Ribbon Knots of 1-Fusion, the Jones Polynomial, and the Casson-Walker Invariant

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ABSTRACT

We give an explicit formula for the Casson-Walker invariant of double branched covers of S^3 branched along ribbon knots of 1-fusion.

Key words: Ribbon knots of 1-fusion, Jones polynomial, Casson-Walker invariant.

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Introduction

Casson introduced an integer-valued invariant for oriented integral homology spheres via constructions on representation spaces, which is called the Casson invariant ([1]). Walker extended the Casson invariant to rational homology spheres, which is called the Casson-Walker invariant ([14]). There has been a big deal of work on these invariants. In particular, Mullins gives a relation between the Jones polynomial of a link with non-zero determinant and the Casson-Walker invariant of its double branched cover of S^3 ([11, Theorem 5.1]).

In this paper, we give an explicit formula for the Casson-Walker invariant of double branched covers of S^3 branched along ribbon knots of 1-fusion (Theorem 1.17). To do this we consider the Jones polynomial of ribbon knots of 1-fusion. Giving an explicit formula for the Jones polynomial is extremely difficult, but we succeed in obtaining a

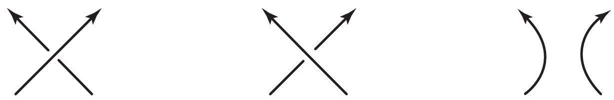

Figure 1

formula for its first derivative at -1 (Proposition 1.14), which extends a formula in [12] (see Example 1.9 in section 1). This formula, together with [11, Theorem 5.1] gives a formula for the Casson-Walker invariant. In [10], we discuss the Casson invariant of homology spheres of Mazur type by using this formula for the Casson invariant.

Our formula for the first derivative has independent interest, since we obtain an application as follows: In [9], we define the *ribbon number* of a ribbon knot, the minimal number of ribbon singularities needed for a ribbon disk bounded by the ribbon knot, and by using this formula we determine the ribbon number of the Kinoshita-Terasaka knot.

1. Definitions and results

Definition 1.1. The Jones polynomial $J_L(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ is an invariant of an oriented link L in S^3, defined by the following formulas:

$$t^{-1}J_{L_+}(t) - tJ_{L_-}(t) = (t^{1/2} - t^{-1/2})J_{L_0}(t),$$
$$J_O(t) = 1,$$

where L_+, L_-, L_0 are three oriented links, which are identical except near one point where they are as shown in figure 1 and O denotes the trivial knot ([4]).

Definition 1.2. A *band sum* of K_0 and K_1, two separable components of a link in S^3, is obtained as follows: Embed $I \times I$ in S^3 by a homeomorphism b such that

(i) $b(I \times I) \cap (K_0 \cup K_1) = b(I \times \{0, 1\})$,

(ii) $b(I \times \{0\}) \subset K_0$; $b(I \times \{1\}) \subset K_1$.

The band sum of K_0 and K_1 along b is the knot

$$(K_0 - b(I \times \{0\})) \cup (K_1 - b(I \times \{1\})) \cup b(\{0, 1\} \times I),$$

denoted by $K_0 \#_b K_1$ (cf. [3]).

Definition 1.3. A *ribbon disk* is an immersed 2-disk of D^2 into S^3 with only transverse double points such that the singular set consists of ribbon singularities, that is, the preimage of each ribbon singularity consists of a properly embedded arc in D^2 and an embedded arc interior to D^2. A knot is a *ribbon knot* if it bounds a ribbon disk in S^3 (cf. [6]).

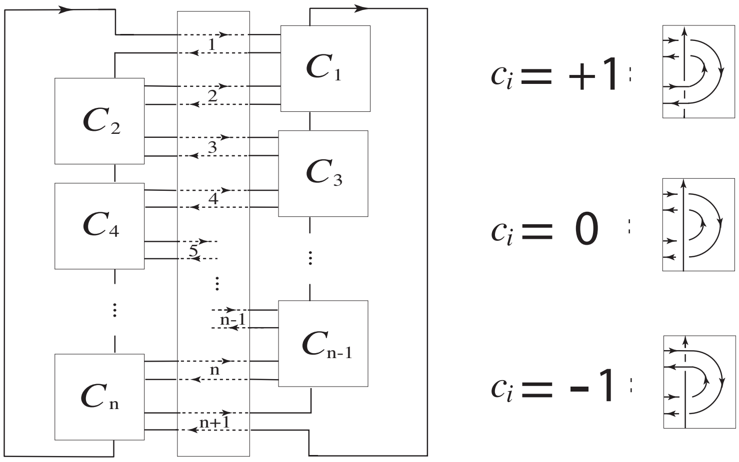

Figure 2: D_K

Definition 1.4. We call a knot K in S^3 a ribbon knot of *1-fusion*, if it has a knot diagram D_K as described in figure 2 (and figure 3), where n is even and each small rectangle named C_i is determined by $c_i \in \{-1, 0, +1\}$ $(i = 1, 2, \ldots, n)$ and there are disjointly embedded $(n + 1)$ subbands inside the "big rectangle", being knotted, twisted and mutually linked (cf. [8]). We call the diagram D_K *1-fusion diagram* of K.

Let α_i $(i = 1, 2, \ldots, n + 1)$ denote the (right hand full) twisting number of i-th subband inside the "big rectangle", and let $\alpha_{i,j}$ $(i < j)$ denote the relative linking number of i-th subband and j-th subband inside the "big rectangle". That is: Direct the subbands from left to right and attach a sign to each crossing of different subbands, as shown in figure 4. Then $\alpha_{i,j}$ is half the sum of the signs of the crossings of i-th and j-th subband. (See figure 3, where $(c_1, c_2) = (+1, +1)$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 0$,

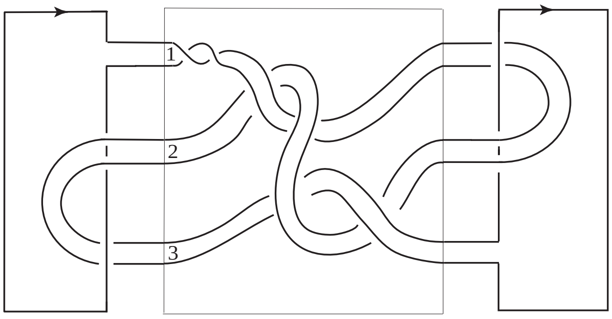

Figure 3: An example of D_K

Figure 4

$\alpha_{1,2} = 1$, $\alpha_{1,3} = 0$, and $\alpha_{2,3} = 1$.)

Remark 1.5. D_K gives a ribbon disk bounded by K.

Remark 1.6. A ribbon knot of 1-fusion is a band sum of 2-component trivial link and viceversa.

Remark 1.7. For any Laurent polynomial $f(t)$ with $f(1) = \pm 1$, there exists a ribbon knot of 1-fusion whose Alexander polynomial is $f(t)f(t^{-1})$ ([13]).

Remark 1.8. Let K be a ribbon knot of 1-fusion in Definition 1.4. The Alexander polynomial of K is written as $f(t)f(t^{-1})$, where $f(t) = \sum_{i=1}^{n/2}(t^{\phi(i)} - t^{\psi(i)}) + 1$, $\phi(i) = \sum_{j=2i-1}^{n}(-1)^j c_j$, and $\psi(i) = \sum_{j=2i}^{n}(-1)^j c_j$ ([8]).

The main proposition of this paper is to express $J'_K(-1)$ of a ribbon knot of 1-fusion K in Definition 1.4 by using data of D_K, which are $c_i (1 \leq i \leq n)$, α_i $(1 \leq i \leq n+1)$ and $\alpha_{i,j}$ $(1 \leq i < j \leq n+1)$. Before stating the main proposition, we give some examples.

Example 1.9. If K has the 1-fusion diagram with $(c_1, c_2) = (+1, +1)$ as shown in the left diagram of figure 5 (K is called 6_1-like ribbon knot in [12]), then Sakai shows

$$J'_K(-1) = 16(\alpha_1 + \alpha_2 + \alpha_3 - \alpha_{1,2} - \alpha_{1,3} - \alpha_{2,3}) - 8.$$

Remark 1.10. It is well known that the Alexander polynomial of ribbon knots is of the form $f(t)f(t^{-1})$, where $f(t)$ is a Laurent polynomial ([2]). Then it is natural to ask whether the Jones polynomial of ribbon knots has some properties reflecting the knots being ribbon. There are few works in this direction. In [12], Sakai also shows

$$J'''_K(1) = -72(\alpha_1 + \alpha_2 + \alpha_3 - \alpha_{1,2} - \alpha_{1,3} - \alpha_{2,3})$$

and have

$$2J'''_K(1) = -9J'_K(-1) - 72.$$

In the process of extending these formulas, we have succeed in giving formulas for $J'(-1)$ and $J'''(1)$ for ribbon knots of 1-fusion.

Example 1.11. If K has the 1-fusion diagram with $(c_1, c_2, c_3, c_4) = (+1, +1, 0, +1)$ as shown in the middle diagram of figure 5, then

$$J'_K(-1) = 48(\alpha_{2,3} - \alpha_{2,4} - \alpha_{3,5} + \alpha_{4,5}) - 24.$$

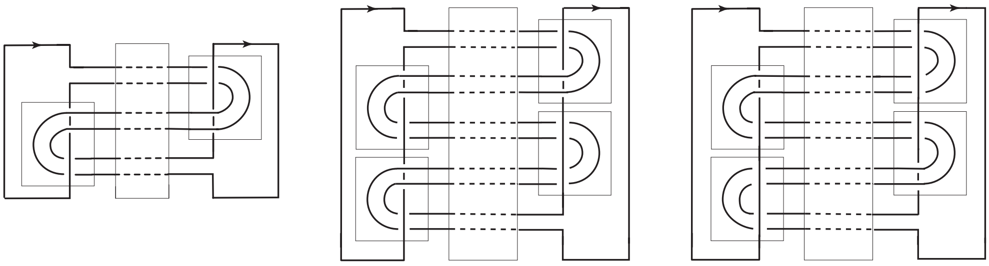

Figure 5

Example 1.12. If K has the 1-fusion diagram with $(c_1, c_2, c_3, c_4) = (0, +1, +1, 0)$ as shown in the right diagram of figure 5, then $J'_K(-1) = 0$. Note that the Alexander polynomial of K is 1. The cases $(c_1, c_2, c_3, c_4) = (0, +1, -1, 0)$, $(0, -1, +1, 0)$, $(0, -1, -1, 0)$ are the same.

We use the following convention:

$$S = \emptyset \Longrightarrow \sum_{i \in S} a_i = 0 \quad \text{and} \quad \prod_{i \in S} a_i = 1.$$

Now we define the following integers determined by $c_i \in \{-1, 0, +1\}$ $(1 \leq i \leq n)$: For $1 \leq p, q, r, s \leq n + 1$,

$$\begin{aligned} f(p,q) &= \prod_{i=p}^{q} (-1)^{-c_i}, \\ g(q,r) &= 2|c_q| \prod_{j=q+1}^{r} (-1)^{-c_j}, \end{aligned}$$

and

$$\begin{aligned} v(p,q) &= \sum_{k=p}^{q} c_k \prod_{i=p}^{q} (-1)^{-c_i}, \\ w(p,q,r) &= \left(-4|c_q| \sum_{i=p}^{q-1} c_i + 2|c_q| \sum_{i=q+1}^{r} c_i - c_q \right) \prod_{j=q+1}^{r} (-1)^{-c_j}, \\ x(p,q) &= 2v(p,q)|c_{q+1}| - f(p,q)c_{q+1}, \\ y(p,q,r) &= 2w(p,q,r)|c_{r+1}| - g(q,r)c_{r+1}. \end{aligned}$$

We also define the following integers determined by α_i $(1 \leq i \leq n+1)$ and $\alpha_{i,j}$

$(1 \leq i < j \leq n+1)$:

$$l(p,q) = -\sum_{i=p}^{q}\alpha_i - 2\sum_{i=p}^{q-1}\sum_{j=i+1}^{q}(-1)^{j-i}\alpha_{i,j},$$

$$l(p,q,r,s) = -\sum_{i=p}^{q}\alpha_i - \sum_{i=r}^{s}\alpha_i - 2\sum_{i=p}^{q-1}\sum_{j=i+1}^{q}(-1)^{j-i}\alpha_{i,j} - 2\sum_{i=r}^{s-1}\sum_{j=i+1}^{s}(-1)^{j-i}\alpha_{i,j}$$
$$- 2(-1)^{p+r-1}\sum_{i=p}^{q}\sum_{j=r}^{s}(-1)^{j-i}\alpha_{i,j}.$$

Remark 1.13. As will be seen in section 2, $l(p,q)$ is the linking number of 2-component link $L(p,q)$ $(R(p,q))$ defined in section 2 and $l(p,q,r,s)$ is the linking number of 2-component link $L(p,q,r,s)$ $(R(p,q,r,s)$, $LL(p,q,r,s)$ or $RR(p,q,r,s))$ defined in section 2.

Now we state the main proposition of this paper:

Proposition 1.14. *Let K and D_K be a ribbon knot of 1-fusion and its 1-fusion diagram as in Definition 1.4 and let $J_K(t)$ be the Jones polynomial of K. Then we have*

$$J'_K(-1) = \sum_{i=1}^{45}E_i + \sum_{i=1}^{13}F_i,$$

where each E_i is expressed by $c_1, c_2, \ldots, c_n$, and α_i $(1 \leq i \leq n+1)$, and $\alpha_{i,j}$ $(1 \leq i < j \leq n+1)$ and each F_i is expressed only by $c_1, c_2, \cdots, c_n$ as follows:

$$E_1 = 2\sum_{h=1}^{n/2} g(2h-1,n)l(2h,n+1),$$

$$E_2 = -4\sum_{k=1}^{n/2}\sum_{h=1}^{k-1}\sum_{r=k}^{n/2} g(2h-1,2k-2)|c_{2k-1}|g(2r,n)l(2h,2k-1),$$

$$E_3 = -4\sum_{k=1}^{n/2}\sum_{h=1}^{k-1}\sum_{r=k+1}^{n/2} g(2h,2k-2)|c_{2k-1}|g(2r-1,n)l(2r,n+1),$$

$$E_4 = -4\sum_{k=1}^{n/2}\sum_{h=1}^{k-1}\sum_{r=k}^{n/2} g(2h,2k-2)|c_{2k-1}|g(2r,n)l(2h+1,2k-1,2r+1,n+1),$$

$$E_5 = -4\sum_{k=1}^{n/2}\sum_{h=1}^{k-1} g(2h,2k-2)|c_{2k-1}|f(2k,n)l(2k,n+1),$$

$$E_6 = -4\sum_{k=1}^{n/2}\sum_{r=k+1}^{n/2} f(1,2k-2)|c_{2k-1}|g(2r-1,n)l(2r,n+1),$$

$$E_7 = -4\sum_{k=1}^{n/2}\sum_{r=k}^{n/2} f(1,2k-2)|c_{2k-1}|g(2r,n)l(1,2k-1,2r+1,n+1),$$

$$E_8 = -4\sum_{k=1}^{n/2} f(1,2k-2)|c_{2k-1}|f(2k,n)l(2k,n+1),$$

$$E_9 = -4\sum_{k=1}^{n/2}\sum_{h=1}^{k}\sum_{r=k+1}^{n/2} g(2h-1,2k-1)|c_{2k}|g(2r-1,n)l(2r,n+1),$$

$$E_{10} = -4\sum_{k=1}^{n/2}\sum_{h=1}^{k}\sum_{r=k+1}^{n/2} g(2h-1,2k-1)|c_{2k}|g(2r,n)l(2h,2k,2r+1,n+1),$$

$$E_{11} = -4\sum_{k=1}^{n/2}\sum_{h=1}^{k} g(2h-1,2k-1)|c_{2k}|f(2k+1,n)l(2h,n+1),$$

$$E_{12} = -4\sum_{k=1}^{n/2}\sum_{h=1}^{k-1}\sum_{r=k+1}^{n/2} g(2h,2k-1)|c_{2k}|g(2r,n)l(2h+1,2k),$$

$$E_{13} = -4\sum_{k=1}^{n/2}\sum_{h=1}^{k-1} g(2h,2k-1)|c_{2k}|f(2k+1,n)l(2h+1,2k),$$

$$E_{14} = -4\sum_{k=1}^{n/2}\sum_{r=k+1}^{n/2} f(1,2k-1)|c_{2k}|g(2r,n)l(1,2k),$$

$$E_{15} = -4\sum_{k=1}^{n/2} f(1,2k-1)|c_{2k}|f(2k+1,n)l(1,2k),$$

$$E_{16} = 4\sum_{j=1}^{n/2}\sum_{h=1}^{j-1} g(2h-1,2j-2)|c_{2j-1}|l(2h,2j-1),$$

$$E_{17} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k-1}\sum_{r=k}^{j-1} g(2h-1,2k-2)|c_{2k-1}|g(2r,2j-2)|c_{2j-1}|l(2h,2k-1),$$

$$E_{18} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k-1}\sum_{r=k+1}^{j-1} g(2h,2k-2)|c_{2k-1}|g(2r-1,2j-2)|c_{2j-1}|l(2r,2j-1),$$

$$E_{19} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k-1}\sum_{r=k}^{j-1} g(2h,2k-2)|c_{2k-1}|g(2r,2j-2)|c_{2j-1}| \times l(2h+1,2k-1,2r+1,2j-1),$$

$$E_{20} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k-1} g(2h,2k-2)|c_{2k-1}|f(2k,2j-2)|c_{2j-1}|l(2k,2j-1),$$

$$E_{21} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{r=k+1}^{j-1} f(1,2k-2)|c_{2k-1}|g(2r-1,2j-2)|c_{2j-1}|l(2r,2j-1),$$

$$E_{22} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{r=k}^{j-1} f(1,2k-2)|c_{2k-1}|g(2r,2j-2)|c_{2j-1}|l(1,2k-1,2r+1,2j-1),$$

$$E_{23} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1} f(1,2k-2)|c_{2k-1}|f(2k,2j-2)|c_{2j-1}|l(2k,2j-1),$$

$$E_{24} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k}\sum_{r=k+1}^{j-1} g(2h-1,2k-1)|c_{2k}|g(2r-1,2j-2)|c_{2j-1}|l(2r,2j-1),$$

$$E_{25} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k}\sum_{r=k+1}^{j-1} g(2h-1,2k-1)|c_{2k}|g(2r,2j-2)|c_{2j-1}| \\ \times l(2h,2k,2r+1,2j-1),$$

$$E_{26} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k} g(2h-1,2k-1)|c_{2k}|f(2k+1,2j-2)|c_{2j-1}|l(2h,2j-1),$$

$$E_{27} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k-1}\sum_{r=k+1}^{j-1} g(2h,2k-1)lvertc_{2k}rvertg(2r,2j-2)|c_{2j-1}|l(2h+1,2k),$$

$$E_{28} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k-1} g(2h,2k-1)|c_{2k}|f(2k+1,2j-2)|c_{2j-1}|l(2h+1,2k),$$

$$E_{29} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{r=k+1}^{j-1} f(1,2k-1)|c_{2k}|g(2r,2j-2)|c_{2j-1}|l(1,2k),$$

$$E_{30} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1} f(1,2k-1)|c_{2k}|f(2k+1,2j-2)|c_{2j-1}|l(1,2k),$$

$$E_{31} = 4\sum_{j=1}^{n/2}\sum_{h=1}^{j-1} g(2h,2j-1)|c_{2j}|l(2h+1,2j),$$

$$E_{32} = 4\sum_{j=1}^{n/2} f(1,2j-1)|c_{2j}|l(1,2j),$$

$$E_{33} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j}\sum_{h=1}^{k-1}\sum_{r=k+1}^{j} g(2h-1,2k-2)|c_{2k-1}|g(2r-1,2j-1)|c_{2j}|l(2h,2k-1),$$

$$E_{34} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j}\sum_{h=1}^{k-1} g(2h-1,2k-2)|c_{2k-1}|f(2k,2j-1)|c_{2j}|l(2h,2k-1),$$

$$E_{35} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j}\sum_{h=1}^{k-1}\sum_{r=k+1}^{j} g(2h,2k-2)|c_{2k-1}|g(2r-1,2j-1)|c_{2j}| \times l(2h+1,2k-1,2r,2j),$$

$$E_{36} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j}\sum_{h=1}^{k-1}\sum_{r=k}^{j-1} g(2h,2k-2)|c_{2k-1}|g(2r,2j-1)|c_{2j}|l(2r+1,2j),$$

$$E_{37} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j}\sum_{h=1}^{k-1} g(2h,2k-2)|c_{2k-1}|f(2k,2j-1)|c_{2j}|l(2h+1,2j),$$

$$E_{38} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{r=k+1}^{j} f(1,2k-2)|c_{2k-1}|g(2r-1,2j-1)|c_{2j}|l(1,2k-1,2r,2j),$$

$$E_{39} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{r=k}^{j-1} f(1,2k-2)|c_{2k-1}|g(2r,2j-1)|c_{2j}|l(2r+1,2j),$$

$$E_{40} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j} f(1,2k-2)|c_{2k-1}|f(2k,2j-1)|c_{2j}|l(1,2j),$$

$$E_{41} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k}\sum_{r=k+1}^{j} g(2h-1,2k-1)|c_{2k}|g(2r-1,2j-1)|c_{2j}|l(2h,2k,2r,2j),$$

$$E_{42} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k}\sum_{r=k+1}^{j-1} g(2h-1,2k-1)|c_{2k}|g(2r,2j-1)|c_{2j}|l(2r+1,2j),$$

$$E_{43} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k} g(2h-1,2k-1)|c_{2k}|f(2k+1,2j-1)|c_{2j}|l(2k+1,2j),$$

$$E_{44} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{h=1}^{k-1}\sum_{r=k+1}^{j} g(2h,2k-1)|c_{2k}|g(2r-1,2j-1)|c_{2j}|l(2h+1,2k),$$

$$E_{45} = -8\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{r=k+1}^{j} f(1,2k-1)|c_{2k}|g(2r-1,2j-1)|c_{2j}|l(1,2k),$$

$$F_1 = \sum_{h=1}^{n/2} w(1,2h,n), \quad F_2 = v(1,n),$$

$$F_3 = \sum_{j=1}^{n/2} \sum_{h=1}^{j-1} y(1,2h,2j-2), \quad F_4 = \sum_{j=1}^{n/2} x(1,2j-2),$$

$$F_5 = \sum_{j=1}^{n/2} \sum_{h=1}^{j} y(1,2h-1,2j-1), \quad F_6 = 2\sum_{j=1}^{n/2} c_{2j-1},$$

$$F_7 = -4\sum_{j=1}^{n/2} \sum_{k=1}^{j-1} \sum_{h=1}^{k-1} g(2h,2k-2)|c_{2k-1}|c_{2j-1},$$

$$F_8 = -4\sum_{j=1}^{n/2} \sum_{k=1}^{j-1} f(1,2k-2)|c_{2k-1}|c_{2j-1},$$

$$F_9 = -4\sum_{j=1}^{n/2} \sum_{k=1}^{j-1} \sum_{h=1}^{k} g(2h-1,2k-1)|c_{2k}|c_{2j-1},$$

$$F_{10} = 2\sum_{j=1}^{n/2} c_{2j}, \quad F_{11} = -4\sum_{j=1}^{n/2} \sum_{k=1}^{j} \sum_{h=1}^{k-1} g(2h,2k-2)|c_{2k-1}|c_{2j},$$

$$F_{12} = -4\sum_{j=1}^{n/2} \sum_{k=1}^{j} f(1,2k-2)|c_{2k-1}|c_{2j},$$

$$F_{13} = -4\sum_{j=1}^{n/2} \sum_{k=1}^{j-1} \sum_{h=1}^{k} g(2h-1,2k-1)|c_{2k}|c_{2j}.$$

By substituting $l(p,q)$ and $l(p,q,r,s)$ into each E_i and expanding them, we obtain

Theorem 1.15. *Let K and D_K be a ribbon knot of 1-fusion and its 1-fusion diagram as in Proposition 1.14 and let $J_K(t)$ be the Jones polynomial of K. Then $J'_K(-1)$ is a linear expression of α_i and $\alpha_{i,j}$*

$$J'_K(-1) = \sum_{1\le i\le n+1} A_i\alpha_i + \sum_{1\le i<j\le n+1} A_{i,j}\alpha_{i,j} + B,$$

where each A_i, $A_{i,j}$ and B is expressed by $c_1, c_2, \cdots, c_n$. To be more precise,

$$\sum_{i=1}^{45} E_i = \sum_{1\le i\le n+1} A_i\alpha_i + \sum_{1\le i<j\le n+1} A_{i,j}\alpha_{i,j}, \quad \sum_{i=1}^{13} F_i = B.$$

Remark 1.16. In the appendix, we discuss some properties of A_i, $A_{i,j}$, and B.

By [11, Theorem 5.1] we obtain the following theorem from Proposition 1.14.

Theorem 1.17. *Let K and D_K be a ribbon knot of 1-fusion and its 1-fusion diagram as in Proposition 1.14. Let Σ_K be a double branched cover of S^3 branched along K. Then $\lambda_{CW}(\Sigma_K)$, the Casson-Walker invariant of Σ_K, is written as follows*

$$\lambda_{CW}(\Sigma_K) = -\frac{1}{6M}\Big(\sum_{i=1}^{45} E_i + \sum_{i=1}^{13} F_i\Big),$$

where $M = (f(-1))^2 = (\sum_{i=1}^{n/2}(-1)^{\sum_{j=2i}^{n}(-1)^j c_j}((-1)^{c_{2i-1}} - 1) + 1)^2$ (Remark 1.8), E_i and F_i are in Proposition 1.14.

In particular, when Σ_K is an integral homology sphere that gives $M = 1$, the Casson invariant $\lambda = \lambda_{CW}/2$ and

$$\lambda(\Sigma_K) = -\frac{1}{12}\Big(\sum_{i=1}^{45} E_i + \sum_{i=1}^{13} F_i\Big).$$

2. Some links associated to 1-fusion diagram

Let D_K be the 1-fusion diagram in Definition 1.4. We shall introduce important 2-component links associated to D_K and prove a lemma for their Jones polynomials.

From now on we often denote a link and its diagram by the same symbol.

Let p, q be the integers satisfying $1 \le p < q \le n+1$.

Definition of $L(p,q)$. Suppose that p is even and q is odd. We define a 2-component link $L(p,q)$ obtained from D_K as follows (figure 6): We erase the outside of the big rectangle of D_K and erase subbands except i-th subbands ($p \le i \le q$). Then we add subbands in the trivial manner as shown in the last picture in figure 6.

Definition of $R(p,q)$. Suppose that p is odd and q is even. We define a 2-component link $R(p,q)$ obtained from D_K as follows (see the left diagram in figure 7, where $n = 10$, $p = 3$, and $q = 4$): We erase the outside of the big rectangle of D_K and erase subbands except i-th subbands ($p \le i \le q$). Then we add subbands in the trivial manner as shown in the figure.

Let p, q, r, s be the integers satisfying $1 \le p < q < r < s \le n+1$.

Definition of $L(p,q,r,s)$. Suppose that p and q are even and r and s are odd. We define a 2-component link $L(p,q,r,s)$ obtained from D_K as follows (see the middle diagram in figure 7, where $n = 10$, $p = 2$, $q = 4$, $r = 7$, and $s = 9$): We erase the outside of the big rectangle of D_K and erase subbands except i-th subbands ($p \le i \le q$) and j-th subbands ($r \le j \le s$). Then we add subbands in the trivial manner as shown in the figure.

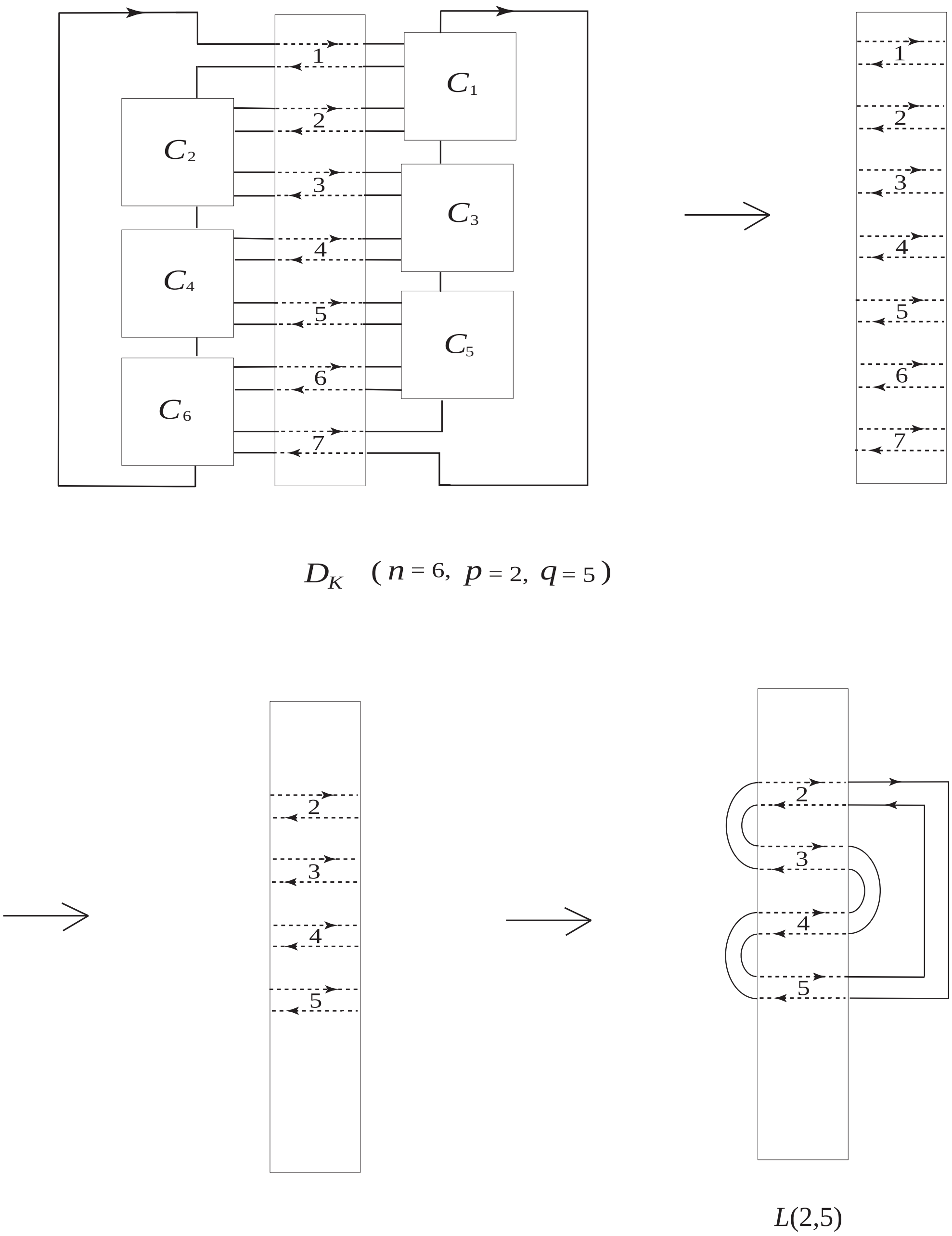

Figure 6

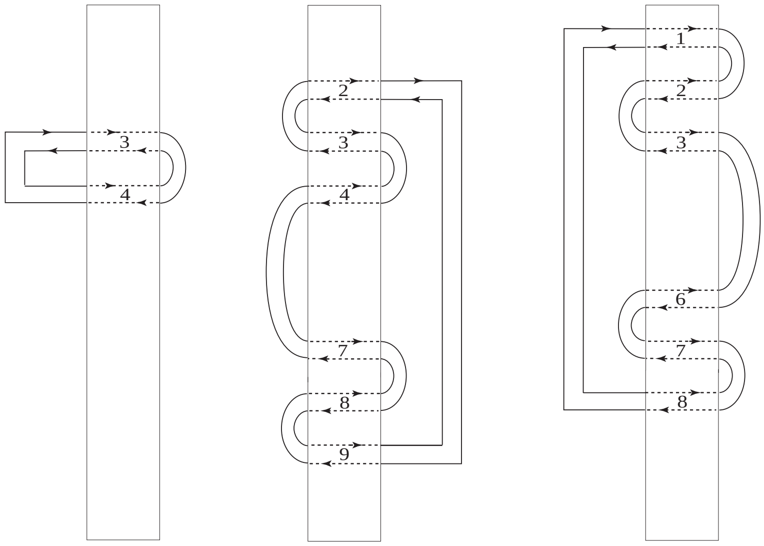

Figure 7

Definition of $R(p,q,r,s)$**.** Suppose that p and q are odd and r and s are even. We define a 2-component link $R(p,q,r,s)$ obtained from D_K as follows (see the right diagram in figure 7, where $n = 10$, $p = 1$, $q = 3$, $r = 6$, and $s = 8$): We erase the outside of the big rectangle of D_K and erase subbands except i-th subbands ($p \leq i \leq q$) and j-th subbands ($r \leq j \leq s$). Then we add subbands in the trivial manner as shown in the figure.

Definition of $LL(p,q,r,s)$**.** Suppose that p, q, r, and s are even. We define a 2-component link $LL(p,q,r,s)$ obtained from D_K as follows (see the left diagram in figure 8, where $n = 10$, $p = 2$, $q = 4$, $r = 8$, and $s = 10$): We erase the outside of the big rectangle of D_K and erase subbands except i-th subbands ($p \leq i \leq q$) and j-th subbands ($r \leq j \leq s$). Then we add subbands in the trivial manner as shown in the figure.

Definition of $RR(p,q,r,s)$**.** Suppose that p, q, r, and s are odd. We define a 2-component link $LL(p,q,r,s)$ obtained from D_K as follows (see the right diagram in figure 8, where $n = 10$, $p = 1$, $q = 3$, $r = 7$, and $s = 9$): We erase the outside of the big rectangle of D_K and erase subbands except i-th subbands ($p \leq i \leq q$) and j-th subbands ($r \leq j \leq s$). Then we add subbands in the trivial manner as shown in the figure.

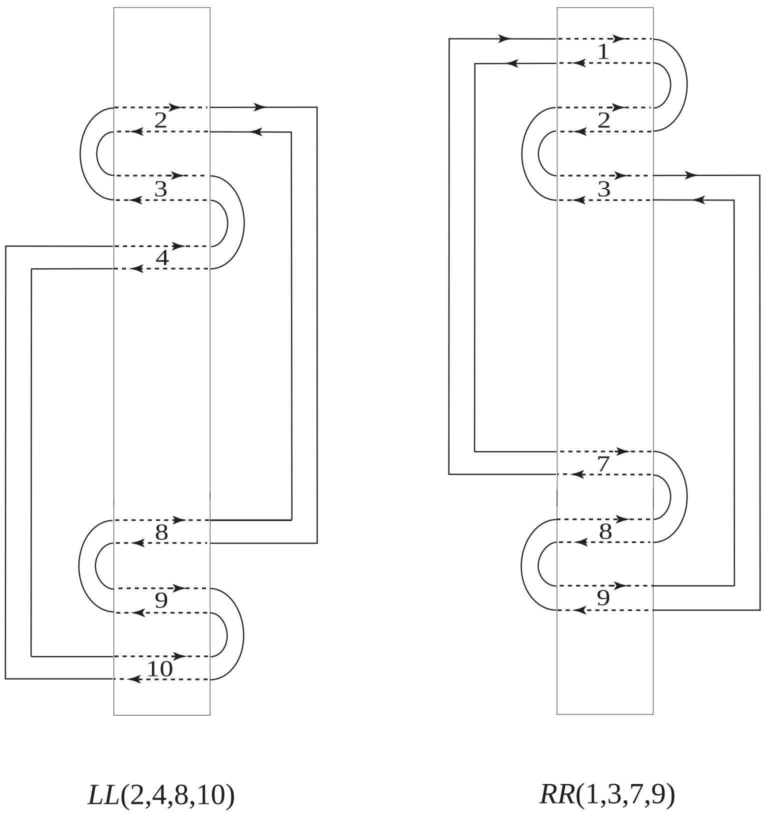

Figure 8

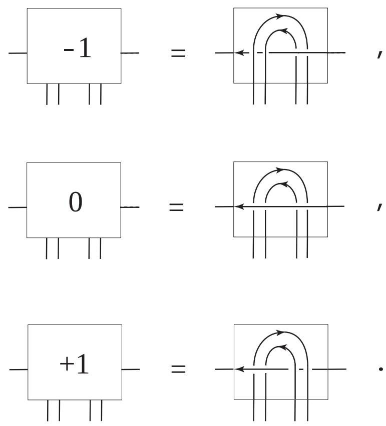

Figure 9

The following lemma is used in section 6.

Lemma 2.1. *The following formulas hold:*

$$J_{L(p,q)}(-1) = J_{R(p,q)}(-1) = (2i)l(p,q),$$
$$J_{L(p,q,r,s)}(-1) = J_{R(p,q,r,s)}(-1) = (2i)l(p,q,r,s),$$
$$J_{LL(p,q,r,s)}(-1) = J_{RR(p,q,r,s)}(-1) = (2i)l(p,q,r,s).$$

For $l(p,q)$ and $l(p,q,r,s)$ see before Proposition 1.14.

Proof. Recall that $J_L(-1) = \Delta_L(-1)$, where Δ denotes the normalized Alexander polynomial ([4], cf. [6,7]). $L(p,q)$ bounds an annulus as a Seifert surface. Its Seifert matrix is 1×1-matrix whose entry is $-l(p,q)$. Hence $J_{L(p,q)}(-1) = \Delta_{L(p,q)}(-1) = (-1)^{-\frac{1}{2}}(-2)l(p,q) = (2i)l(p,q)$. □

3. A formula for the Jones polynomial

The following proposition is useful for calculating the Jones polynomial.

Proposition 3.1. *Let L be a link diagram which has c as the framed part on the left in figure 9, where $c \in \{-1, 0, +1\}$. Then we have*

$$J_L(t) = (1-t^c)J_{L_1}(t) + t^{-\frac{c}{2}}(1-t^{2c})J_{L_2}(t) + t^{-c}J_{L_3}(t) + (1-t^c)J_{L_4}(t) + t^{-\frac{c}{2}}(1-t^c)J_{L_5}(t),$$

where L_1, L_2, L_3, L_4, L_5 are the following oriented link diagrams obtained from L, which are identical with L except inside of the rectangle (figure 10).

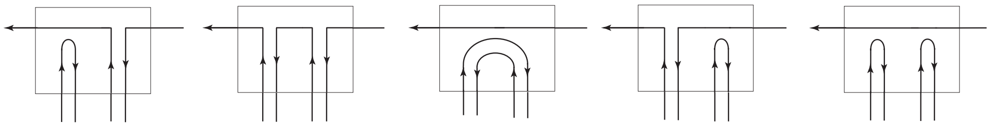

Figure 10

Proof. We apply the recursive formula of the Kauffman bracket ([5]) to four crossings in the rectangle of L. If $c = -1$, then we have

$$\langle L\rangle = (2 + A^2 d)\langle L_1\rangle + (A^2 + B^2 + A^4 d)\langle L_2\rangle + B^4\langle L_3\rangle + (2 + A^2 d)\langle L_4\rangle + (2B^2 + d)\langle L_5\rangle,$$

where $d = -(A^2 + B^2)$, $B = A^{-1}$. Here we notice that the writhes of L and these five diagrams are the same, so we have

$$J_L(t) = (1 - t^{-1})J_{L_1}(t) + t^{\frac{1}{2}}(1 - t^{-2})J_{L_2}(t) + tJ_{L_3}(t) + (1 - t^{-1})J_{L_4}(t) + t^{\frac{1}{2}}(1 - t^{-1})J_{L_5}(t).$$

The case $c = +1$ is similar.

If $c = 0$, L is isotopic to L_3, so we have $J_L(t) = J_{L_3}(t)$. □

Remark 3.2. $J_L(t)$ is obtained by weighting $J_{L_1}(t)$, $J_{L_2}(t)$, $J_{L_3}(t)$, $J_{L_4}(t)$, and $J_{L_5}(t)$ and adding them. Note that the weight of $J_{L_1}(t)$ and the weight of $J_{L_4}(t)$ are the same.

4. Calculation of the Jones polynomial

To apply Proposition 3.1 to the 1-fusion diagram D_K in Proposition 1.14 we introduce some links obtained from D_K.

We denote D_K by $[C_1, \ldots, C_n]$.

4.1. A notation

We denote by $[X_1, \ldots, X_i, C_{i+1}, \ldots, C_n]$ the diagram obtained from $[C_1, \ldots, C_n]$ by changing C_1 to X_1, …, C_i to X_i, where $X_1, \ldots, X_i \in \{S, U, T, P, Q\}$ and S, T, U, P, and Q are the figures as shown in the following table (see figure 11). Note that figures T in X_{odd} and in X_{even} are the same. So are U and Q, but for S and P they are not same. For example, $[U, U, S, S, S, U]$ is in figure 12, $[S, U, T, S, U, S]$ is in figure 13 and $[U, S, U, T, S, U, P, C_8]$ is in figure 14. Note that the diagram in figure 12 gives a split

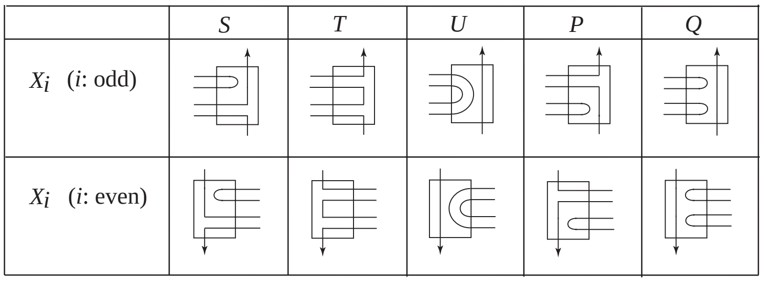

Figure 11

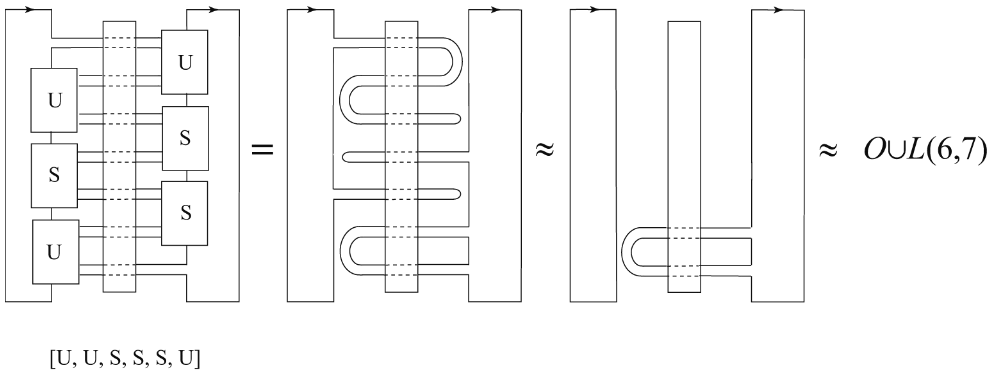

Figure 12

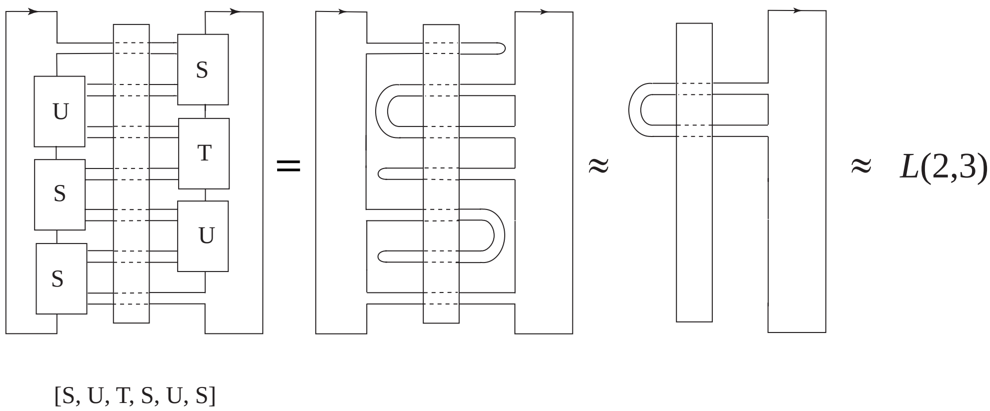

Figure 13

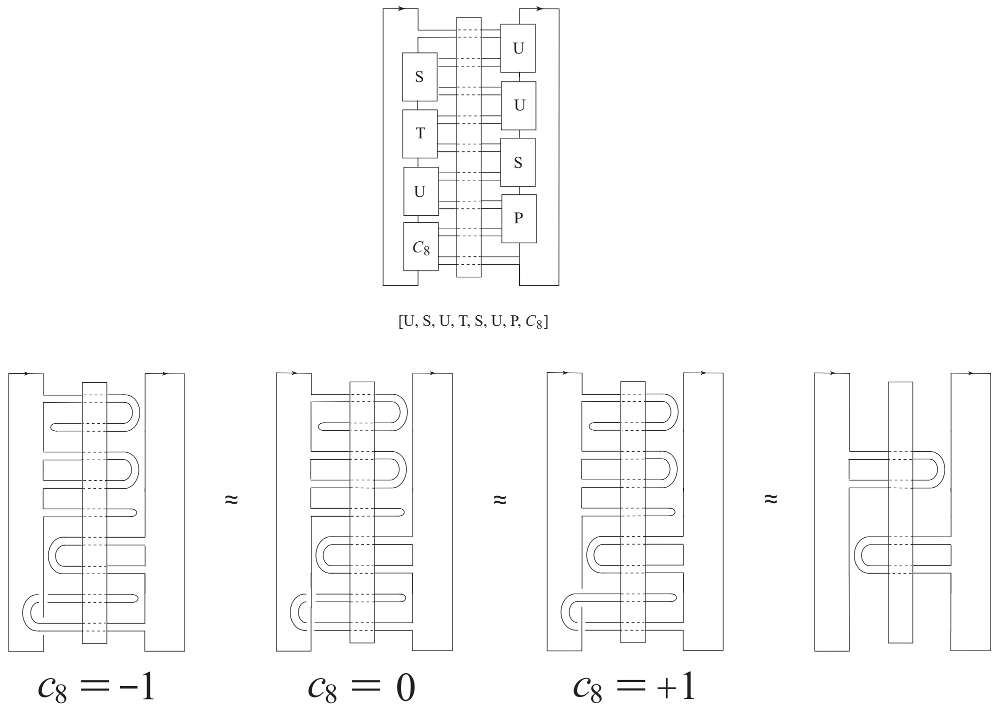

Figure 14

link which consists of the trivial knot and $L(6,7)$ defined in section 2.

When $i = 1$, $[X_1, \dots, X_{i-1}, Y, C_{i+1}, \dots, C_n]$ is nothing but $[Y, C_2, \dots, C_n]$.

We denote $[X_1, \dots, X_{i-1}, P, C_{i+1}, \dots, C_n]$, $[X_1, \dots, X_{i-1}, Q, C_{i+1}, \dots, C_n]$ by $[X_1, \dots, X_{i-1}, P]$, $[X_1, \dots, X_{i-1}, Q]$ respectively, since their link types do not depend on the values of $c_{i+1}, \dots, c_n$. For example, as it is seen from figure 14 the link type of $[U, S, U, T, S, U, P, C_8]$ does not depend on the value of c_8.

4.2. Symbols

We prepare the following symbols by the connection with Proposition 3.1. Let $S(c) = 1 - t^c$, $T(c) = t^{-\frac{c}{2}}(1 - t^{2c})$, $U(c) = t^{-c}$, $P(c) = 1 - t^c$, $Q(c) = t^{-\frac{c}{2}}(1 - t^c)$, where $c \in \{-1, 0, +1\}$.

Note that $S(c) = P(c)$ (Remark 3.2).

4.3. A notation

We denote $\sum_{X_1 \in J} \sum_{X_2 \in J} \cdots \sum_{X_m \in J} F(X_1, \dots, X_m)$ by $\sum_{X_i \in J} F(X_1, \dots, X_m)$. From now on we use this convention.

4.4. Calculations

We apply Proposition 3.1 to C_1 in $[C_1, \dots, C_n]$ and we have

$$\begin{aligned} J_K(t) &= S(c_1) J_{[S, C_2, \dots, C_n]}(t) + T(c_1) J_{[T, C_2, \dots, C_n]}(t) + U(c_1) J_{[U, C_2, \dots, C_n]}(t) \\ &\quad + P(c_1) J_{[P, C_2, \dots, C_n]}(t) + Q(c_1) J_{[Q, C_2, \dots, C_n]}(t) \\ &= \sum_{X_1 \in \{S, U, T\}} X_1(c_1) J_{[X_1, C_2, \dots, C_n]}(t) + P(c_1) J_{[P]}(t) + Q(c_1) J_{[Q]}(t). \end{aligned}$$

Next we apply Proposition 3.1 to C_2 of $[X_1, C_2, \dots, C_n]$ ($X_1 \in \{S, T, U\}$) and we have

$$\begin{aligned} J_{[X_1, C_2, \dots, C_n]}(t) &= S(c_2) J_{[X_1, S, \dots, C_n]}(t) + T(c_2) J_{[X_1, T, \dots, C_n]}(t) + U(c_2) J_{[X_1, U, \dots, C_n]}(t) \\ &\quad + P(c_2) J_{[X_1, P, C_3, \dots, C_n]}(t) + Q(c_2) J_{[X_1, Q, C_3, \dots, C_n]}(t) \\ &= \sum_{X_2 \in \{S, U, T\}} X_2(c_2) J_{[X_1, X_2, C_3, \dots, C_n]}(t) + P(c_2) J_{[X_1, P]}(t) \\ &\quad + Q(c_2) J_{[X_1, Q]}(t). \end{aligned}$$

So we have

$$
\begin{aligned}
J_K(t) = & \sum_{X_1\in\{S,U,T\}} X_1(c_1) \sum_{X_2\in\{S,U,T\}} X_2(c_2) J_{[X_1,X_2,C_3,\dots,C_n]}(t) \\
& + \sum_{1\le j\le 2} \sum_{X_i\in\{S,U,T\}} \prod_{i=1}^{j-1} X_i(c_i)P(c_j)J_{[X_1,\dots,X_{j-1},P]}(t) \\
& + \sum_{1\le j\le 2} \sum_{X_i\in\{S,U,T\}} \prod_{i=1}^{j-1} X_i(c_i)Q(c_j)J_{[X_1,\dots,X_{j-1},Q]}(t).
\end{aligned}
$$

Note that we use the convention in section 4.3.

We continue this until we come to C_n, and then we have

Proposition 4.1. *Let $[C_1,\dots,C_n]$ be the 1-fusion diagram as above. Then $J_K(t)$ is written as follows:*

$$
\begin{aligned}
J_K(t) = & \sum_{X_i\in\{S,U,T\}} \prod_{i=1}^{n} X_i(c_i) J_{[X_1,\dots,X_n]}(t) \\
& + \sum_{1\le j\le n} \sum_{X_i\in\{S,U,T\}} \prod_{i=1}^{j-1} X_i(c_i)P(c_j)J_{[X_1,\dots,X_{j-1},P]}(t) \\
& + \sum_{1\le j\le n} \sum_{X_i\in\{S,U,T\}} \prod_{i=1}^{j-1} X_i(c_i)Q(c_j)J_{[X_1,\dots,X_{j-1},Q]}(t).
\end{aligned}
$$

Moreover, by decomposing $\sum_{1\le j\le n}$ into the sum of $\sum_{j\ odd}$ and $\sum_{j\ even}$, $J_K(t)$ becomes the sum of five parts:

$$
J_K(t) = J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) \tag{1}
$$

where

$$
\begin{aligned}
J_1(t) = & \sum_{X_i\in\{S,U,T\}} \prod_{i=1}^{n} X_i(c_i) J_{[X_1,\dots,X_n]}(t), \\
J_2(t) = & \sum_{j=1}^{n/2} \sum_{X_i\in\{S,U,T\}} \prod_{i=1}^{2j-2} X_i(c_i)P(c_{2j-1})J_{[X_1,\dots,X_{2j-2},P]}(t),
\end{aligned}
$$

$$J_3(t) = \sum_{j=1}^{n/2} \sum_{X_i \in \{S,U,T\}} \prod_{i=1}^{2j-1} X_i(c_i) P(c_{2j}) J_{[X_1,\ldots,X_{2j-1},P]}(t),$$

$$J_4(t) = \sum_{j=1}^{n/2} \sum_{X_i \in \{S,U,T\}} \prod_{i=1}^{2j-2} X_i(c_i) Q(c_{2j-1}) J_{[X_1,\ldots,X_{2j-2},Q]}(t),$$

$$J_5(t) = \sum_{j=1}^{n/2} \sum_{X_i \in \{S,U,T\}} \prod_{i=1}^{2j-1} X_i(c_i) Q(c_{2j}) J_{[X_1,\ldots,X_{2j-1},Q]}(t).$$

5. Lemmas

To calculate $J'_K(-1)$ we prepare some lemmas.

Lemma 5.1. *Let $c_i \in \{-1, 0, +1\}$. Then we have*

$$\sum_{X_i \in \{S,U\}} \prod_{i=p}^{q} X_i(c_i) = \prod_{i=p}^{q} (t^{-c_i} + 1 - t^{c_i}) \tag{2a}$$

$$\sum_{X_i \in \{S,U\}} \prod_{i=p}^{q} X_i(c_i)|_{t=-1} = 1 \tag{2b}$$

$$\Bigg(\sum_{X_i \in \{S,U\}} \prod_{i=p}^{q} X_i(c_i) \Bigg)' (-1) = -2 \sum_{i=p}^{q} c_i \tag{2c}$$

Proof. (2a) follows from

$$\begin{aligned} &\{S(c_p) + U(c_p)\}\{S(c_{p+1}) + U(c_{p+1})\} \cdots \{S(c_q) + U(c_q)\} \\ &\qquad = \sum_{X_p \in \{S,U\}} \sum_{X_{p+1} \in \{S,U\}} \cdots \sum_{X_q \in \{S,U\}} \prod_{i=p}^{q} X_i(c_i) \\ &\qquad = \sum_{X_i \in \{S,U\}} \prod_{i=p}^{q} X_i(c_i) \end{aligned}$$

and

$$S(c_i) + U(c_i) = (1 - t^{c_i}) + t^{-c_i} = t^{-c_i} + 1 - t^{c_i}.$$

(2b) is trivial from (2a).

(2c) follows from (2a) and

$$\Bigg(\prod_{i=p}^{q} (t^{-c_i} + 1 - t^{c_i}) \Bigg)' (-1) = \sum_{i=p}^{q} (-c_i(-1)^{-c_i-1} - c_i(-1)^{c_i-1}) = \sum_{i=p}^{q} (-2c_i). \qquad \square$$

The following are lemmas for calculations with Jones polynomials, so we adopt the convention $(-1)^{\frac{1}{2}} = -i$.

We often use the following in the proof of the lemmas:

$$\{(1+t)^2 f(t)\}'(-1) = 0,$$
$$\{(1+t)f(t)\}'(-1) = f(-1).$$

Lemma 5.2. *Let $G(t)$ be a Laurent polynomial. Let $d(t)$ denote $-t^{-\frac{1}{2}}(1+t)$. Let $S(c)$, etc., be as in section 4.2. Then the following holds.*

$$\{G(t)d(t)\}'(-1) = -iG(-1) \tag{3a}$$
$$\{T(c)G(t)\}'(-1) = 2|c|iG(-1) \tag{3b}$$
$$S(c)|_{t=-1} = P(c)|_{t=-1} = 2|c| \tag{3c}$$
$$\{T(c)d(t)G(t)\}'(-1) = 0 \tag{3d}$$
$$(\{S(c)\}'(-1) = \{P(c)\}'(-1) = -c \tag{3e}$$
$$Q(c)|_{t=-1} = 2ci \tag{3f}$$
$$\{Q(c)\}'(-1) = 0 \tag{3g}$$

Let

$$V(p,q) = \prod_{i=p}^{q} U(c_i),$$
$$W(p,q,r) = \sum_{X_i \in \{S,U\}} \prod_{i=p}^{q-1} X_i(c_i) S(c_q) \prod_{j=q+1}^{r} U(c_j).$$

Then we have

Lemma 5.3.

$$V(p,q)|_{t=-1} = f(p,q) \tag{4a}$$
$$W(p,q,r)|_{t=-1} = g(q,r) \tag{4b}$$
$$\{V(p,q)\}'(-1) = v(p,q) \tag{4c}$$
$$\{W(p,q,r)\}'(-1) = w(p,q,r) \tag{4d}$$
$$\{V(p,q)P(c_{q+1})\}'(-1) = x(p,q) \tag{4e}$$
$$\{W(p,q,r)P(c_{r+1})\}'(-1) = y(p,q,r) \tag{4f}$$

Note that the right hand side of (4b) *does not depend on p.*

Proof. (4a) follows from

$$V(p,q) = \prod_{i=p}^{q} U(c_i) = \prod_{i=p}^{q} t^{-c_i}$$

and

$$V(p,q)|_{t=-1} = \prod_{i=p}^{q}(-1)^{-c_i} = f(p,q).$$

(4b) follows from (2b), (3c), and (4a). (4c) follows from

$$\prod_{i=p}^{q} U(c_i) = \prod_{i=p}^{q} t^{-c_i} = t^{-\sum_{i=p}^{q} c_i}.$$

(4d) follows from (2b), (2c), (3c), (3e), (4a), and (4c). □

6. Proof of Proposition 1.14

Now we begin to calculate $J'_K(-1)$.

Note that the first derivative at -1 of a Laurent polynomial which has $(1+t)^2$ as a factor is 0. If at least two of X_i's in a term of (1) in Proposition 3.1 are T, then the first derivative at -1 of the term is 0. In fact, let X_i and X_j be T. Then the term has $T(c_i)T(c_j) = t^{-\frac{c_i}{2}}(1-t^{2c_i})t^{-\frac{c_j}{2}}(1-t^{2c_j})$ as a factor. So if $c_i \neq 0$ and $c_j \neq 0$, the term has $(1+t)^2$ as a factor. If at least one of c_i or c_j is 0, then the term is 0.

Thus, to calculate $J'_K(-1)$, it is enough to consider the terms of (1) *in Proposition 4.1 without T and with a single T and calculation proceeds as follows:* In [i-0] ($1 \leq i \leq 5$), we consider the terms in $J_i(t)$ without T. In [i-1], we consider the terms in $J_i(t)$ with a single T. Moreover, [i-1] divides into [i-1-odd] and [i-1-even] in terms of position of the T being right or left. In [i-1-odd], we consider the terms with a single T which appears in X_{odd} (i.e. in the right). In [i-1-even], we consider the terms with a single T which appears in X_{even} (i.e. in the left).

6.1. We consider the part $J_1(t)$ in Proposition 4.1.

[1-0] Picking up the terms without T from $J_1(t)$, we obtain

$$\sum_{X_i \in \{S,U\}} \prod_{i=1}^{n} X_i(c_i) J_{[X_1,\dots,X_n]}(t).$$

We divide these terms into three groups by the link type of

$$[X_1,\dots,X_n] \qquad (X_i \in \{S,U\});$$

- **(1)** $[X_1,\dots,X_{2h-2},S,U,\dots,U]$ $(1 \leq h \leq n/2)$. Precisely $X_{2h-1} = S$, $X_i \in \{S,U\}$ $(1 \leq i \leq 2h-2)$, $X_j = U$ $(2h \leq j \leq n)$.
- **(2)** $[X_1,\dots,X_{2h-1},S,U,\dots,U]$ $(1 \leq h \leq n/2)$. Precisely $X_{2h} = S$, $X_i \in \{S,U\}$ $(1 \leq i \leq 2h-1)$, $X_j = U$ $(2h+1 \leq j \leq n)$.

(3) $[U, \ldots, U]$. Precisely $X_i = U$ $(1 \leq i \leq n)$.

The link type of $[X_1, \ldots, X_n]$ is as follows, where O is the trivial knot and $\cup$ means the split sum. Note that in group **1** the link type does not depend on $X_1, \cdots, X_{2h-2}$:

1	$O \cup L(2h, n+1)$
2	O
3	O

For example $[U, U, S, S, S, U] = O \cup L(6, 7)$ (see figure 12).

The derivative at -1 of the sum of the terms in each group is calculated as follows:

(1) The sum of the terms in this group is

$$\sum_{h=1}^{n/2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2h-2} X_i(c_i) S(c_{2h-1}) \prod_{j=2h}^{n} U(c_j) J_{O \cup L(2h,n+1)}(t) = \sum_{h=1}^{n/2} W(1, 2h-1, n) J_{L(2h,n+1)}(t) d(t).$$

By using (3a) and (4b), the derivative at -1 is E_1.

(2) The sum is

$$\sum_{h=1}^{n/2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2h-1} X_i(c_i) S(c_{2h}) \prod_{j=2h+1}^{n} U(c_j) J_O(t) = \sum_{h=1}^{n/2} W(1, 2h, n).$$

By using (4d), the derivative is F_1.

(3) The term is

$$\prod_{i=1}^{n} U(c_i) J_O(t) = V(1, n).$$

By using (4c), the derivative is F_2.

[**1-1**] Picking up the terms with a single T from $J_1(t)$, we obtain

$$\sum_{l=1}^{n} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{l-1} X_i(c_i) T(c_l) \prod_{i=l+1}^{n} X_i(c_i) J_{[X_1, \ldots, X_{l-1}, T, X_{l+1}, \ldots, X_n]}(t).$$

We divide $\sum_{1 \leq l \leq n}$ into two parts $\sum_{l \text{ odd}}$ ([**1-1-odd**]) and $\sum_{l \text{ even}}$ ([**1-1-even**]).

[1-1-odd] We consider the following terms (in which l is odd).

$$\sum_{k=1}^{n/2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2k-2} X_i(c_i) T(c_{2k-1}) \prod_{i=2k}^{n} X_i(c_i) J_{[X_1,\ldots,X_{2k-2},T,X_{2k},\ldots,X_n]}(t).$$

We divide these terms into nine groups by the link type of

$$[X_1, \ldots, X_{2k-2}, T, X_{2k}, \ldots, X_n] \qquad (X_i \in \{S, U\});$$

$[X_1, \ldots, X_{2k-2}]$ is divided into three groups:
(1) $[X_1, \ldots, X_{2h-2}, S, U, \ldots, U]$ $(1 \le h \le k-1)$: $g(2h-1, 2k-2)$,
(2) $[X_1, \ldots, X_{2h-1}, S, U, \ldots, U]$ $(1 \le h \le k-1)$: $g(2h, 2k-2)$,
(3) $[U, \ldots, U]$: $f(1, 2k-2)$.

$[X_{2k}, \ldots, X_n]$ is divided into three groups:
(a) $[X_{2k}, \ldots, X_{2r-2}, S, U, \ldots, U]$ $(k+1 \le r \le n/2)$: $g(2r-1, n)$,
(b) $[X_{2k}, \ldots, X_{2r-1}, S, U, \ldots, U]$ $(k \le r \le n/2)$: $g(2r, n)$,
(c) $[U, \ldots, U]$: $f(2k, n)$.
(The reason why g, f are written there will be found in **(1,b)**.)

Then we have nine groups **(1,a)**–**(3,c)**. The link type of $[X_1, \ldots, X_{2k-2}, T, X_{2k}, \ldots, X_n]$ is as follows, where each link in **(1,a)** and **(1,c)** is a split link which consists of the trivial knot and some link:

	a	**b**	**c**
1		$L(2h, 2k-1)$	
2	$L(2r, n+1)$	$RR(2h+1, 2k-1, 2r+1, n+1)$	$L(2k, n+1)$
3	$L(2r, n+1)$	$RR(1, 2k-1, 2r+1, n+1)$	$L(2k, n+1)$

As an example of **(1,b)**, $[S, U, T, S, U, S] = L(2, 3)$ (see figure 13).

The derivative at -1 of the sum of the terms in each group is calculated as follows:
(1,b): The sum of the terms is

$$\sum_{k=1}^{n/2} \sum_{h=1}^{k-1} \sum_{r=k}^{n/2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2h-2} X_i(c_i) S(c_{2h-1}) \prod_{j=2h}^{2k-2} U(c_j) T(c_{2k-1}) \cdot \prod_{i=2k}^{2r-1} X_i(c_i) S(c_{2r}) \prod_{j=2r+1}^{n} U(c_j) \times J_{L(2h,2k-1)}(t)$$

$$= \sum_{k=1}^{n/2} \sum_{h=1}^{k-1} \sum_{r=k}^{n/2} W(1, 2h-1, 2k-2) T(c_{2k-1}) W(2k, 2r, n) J_{L(2h,2k-1)}(t).$$

By using (3b) and (4b), the derivative is

$$\sum_{k=1}^{n/2}\sum_{h=1}^{k-1}\sum_{r=k}^{n/2} g(2h-1,2k-2)(2i)|c_{2k-1}|g(2r,n)(2i)l(2h,2k-1)$$
$$= -4\sum_{k=1}^{n/2}\sum_{h=1}^{k-1}\sum_{r=k}^{n/2} g(2h-1,2k-2)|c_{2k-1}|g(2r,n)l(2h,2k-1).$$

This is E_2.

As we see from this calculation, we can calculate the derivative of the terms with a single T automatically by the following procedure:

$$[\underbrace{X_1,\dots,X_{2h-2},S,U,\dots,U}_{\mathbf{1}},T,\underbrace{X_{2k},\dots,X_{2r-1},S,U,\dots,U}_{\mathbf{b}}] = L(2h,2k-1)$$
$$\swarrow \qquad \downarrow \qquad \downarrow \qquad \swarrow$$
$$g(2h-1,2k-2) \qquad (2i)|c_{2k-1}| \qquad g(2r,n) \qquad (2i)l(2h,2k-1)$$
$$\Downarrow$$
$$-4g(2h-1,2k-2)|c_{2k-1}|g(2r,n)l(2h,2k-1).$$

By summing up we have E_2.

By using this procedure we have the following, where each link in **(1,a)** and **(1,c)** is a split link which consists of the trivial knot and some link, and hence by using (3d) the derivative of the terms is 0:

	a: $g(2r-1,n)$	**b**: $g(2r,n)$	**c**: $f(2k,n)$
1: $g(2h-1,2k-2)$	0	E_2	0
2: $g(2h,2k-2)$	E_3	E_4	E_5
3: $f(1,2k-2)$	E_6	E_7	E_8

[1-1-even] We consider the following terms (in which l is even).

$$\sum_{k=1}^{n/2}\sum_{X_i\in\{S,U\}}\prod_{i=1}^{2k-1} X_i(c_i)T(c_{2k})\prod_{i=2k+1}^{n} X_i(c_i)J_{[X_1,\dots,X_{2k-1},T,X_{2k+1},\dots,X_n]}(t).$$

We divide these terms into nine groups by the link type of

$$[X_1,\dots,X_{2k-1},T,X_{2k+1},\dots,X_n] \qquad (X_i\in\{S,U\}).$$

$[X_1,\dots,X_{2k-1}]$ is divided into three groups:

(1) $[X_1, \dots, X_{2h-2}, S, U, \dots, U]$ $(1 \le h \le k)$: $g(2h-1, 2k-1)$,
(2) $[X_1, \dots, X_{2h-1}, S, U, \dots, U]$ $(1 \le h \le k-1)$: $g(2h, 2k-1)$,
(3) $[U, \dots, U]$: $f(1, 2k-1)$.

$[X_{2k+1}, \dots, X_n]$ is divided into three groups:

(a) $[X_{2k+1}, \dots, X_{2r-2}, S, U, \dots, U]$ $(k+1 \le r \le n/2)$: $g(2r-1, n)$,
(b) $[X_{2k+1}, \dots, X_{2r-1}, S, U, \dots, U]$ $(k+1 \le r \le n/2)$: $g(2r, n)$,
(c) $[U, \dots, U]$: $f(2k+1, n)$.

Then we have nine groups **(1,a)**–**(3,c)**. The link type of $[X_1, \dots, X_{2k-1}, T, X_{2k+1}, \dots, X_n]$ is as follows, where each link in **(2,a)** and **(3,a)** is a split link which consists of the trivial knot and some link:

	a	**b**	**c**
1	$L(2r, n+1)$	$L(2h, 2k, 2r+1, n+1)$	$L(2h, n+1)$
2		$R(2h+1, 2k)$	$R(2h+1, 2k)$
3		$R(1, 2k)$	$R(1, 2k)$

The derivative at -1 of the sum of the terms in each group is as follows:

	a: $g(2r-1, n)$	**b**: $g(2r, n)$	**c**: $f(2k+1, n)$
1: $g(2h-1, 2k-1)$	E_9	E_{10}	E_{11}
2: $g(2h, 2k-1)$	0	E_{12}	E_{13}
3: $f(1, 2k-1)$	0	E_{14}	E_{15}

Each link in **(2,a)** and **(3,a)** is a split link which consists of the trivial knot and some link. By using (3d) the derivative of the terms is 0.

6.2. We consider the part $J_2(t)$ in Proposition 4.1.

[2-0] Picking up the terms without T from $J_2(t)$, we obtain

$$\sum_{j=1}^{n/2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2j-2} X_i(c_i) P(c_{2j-1}) J_{[X_1, \dots, X_{2j-2}, P]}(t).$$

We divide these terms into three groups by the link type of

$$[X_1, \cdots, X_{2j-2}, P] \qquad (X_i \in \{S, U\}):$$

(1) $[X_1, \dots, X_{2h-2}, S, U, \dots, U, P] = O \cup L(2h, 2j-1) \quad (1 \le h \le j-1)$,
(2) $[X_1, \dots, X_{2h-1}, S, U, \dots, U, P] = O \quad (1 \le h \le j-1)$,
(3) $[U, \cdots, U, P] = O$.

The sum of the terms in each group and its derivative at -1 are as follows:

(1) $\sum_{j=1}^{n/2}\sum_{h=1}^{j-1} W(1,2h-1,2j-2)P(c_{2j-1})J_{L(2h,2j-1)}(t)d(t)$. The derivative is E_{16}.

(2) $\sum_{j=1}^{n/2}\sum_{h=1}^{j-1} W(1,2h,2j-2)P(c_{2j-1})$. By using (4f), the derivative is F_3.

(3) $\sum_{j=1}^{n/2} V(1,2j-2)P(c_{2j-1})$. By using (4e), the derivative is F_4.

[**2-1**] Picking up the terms with a single T from $J_2(t)$, we obtain

$$\sum_{j=1}^{n/2}\sum_{l=1}^{2j-2}\sum_{X_i\in\{S,U\}}\prod_{i=1}^{l-1} X_i(c_i)T(c_l) \cdot \prod_{i=l+1}^{2j-2} X_i(c_i)P(c_{2j-1})J_{[X_1,\dots,X_{l-1},T,X_{l+1},\dots,X_{2j-2},P]}(t).$$

[**2-1-odd**] We consider the following terms (in which l is odd).

$$\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{X_i\in\{S,U\}}\prod_{i=1}^{2k-2} X_i(c_i)T(c_{2k-1}) \cdot \prod_{i=2k}^{2j-2} X_i(c_i)P(c_{2j-1})J_{[X_1,\dots,X_{2k-2},T,X_{2k},\dots,X_{2j-2},P]}(t).$$

We divide these terms into nine groups by the link type of

$$[X_1,\dots,X_{2k-2},T,X_{2k},\dots,X_{2j-2},P] \qquad (X_i\in\{S,U\}):$$

$[X_1,\dots,X_{2k-2}]$ is divided into three groups:

(1) $[X_1,\dots,X_{2h-2},S,U,\dots,U]$ $(1\le h\le k-1)$; $g(2h-1,2k-2)$,
(2) $[X_1,\dots,X_{2h-1},S,U,\dots,U]$ $(1\le h\le k-1)$; $g(2h,2k-2)$,
(3) $[U,\dots,U]$; $f(1,2k-2)$.

$[X_{2k},\dots,X_{2j-2}]$ is divided into three groups:

(a) $[X_{2k},\dots,X_{2r-2},S,U,\dots,U]$ $(k+1\le r\le j-1)$; $g(2r-1,2j-2)$,
(b) $[X_{2k},\dots,X_{2r-1},S,U,\dots,U]$ $(k\le r\le j-1)$; $g(2r,2j-2)$,
(c) $[U,\dots,U]$; $f(2k,2j-2)$.

The link type of $[X_1,\dots,X_{2k-2},T,X_{2k},\dots,X_{2j-2},P]$ is as follows, where each link in **(1,a)** and **(1,c)** is a split link which consists of the trivial knot and some link:

	a	**b**	**c**
1		$L(2h, 2k-1)$	
2	$L(2r, 2j-1)$	$RR(2h+1, 2k-1, 2r+1, 2j-1)$	$L(2k, 2j-1)$
3	$L(2r, 2j-1)$	$RR(1, 2k-1, 2r+1, 2j-1)$	$L(2k, 2j-1)$

The derivative at -1 of the sum of the terms in each group is as follows:

	a: $g(2r-1, 2j-2)$	**b**: $g(2r, 2j-2)$	**c**: $f(2k, 2j-2)$
1: $g(2h-1, 2k-2)$	0	E_{17}	0
2: $g(2h, 2k-2)$	E_{18}	E_{19}	E_{20}
3: $f(1, 2k-2)$	E_{21}	E_{22}	E_{23}

Each link in **(1,a)** and **(1,c)** is a split link which consists of the trivial knot and some link. By using (3d) the derivative of the terms is 0.

[**2-1-even**] We consider the following terms (in which l is even).

$$\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{X_i\in\{S,U\}}\prod_{i=1}^{2k-1} X_i(c_i)T(c_{2k}) \cdot \prod_{i=2k+1}^{2j-2} X_i(c_i)P(c_{2j-1})J_{[X_1,\dots,X_{2k-1},T,X_{2k+1},\dots,X_{2j-2},P]}(t).$$

We divide these terms into nine groups by the link type of

$$[X_1,\dots,X_{2k-1},T,X_{2k+1},\dots,X_{2j-2},P] \qquad (X_i\in\{S,U\}):$$

$[X_1,\dots,X_{2k-1}]$ is divided into three groups:

(1) $[X_1,\dots,X_{2h-2},S,U,\dots,U]$ $(1\le h\le k)$: $g(2h-1,2k-1)$,
(2) $[X_1,\dots,X_{2h-1},S,U,\dots,U]$ $(1\le h\le k-1)$: $g(2h,2k-1)$,
(3) $[U,\cdots,U]$: $f(1,2k-1)$.

$[X_{2k+1},\dots,X_{2j-2}]$ is divided into three groups:

(a) $[X_{2k+1},\dots,X_{2r-2},S,U,\dots,U]$ $(k+1\le r\le j-1)$: $g(2r-1,2j-2)$,
(b) $[X_{2k+1},\dots,X_{2r-1},S,U,\dots,U]$ $(k+1\le r\le j-1)$: $g(2r,2j-2)$,
(c) $[U,\dots,U]$: $f(2k+1,2j-2)$.

The link type of $[X_1,\dots,X_{2k-1},T,X_{2k+1},\dots,X_{2j-2},P]$ is as follows, where each link in **(2,a)** and **(3,a)** bounds a disconnected Seifert surface:

	a	**b**	**c**
1	$L(2r, 2j-1)$	$L(2h, 2k, 2r+1, 2j-1)$	$L(2h, 2j-1)$
2		$R(2h+1, 2k)$	$R(2h+1, 2k)$
3		$R(1, 2k)$	$R(1, 2k)$

The derivative at -1 of the sum of the terms in each group is as follows:

	a: $g(2r-1, 2j-2)$	**b**: $g(2r, 2j-2)$	**c**: $f(2k+1, 2j-2)$
1: $g(2h-1, 2k-1)$	E_{24}	E_{25}	E_{26}
2: $g(2h, 2k-1)$	0	E_{27}	E_{28}
3: $f(1, 2k-1)$	0	E_{29}	E_{30}

Each link in **(2,a)** and **(3,a)** bounds a disconnected Seifert surface (see figure 14 for an example of **(2,a)**), so the Alexander polynomial is 0 (cf. [7, Proposition 6.14]), and then the Jones polynomial evaluated at -1 is 0. Hence the derivative is 0.

6.3. We consider the part $J_3(t)$ in Proposition 4.1.

[**3-0**] Picking up the terms without T from $J_3(t)$, we obtain

$$\sum_{j=1}^{n/2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2j-1} X_i(c_i) P(c_{2j}) J_{[X_1,\ldots,X_{2j-1},P]}(t).$$

We divide these terms into three groups by the link type of

$$[X_1, \ldots, X_{2j-1}, P] \qquad (X_i \in \{S, U\}):$$

(1) $[X_1, \ldots, X_{2h-2}, S, U, \ldots, U, P] = O$ $(1 \le h \le j)$,

(2) $[X_1, \ldots, X_{2h-1}, S, U, \ldots, U, P] = O \cup L(2h+1, 2j)$ $(1 \le h \le j-1)$,

(3) $[U, \ldots, U, P] = O \cup L(1, 2j)$.

The sum of the terms in each group and its derivative at -1 are as follows:

(1) $\sum_{j=1}^{n/2} \sum_{h=1}^{j} W(1, 2h-1, 2j-1) P(c_{2j})$. The derivative is F_5.

(2) $\sum_{j=1}^{n/2} \sum_{h=1}^{j-1} W(1, 2h, 2j-1) P(c_{2j}) J_{L(2h+1,2j)}(t) d(t)$. The derivative is E_{31}.

(3) $\sum_{j=1}^{n/2} V(1, 2j-1) P(c_{2j}) J_{L(1,2j)}(t) d(t)$. The derivative is E_{32}.

[**3-1**] Picking up the terms with a single T from $J_3(t)$, we obtain

$$\sum_{j=1}^{n/2} \sum_{l=1}^{2j-1} \sum_{X_i \in \{S,U,\}} \prod_{i=1}^{l-1} X_i(c_i) T(c_l) \cdot \prod_{i=l+1}^{2j-1} X_i(c_i) P(c_{2j}) J_{[X_1,\ldots,X_{l-1},T,X_{l+1},\ldots,X_{2j-1},P]}(t).$$

[3-1-odd] We consider the following terms (in which l is odd).

$$\sum_{j=1}^{n/2}\sum_{k=1}^{j}\sum_{X_i\in\{S,U\}}\prod_{i=1}^{2k-2}X_i(c_i)T(c_{2k-1}) \cdot \prod_{i=2k}^{2j-1}X_i(c_i)P(c_{2j})J_{[X_1,\dots,X_{2k-2},T,X_{2k},\dots,X_{2j-1},P]}(t).$$

We divide these terms into nine groups by the link type of

$$[X_1,\dots,X_{2k-2},T,X_{2k},\dots,X_{2j-1},P] \qquad (X_i\in\{S,U\}):$$

$[X_1,\dots,X_{2k-2}]$ is divided into three groups:

(1) $[X_1,\dots,X_{2h-2},S,U,\dots,U]$ $(1\le h\le k-1)$: $g(2h-1,2k-2)$,
(2) $[X_1,\dots,X_{2h-1},S,U,\dots,U]$ $(1\le h\le k-1)$: $g(2h,2k-2)$,
(3) $[U,\dots,U]$: $f(1,2k-2)$.

$[X_{2k},\dots,X_{2j-1}]$ is divided into three groups:

(a) $[X_{2k},\dots,X_{2r-2},S,U,\dots,U]$ $(k+1\le r\le j)$: $g(2r-1,2j-1)$,
(b) $[X_{2k},\dots,X_{2r-1},S,U,\dots,U]$ $(k\le r\le j-1)$: $g(2r,2j-1)$,
(c) $[U,\cdots,U]$: $f(2k,2j-1)$.

The link type of $[X_1,\dots,X_{2k-1},T,X_{2k+1},\dots,X_{2j-1},P]$ is as follows, where each link in **(1,b)** bounds a disconnected Seifert surface:

	a	**b**	**c**
1	$L(2h,2k-1)$		$L(2h,2k-1)$
2	$R(2h+1,2k-1,2r,2j)$	$R(2r+1,2j)$	$R(2h+1,2j)$
3	$R(1,2k-1,2r,2j)$	$R(2r+1,2j)$	$R(1,2j)$

The derivative at -1 of the sum of the terms in each group is as follows:

	a: $g(2r-1,2j-1)$	**b**: $g(2r,2j-1)$	**c**: $f(2k,2j-1)$
1: $g(2h-1,2k-2)$	E_{33}	0	E_{34}
2: $g(2h,2k-2)$	E_{35}	E_{36}	E_{37}
3: $f(1,2k-2)$	E_{38}	E_{39}	E_{40}

Each link in **(1,b)** bounds a disconnected Seifert surface, so the Alexander polynomial is 0, and then the Jones polynomial evaluated at -1 is 0. Hence the derivative is 0.

[3-1-even] We consider the following terms (in which l is even).

$$\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{X_i\in\{S,U\}}\prod_{i=1}^{2k-1}X_i(c_i)T(c_{2k}) \cdot \prod_{i=2k+1}^{2j-1}X_i(c_i)P(c_{2j})J_{[X_1,\dots,X_{2k-1},T,X_{2k+1},\dots,X_{2j-1},P]}(t).$$

We divide these terms into nine groups by the link type of

$$[X_1,\dots,X_{2k-1},T,X_{2k+1},\dots,X_{2j-1},P] \qquad (X_i\in\{S,U\}):$$

$[X_1,\dots,X_{2k-1}]$ is divided into three groups:

(1) $[X_1,\dots,X_{2h-2},S,U,\dots,U]$ $(1\le h\le k)$: $g(2h-1,2k-1)$,

(2) $[X_1,\dots,X_{2h-1},S,U,\dots,U]$ $(1\le h\le k-1)$: $g(2h,2k-1)$,

(3) $[U,\dots,U]$: $f(1,2k-1)$.

$[X_{2k+1},\dots,X_{2j-1}]$ is divided into three groups:

(a) $[X_{2k+1},\dots,X_{2r-2},S,U,\dots,U]$ $(k+1\le r\le j)$: $g(2r-1,2j-1)$,

(b) $[X_{2k+1},\dots,X_{2r-1},S,U,\dots,U]$ $(k+1\le r\le j-1)$: $g(2r,2j-1)$,

(c) $[U,\cdots,U]$: $f(2k+1,2j-1)$.

The link type of $[X_1,\dots,X_{2k-2},T,X_{2k},\dots,X_{2j-1},P]$ is as follows, where each link in **(2,b)**, **(2,c)**, **(3,b)**, **(3,c)** is a split link which consists of the trivial knot and some link:

	a	**b**	**c**
1	$LL(2h,2k,2r,2j)$	$R(2r+1,2j)$	$R(2k+1,2j)$
2	$R(2h+1,2k)$		
3	$R(1,2k)$		

The derivative at -1 of the sum of the terms in each group is as follows:

	a: $g(2r-1,2j-1)$	**b**: $g(2r,2j-1)$	**c**: $f(2k+1,2j-1)$
1: $g(2h-1,2k-1)$	E_{41}	E_{42}	E_{43}
2: $g(2h,2k-1)$	E_{44}	0	0
3: $f(1,2k-1)$	E_{45}	0	0

Each link in **(2,b)**, **(2,c)**, **(3,b)**, **(3,c)** is a split link which consists of the trivial knot and some link. By using (3d) the derivative of the terms is 0.

6.4. We consider the part $J_4(t)$ in Proposition 4.1.

[4-0] Picking up the terms without T from $J_4(t)$, we obtain

$$\sum_{j=1}^{n/2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2j-2} X_i(c_i) Q(c_{2j-1}) J_{[X_1,\ldots,X_{2j-2},Q]}(t).$$

The link type of $[X_1, \ldots, X_{2j-2}, Q]$, $X_i \in \{S, U\}$, is the 2-component trivial link. So we have

$$\sum_{j=1}^{n/2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2j-2} X_i(c_i) Q(c_{2j-1}) d(t).$$

By using (3a), (2b), and (3f), the derivative at -1 is F_6.

[4-1] Picking up the terms with a single T from $J_4(t)$, we obtain

$$\sum_{j=1}^{n/2} \sum_{l=1}^{2j-2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{l-1} X_i(c_i) T(c_l) \cdot \prod_{i=l+1}^{2j-2} X_i(c_i) Q(c_{2j-1}) J_{[X_1,\ldots,X_{l-1},T,X_{l+1},\ldots,X_{2j-2},Q]}(t).$$

[4-1-odd] We consider the following terms (in which l is odd).

$$\sum_{j=1}^{n/2} \sum_{k=1}^{j-1} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2k-2} X_i(c_i) T(c_{2k-1}) \cdot \prod_{i=2k}^{2j-2} X_i(c_i) Q(c_{2j-1}) J_{[X_1,\ldots,X_{2k-2},T,X_{2k},\ldots,X_{2j-2},Q]}(t).$$

We divide these terms into three groups by the link type of

$$[X_1, \ldots, X_{2k-2}, T, X_{2k}, \ldots, X_{2j-2}, Q] \qquad (X_i \in \{S, U\}):$$

(1) $[X_1, \ldots, X_{2h-2}, S, U, \ldots, U, T, X_{2k}, \ldots, X_{2j-2}, Q]$ $(1 \leq h \leq k-1)$ is a split link which consists of the trivial knot and some link. By using (3d) the derivative of the terms is 0.

(2) $[X_1, \ldots, X_{2h-1}, S, U, \ldots, U, T, X_{2k}, \ldots, X_{2j-2}, Q] = O$ $(1 \leq h \leq k-1)$. So the sum of the terms is

$$\sum_{j=1}^{n/2} \sum_{k=1}^{j-1} \sum_{h=1}^{k-1} W(1, 2h, 2k-2) T(c_{2k-1}) \prod_{i=2k}^{2j-2} (t^{-c_i} + 1 - t^{c_i}) Q(c_{2j-1}).$$

The derivative is F_7.

(3) $[U, \dots, U, T, X_{2k}, \dots, X_{2j-2}, Q] = O$. So the sum of the terms is

$$\sum_{j=1}^{n/2} \sum_{k=1}^{j-1} V(1, 2k-2) T(c_{2k-1}) \prod_{i=2k}^{2j-2} (t^{-c_i} + 1 - t^{c_i}) Q(c_{2j-1}).$$

The derivative is F_8.

[**4-1-even**] We consider the following terms (in which l is even).

$$\sum_{j=1}^{n/2} \sum_{k=1}^{j-1} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2k-1} X_i(c_i) T(c_{2k}) \cdot \prod_{i=2k+1}^{2j-2} X_i(c_i) Q(c_{2j-1}) J_{[X_1, \dots, X_{2k-1}, T, X_{2k+1}, \dots, X_{2j-2}, Q]}(t).$$

We divide these terms into three groups by the link type of

$$[X_1, \dots, X_{2k-1}, T, X_{2k+1}, \dots, X_{2j-2}, Q] \qquad (X_i \in \{S, U\}):$$

(1) $[X_1, \dots, X_{2h-2}, S, U, \dots, U, T, X_{2k+1}, \dots, X_{2j-2}, Q] = O$ $(1 \le h \le k)$. As in **(2)** in [**4-1-odd**], the derivative of the terms is F_9.

(2) $[X_1, \dots, X_{2h-1}, S, U, \dots, U, T, X_{2k+1}, \dots, X_{2j-2}, Q]$ $(1 \le h \le k-1)$ is a split link which consists of the trivial knot and some link. By using (3d) the derivative of the terms is 0.

(3) $[U, \dots, U, T, X_{2k+1}, \dots, X_{2j-2}, Q]$ is a split link which consists of the trivial knot and some link. By using (3d) the derivative of the terms is 0.

6.5. We consider the part $J_5(t)$ in Proposition 4.1.

[**5-0**] Picking up the terms without T from $J_5(t)$, we obtain

$$\sum_{j=1}^{n/2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2j-1} X_i(c_i) Q(c_{2j}) J_{[X_1, \dots, X_{2j-1}, Q]}(t).$$

The link type of $[X_1, \dots, X_{2j-1}, Q]$, $X_i \in \{S, U\}$ is the 2-component trivial link. So we have

$$\sum_{j=1}^{n/2} \sum_{X_i \in \{S,U\}} \prod_{i=1}^{2j-1} X_i(c_i) Q(c_{2j}) d(t).$$

By using (3a), (2b), and (3f), the derivative is F_{10}.

[5-1] Picking up the terms with a single T from $J_5(t)$, we obtain

$$\sum_{j=1}^{n/2}\sum_{l=1}^{2j-1}\sum_{X_i\in\{S,U\}}\prod_{i=1}^{l-1}X_i(c_i)T(c_l) \cdot\prod_{i=l+1}^{2j-1}X_i(c_i)Q(c_{2j})J_{[X_1,\dots,X_{l-1},T,X_{l+1},\dots,X_{2j-1},Q]}(t).$$

[5-1-odd] We consider the following terms (in which l is odd).

$$\sum_{j=1}^{n/2}\sum_{k=1}^{j}\sum_{X_i\in\{S,U\}}\prod_{i=1}^{2k-2}X_i(c_i)T(c_{2k-1}) \cdot\prod_{i=2k}^{2j-1}X_i(c_i)Q(c_{2j})J_{[X_1,\dots,X_{2k-2},T,X_{2k},\dots,X_{2j-1},Q]}(t).$$

We divide these terms into three groups by the link type of

$$[X_1,\dots,X_{2k-2},T,X_{2k},\dots,X_{2j-1},Q] \qquad (X_i\in\{S,U\}):$$

(1) $[X_1,\dots,X_{2h-2},S,U,\dots,U,T,X_{2k},\dots,X_{2j-1},Q]$ $(1\le h\le k-1)$ is a split link which consists of the trivial knot and some link. By using (3d) the derivative of the terms is 0.

(2) $[X_1,\dots,X_{2h-1},S,U,\dots,U,T,X_{2k},\dots,X_{2j-1},Q]=O$ $(1\le h\le k-1)$. The derivative of the terms is F_{11}.

(3) $[U,\dots,U,T,X_{2k},\dots,X_{2j-1},Q]=O$. The derivative of the terms is F_{12}.

[5-1-even] We consider the following terms (in which l is even).

$$\sum_{j=1}^{n/2}\sum_{k=1}^{j-1}\sum_{X_i\in\{S,U\}}\prod_{i=1}^{2k-1}X_i(c_i)T(c_{2k}) \cdot\prod_{i=2k+1}^{2j-2}X_i(c_i)Q(c_{2j-1})J_{[X_1,\dots,X_{2k-1},T,X_{2k+1},\dots,X_{2j-2},Q]}(t).$$

We divide these terms into three groups by the link type of

$$[X_1,\dots,X_{2k-1},T,X_{2k+1},\dots,X_{2j-2},Q] \qquad (X_i\in\{S,U\}):$$

(1) $[X_1,\dots,X_{2h-2},S,U,\dots,U,T,X_{2k+1},\dots,X_{2j-2},Q]=O$ $(1\le h\le k)$. The derivative of the terms is F_{13}.

(2) $[X_1, \dots, X_{2h-1}, S, U, \dots, U, T, X_{2k+1}, \dots, X_{2j-2}, Q]$ $(1 \leq h \leq k-1)$ is a split link which consists of the trivial knot and some link. By using (3d) the derivative of the terms is 0.

(3) $[U, \dots, U, T, X_{2k+1}, \dots, X_{2j-2}, Q]$ is a split link which consists of the trivial knot and some link. By using (3d) the derivative of the terms is 0.

This completes the proof of Proposition 1.14. □

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Appendix: Some properties of the derivative

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Let K, $c_1, \cdots, c_n$, A_i, $A_{i,j}$ and B be as in Proposition 1.14 and Theorem 1.15 in section 1. Then we have

Proposition A. *$A_{i,j}$ is divisible by 16 for any i, j.*

Proposition B. $A_1 = A_2 = \cdots = A_{n+1} = 2\Delta_K(-1) - 2$.

Proposition C. *Moreover, if we suppose that $\Delta_K(-1) = 1$, then*

(i) *$A_{i,j}$ is divisible by 48 for any i, j.*

(ii) *$A_i = 0$ for all i.*

(iii) *B is divisible by 24.*

Proof of Proposition A. By Theorem 1.15 in section 1,

$$\sum_{1 \leq i < j \leq n+1} A_{i,j}\alpha_{i,j} + \sum_{1 \leq i \leq n+1} A_i \alpha_i = \sum_{i=1}^{45} E_i.$$

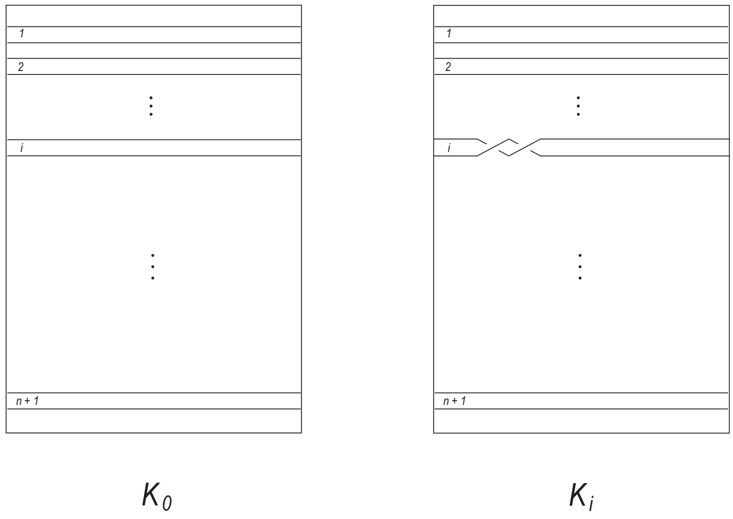

Figure 15

By the explicit formula for E_i in Proposition 1.14 in section 1,

$$\begin{aligned}E_1+E_8 &= 2\sum_{k=1}^{n/2} g(2k-1,n)l(2k,n+1) + (-4)\sum_{k=1}^{n/2} f(1,2k-2)|c_{2k-1}|f(2k,n)l(2k,n+1)\\ &= 4\sum_{k=1}^{n/2}|c_{2k-1}|\prod_{j=2k}^{n}(-1)^{-c_j}l(2k,n+1)\\ &\quad -4\sum_{k=1}^{n/2}\prod_{j=1}^{2k-2}(-1)^{-c_j}|c_{2k-1}|\prod_{j=2k}^{n}(-1)^{-c_j}l(2k,n+1)\\ &= 4\sum_{k=1}^{n/2}\prod_{j=2k}^{n}(-1)^{-c_j}|c_{2k-1}|\{1-\prod_{j=1}^{2k-2}(-1)^{-c_j}\}l(2k,n+1).\end{aligned}$$

By definition of $l(p,q)$, the coefficient of $\alpha_{i,j}$ in $l(p,q)$ has 2 as a factor. Moreover $1-\prod_{j=1}^{2k-2}(-1)^{-c_j}=0$ or 2. Hence the coefficient of $\alpha_{i,j}$ in E_1+E_8 has 16 as a factor. Similarly the coefficient of $\alpha_{i,j}$ in $E_{15}+E_{32}$ has 16 as a factor.

By definition $g(q,r)$ has 2 as a factor and the coefficient of $\alpha_{i,j}$ in $l(p,q)$, $l(p,q,r,s)$ has 2 as a factor. Hence the coefficient of $\alpha_{i,j}$ in E_k $(k\neq 1,8,15,32)$ has 16 as a factor. This completes the proof. □

Proof of Proposition B. Let K_0, K_i $(i=1,\cdots,n+1)$ denote the knot obtained from K as follows: K, K_0, and K_i are identical except for the big rectangle where they are as shown in figure 15. All subbands in the big rectangle of K_0 are untwisted and unlinked. All subbands in the big rectangle of K_i are unlinked. The i-th subband in the big rectangle of K_i is the only one full twisted and the other subbands are untwisted. Then it follows from Theorem 1.15 in section 1 that $J'_{K_0}(-1) = B$, $J'_{K_i}(-1) = B + A_i$. Note that the K_i's are all equivalent. Hence $A_1 = A_2 = \cdots = A_{n+1}$.

By using the skein relation for the Jones polynomial, we have

$$J_{K_0}(t) = t^2 J_{K_1}(t) + (1 - t^2)$$

and

$$J'_{K_0}(-1) = -2J_{K_1}(-1) + J'_{K_1}(-1) + 2.$$

Since the Alexander polynomial of ribbon knot of 1-fusion is determined by $c_1, \dots, c_n$,

$$\Delta_{K_0}(t) = \Delta_{K_1}(t) = \Delta_K(t)$$

and

$$J_{K_1}(-1) = \Delta_{K_1}(-1) = \Delta_K(-1).$$

Hence we have

$$B = -2\Delta_K(-1) + (B + A_1) + 2.$$

That is,

$$A_1 = 2\Delta_K(-1) - 2.$$

□

Proof of Proposition C. (ii) immediately follows from Proposition B. To prove (i) and (iii), we use the following fact that is a consequence to a result in [11].

Let K be a ribbon knot with $\Delta_K(-1) = 1$. Then $J'_K(-1)$ is divisible by 24. (*)

By [11, Theorem 5.1], the Casson invariant of $\Sigma_2(K)$ is equal to $-J'_K(-1)/12$. Since mod 2 reduction of the Casson invariant is equal to the Rochlin invariant ([1]), we have (*).

We return to the proof of Proposition C. Note that K_0 is also a ribbon knot and $\Delta_{K_0}(-1) = \Delta_K(-1) = 1$. Hence by (*) $J'_{K_0}(-1)$ is divisible by 24, and (iii) is proved.

Let $K_{i,j}$ denote the knot obtained from K as follows: $K$and $K_{i,j}$ are identical except for the big rectangle where they are as shown in figure 16. All subbands in the big rectangle of $K_{i,j}$, except the i-th subband and the j-th subband, are untwisted and unlinked. The i-th subband and the j-th subband are linked with relative linking number one.

By Theorem 1.15 in section 1 again, $J'_{K_{i,j}}(-1) = B + A_{i,j}$. By (*), $B + A_{i,j}$ is divisible by 24. Hence by (iii), $A_{i,j}$ is divisible by 24. By Proposition A, $A_{i,j}$ is also divisible by 16. Hence $A_{i,j}$ is divisible by 48. Thus (i) is proved. □

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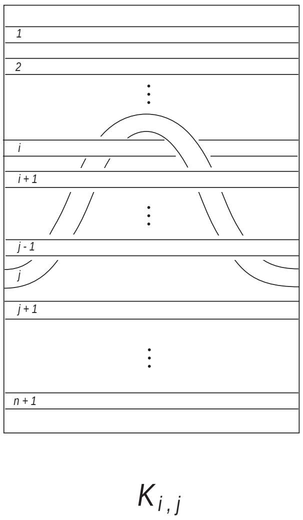

Figure 16

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