 8k, \\ c_2(X) &= \sigma^*c_2(X'), \\ c_3(X) &= \sigma^*c_3(X') + 2k, \end{aligned}$$

and substituting in (5)

$$\begin{aligned} d^2 - 35d &= \sigma^*(7H' \cdot c_1(X')^2 - 21(H')^2 \cdot c_1(X') - c_1(X')^3 \\ &\quad - 7H' \cdot c_2(X') + 2c_1(X') \cdot c_2(X') - c_3(X')) + 20k. \end{aligned} \tag{14}$$

As in the preceding cases, by Proposition 1.3 (i) and (iv), we can write (14) as

$$d^2 - 35d = -\frac{35}{2}(d + k) - 48(2 - 2g(B)) + 20k. \tag{15}$$

By adjunction formula

$$k = 2g - d - 8g(B) + 6$$

that with (15) proves the Proposition. □

As above by Chern classes computations we have the following:

**Proposition 2.10.** *For a scroll in planes in  $\mathbb{P}^6$*

$$\frac{d^2}{12} + O(d) \leq g.$$

We can now conclude.

*Proof of Theorem 2.1.* We may suppose our threefold  $X$  non degenerate. Indeed, if  $X \subset \mathbb{P}^5$  by [2, Theorem 2]  $X$  would have bounded degree. It follows that the general curve section of  $X$ ,  $C \subset \mathbb{P}^4$ , is non-degenerate. Suppose that  $X$  is not contained in a fourfold of degree less than 15. If  $d$  is large enough then  $C$  is not contained in a

surface of  $\mathbb{P}^4$  of degree less than 15. If it were so then by [3, p. 97, Theorem (0.2)],  $X$  would be contained in a fourfold of degree less than 15, that is impossible by our assumption.

By [4, Main Theorem], if  $d$  is large enough,

$$g \leq \frac{d^2}{30} + O(d),$$

which by Propositions 2.5–2.10 concludes the proof. □

### 3. Boundedness for threefolds in a fourfold of bounded degree

Having in mind Theorem 2.1, one may ask, if it is possible to prove a full boundedness result by an argument as [7, Lemma 1, p. 3]. We have not been able to prove such a general statement so far, but only a partial one as in [2, Proposition 5.1, p. 331].

Let then  $X$  be a threefold in  $\mathbb{P}^6$  with ruled hyperplane section such that  $X \subset W \subset \mathbb{P}^6$  where  $\dim(W) = 4$  and  $\deg(W) = \sigma < 15$  fixed. By choosing two hypersurfaces of  $\mathbb{P}^6$  containing  $W$  hence  $X$ , we can define a map of vector bundles on  $X$

$$\mu : N_{X/\mathbb{P}^6} \rightarrow \mathcal{O}_X(\sigma)^{\oplus 2}.$$

Following [9] we will denote by  $D_1(\mu)$  the scheme where  $\mu$  has rank less than or equal to one. First of all let us make a remark.

*Remark 3.1.* If  $X$  is a scroll in lines, quadric fibration, Veronese fibration or scroll in planes and  $S$  denote its generic hyperplane section then by a Chern classes computation and Riemann-Roch we have

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S).$$

Notations are as in section 2. For a scroll in lines, in view of (7),

$$\begin{aligned} 24\chi(\mathcal{O}_X) &= c_1(X) \cdot c_2(X) \\ &= 4H^2 \cdot p^*c_1(B) - 4H \cdot p^*c_1(E) \cdot p^*c_1(B) + 2H \cdot p^*c_2(B) + 2H \cdot p^*c_1(B)^2 \\ &= 4H^2 \cdot p^*c_1(B) - 4H^2 \cdot p^*c_1(B) + 2H \cdot c_2(B) + 2H \cdot p^*c_1(B)^2 \\ &= 2c_2(B) + 2K_B^2 = 24\chi(\mathcal{O}_B), \end{aligned}$$

since  $\chi$  is a birational invariant this is our claim. For a quadric fibration and a Veronese fibration we have respectively

$$\begin{aligned} 24\chi(\mathcal{O}_X) &= c_1(X) \cdot c_2(X) \\ &= 4H^3 - 4H^2 \cdot p^*c_1(E) + 6H^2 \cdot p^*c_1(B) + 2H^2 \cdot p^*L \\ &= 6H^2 \cdot p^*c_1(B) = 24(1 - g(B)), \quad (\text{by (10)}) \end{aligned}$$

$$\begin{aligned} 24\chi(\mathcal{O}_X) &= c_1(X') \cdot c_2(X') \\ &= 9h^3 - 9h^2 \cdot p^*c_1(E) + 12h^2 \cdot p^*c_1(B) \\ &= 12h^2 \cdot p^*c_1(B) = 24(1 - g(B)). \quad (\text{by Proposition 1.3 (i)}) \end{aligned}$$

The claim follows in this cases noting that  $S$  is a birationally ruled surface over a curve of genus  $g(B)$ . As above by a Chern classes computation one can prove the claim for scroll in planes.

**Proposition 3.2.** *Threefolds in  $\mathbb{P}^6$  (with ruled surface section) contained in a fourfold of bounded degree  $\sigma$  such that  $D_1(\mu)$ , for a map  $\mu$  constructed as above, has pure codimension two, are bounded.*

*Proof.* To make the proof clearer we split it in two steps. In Step 1 we prove an upper bound for the sectional genus of  $X$ . Such bound, by the results in section 2, proves the Proposition provided that the degree of the fourfold is not equal to three. In step 2 we prove the Proposition in this particular case by a direct computation. Notations are as in section 2.

Step 1. For a fixed degree  $\sigma \geq 3$  we have

$$g \leq \frac{d^2}{28(\sigma - 2)} + O(d), \tag{16}$$

for a scroll in lines, Veronese fibration, scroll in planes while

$$g \leq \frac{3d^2}{84(\sigma - 2) - 14} + O(d), \tag{17}$$

for a quadric fibration.

Define

$$\gamma = c_1(\mathcal{O}_X(\sigma)^{\oplus 2} - N_{X/\mathbb{P}^6})^2 - c_2(\mathcal{O}_X(\sigma)^{\oplus 2} - N_{X/\mathbb{P}^6}) \in A_1X$$

where  $AX$  denotes the Chow ring of  $X$ . Since by hypothesis  $D_1(\mu)$  has pure codimension 2 by [9, p. 254, Theorem 14.4(d)], we have  $\gamma = [D_1(\mu)]$  in  $A_1X$ , where  $[D_1(\mu)]$  denotes the cycle associated to  $D_1(\mu)$  in  $A_1X$ , hence  $0 \leq \gamma \cdot H$ .

By Proposition 1.2 we can write

$$\gamma = c_2(N_{X/\mathbb{P}^6}) + 3\sigma^2H^2 - 2\sigma H \cdot (7H + K_X),$$

and

$$0 \leq H \cdot c_2(N_{X/\mathbb{P}^6}) + 3\sigma^2H^3 - 2\sigma H^2 \cdot (7H + K_X). \tag{18}$$

By (3), intersecting with  $H$ ,

$$H \cdot c_2(N_{X/\mathbb{P}^6}) = 21H^3 - H \cdot c_2(X) - H \cdot c_1(X) \cdot (7H - c_1(X)).$$

Solving (5) with respect to  $H \cdot c_2(X)$  and substituting in the preceding equality we get

$$H \cdot c_2(N_{X/\mathbb{P}^6}) = \frac{d^2}{7} + 16d + 4H^2 \cdot K_X - \frac{K_X^3}{7} - \frac{48}{7}\chi(\mathcal{O}_X) + \frac{c_3(X)}{7}. \quad (19)$$

In view of (19) and Remark 3.1, the inequality (18) becomes

$$0 \leq \frac{d^2}{7} + (3\sigma^2 - 10\sigma + 8)d + 4(2 - \sigma)(g - 1) + \Delta$$

where we set

$$\Delta = -\frac{K_X^3}{7} - \frac{48}{7}\chi(\mathcal{O}_S) + \frac{c_3(X)}{7}.$$

To find an upper bound for  $g$ , it is sufficient to find one for  $\Delta$ . We proceed by a case by case analysis as in section 2. For a scroll in lines we have

$$\begin{aligned} -K_X^3 &= 8H^3 - 6H^2 \cdot p^*c_1(E) + 6H \cdot p^*c_1(B)^2 \\ &= 6H \cdot p^*c_1(B)^2 + 2d = 6K_B^2 + 2d - 6k, \\ 48\chi(\mathcal{O}_S) &= 48\chi(\mathcal{O}_B) = 4(K_B^2 + c_2(B)), \\ c_3(X) &= 2c_2(B), \end{aligned}$$

hence

$$\Delta = \frac{2}{7}(d + K_B^2 - c_2(B) - 3k) \leq \frac{2}{7}(d + 6)$$

since  $B$  is ruled. This proves Step 1 for a scroll in lines.

For a quadric fibration

$$\begin{aligned} -K_X^3 &= -4H^2 \cdot p^*c_1(E) + 12H^2 \cdot p^*c_1(B) + 8H^2 \cdot p^*L \\ \chi(\mathcal{O}_S) &= 1 - g(B), \\ c_3(X) &= -H^2 \cdot p^*c_1(E) + 2H^2 \cdot p^*c_1(B) + 2H^2 \cdot p^*L. \end{aligned}$$

Then, by definition of  $\Delta$ , (9), and (12), we have

$$\Delta \leq \frac{2}{3}g + \frac{10}{7}d - \frac{2}{3}$$

The same kind of argument proves that  $\Delta$  is less than or equal to a constant for a Veronese fibration and a scroll in planes. This concludes the proof of Step 1.

By (16) and (17) and by Propositions 2.5–2.10, to conclude the proof of the Proposition it is sufficient to prove

*Step 2. Scrolls in lines and quadric fibrations contained in a fourfold  $W \subseteq \mathbb{P}^6$  of degree  $\sigma = 3$  are bounded.*

In this case  $W$  is a variety of minimal degree. By the classification of such varieties, see for example [6],  $W$  is a cone over a rational normal scroll in  $\mathbb{P}^5$  and, (adopting notations of [6])  $W$  is projectively equivalent to  $S(0, 1, 1, 1)$ .

Set  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}$ , then the linear system  $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$  defines a birational morphism  $\mathbb{P}(\mathcal{E}) \rightarrow W$ . Denote by  $\tilde{X}$  the strict transform of  $X$  by this morphism. Since  $X$  is smooth,  $\tilde{X}$  is smooth too.

It is well known that  $\text{Pic}(\mathbb{P}(\mathcal{E})) = \mathbb{Z}h + \mathbb{Z}f$ , where  $f$  is a class of a fibre of the canonical projection  $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$ , and  $h \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ . Write  $\tilde{X} = ah + bf$ ,  $\tilde{H} = h|_{\tilde{X}}$ , and  $\tilde{F} = f|_{\tilde{X}}$ . Then

$$\begin{aligned} \tilde{H}^3 &= \text{deg}(X) = d = 3a + b, \\ \tilde{H}^2 \cdot \tilde{F} &= a. \end{aligned}$$

Since the general curve section of  $\tilde{X}$  by  $\tilde{H}$  has genus equal to  $g$ , the sectional genus of  $X$ , then by the following exact sequence

$$0 \rightarrow T_{\tilde{X}} \rightarrow T_{\mathbb{P}(\mathcal{E})}|_{\tilde{X}} \rightarrow N_{\tilde{X}/\mathbb{P}(\mathcal{E})} \rightarrow 0,$$

and adjunction formula

$$2g - 2 = (1 + b)a + (a - 2)(3a + b).$$

Moreover denote by  $\tilde{S}$  the general surface section of  $\tilde{X}$  by  $\tilde{H}$ , then by the exact sequence

$$0 \rightarrow T_{\tilde{S}} \rightarrow T_{\tilde{X}}|_{\tilde{S}} \rightarrow N_{\tilde{S}/\tilde{X}} \rightarrow 0,$$

we are able to compute

$$c_1(\tilde{S}) = ((3 - a)\tilde{H} - (b + 1)\tilde{F})|_{\tilde{S}} \tag{20}$$

$$c_2(\tilde{S}) = ((a^2 - 3a + 3) \cdot \tilde{H}^2 + (2ab + a - 3b)\tilde{H} \cdot \tilde{F})|_{\tilde{S}} \tag{21}$$

Now observe that  $d \geq 0$ ,  $K_{\tilde{S}}^2 \leq 9$ , and since  $\chi$  is a birational invariant, by Remark 3.1 we have that  $\chi(\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_{\tilde{S}})$ . In view of (20) and (21) we can rewrite the terms in this equality and in the two inequalities above as functions of  $a$  and  $b$  obtaining

$$\begin{aligned} 3a + b &\geq 0, \\ b(3a^2 - 12a + 9) &\leq -3a^3 + 16a^2 - 21a + 9, \\ b(12a^2 - 8a) &= 3a^4 - 10a^3 + 13a^2 - 2a. \end{aligned}$$

The above inequalities imply that  $a$  is bounded. Again writing the terms in the inequality of Proposition 2.5 and of Proposition 2.7 as functions of  $a$  and  $b$  we obtain

$$b^2 + \phi(a, b) \leq 0, \tag{22}$$

where  $\phi(a, b)$  is a polynomial in  $a$  and  $b$  of degree one in  $b$ . Since  $a$  is bounded, by (22)  $b$  is also bounded, hence  $d$  is bounded.  $\square$

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