

# The Homotopy Type of the Space of Degree 0 — Immersed Plane Curves

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## ABSTRACT

The space  $B_i^0 = \text{Imm}^0(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$  of all immersions of rotation degree 0 in the plane modulo reparameterizations has homotopy groups  $\pi_1(B_i^0) = \mathbb{Z}$ ,  $\pi_2(B_i^0) = \mathbb{Z}$ , and  $\pi_k(B_i^0) = 0$  for  $k \geq 3$ .

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## Introduction

For an immersion  $c : S^1 \rightarrow \mathbb{R}^2$  the (rotation) degree is the winding number of  $c' : S^1 \rightarrow \mathbb{R}^2$  around 0. Let  $\text{Imm}^k(S^1, \mathbb{R}^2)$  denote the connected smooth Fréchet manifold of all immersions of degree  $k$ . It was shown in [4] that for  $k \neq 0$  the space  $\text{Imm}^k(S^1, \mathbb{R}^2)$  contains a copy of  $S^1$  as a smooth strong deformation retract and that the infinite dimensional orbifold  $\text{Imm}^k(S^1, \mathbb{R}^2)/\text{Diff}^+(S^1)$  is contractible, where  $\text{Diff}^+(S^1)$  denotes the regular Fréchet Lie group of orientation preserving diffeomorphisms. The proof in [4] consists in expanding the classical proof of the theorem of Whitney and Graustein (see [1, 2, 8]) into the construction of an  $S^1$ -equivariant smooth deformation retraction. For  $k = 0$  this did not work.

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In this paper we treat the case  $k = 0$ . In section 1 we first give simple argument which shows that  $\pi_1(B^0(S^1, \mathbb{R}^2))$  contains  $\mathbb{Z}$ . In section 2 we give a more involved proof that  $\text{Imm}^0(S^1, \mathbb{R}^2)$  is homotopy equivalent to  $S^1$ . In section 3 we show that factoring out  $\text{Diff}^+(S^1)$  gives a fibration with homotopically trivially embedded fiber, and then the homotopy sequence shows that  $\pi_1(B^{0,+}(S^1, \mathbb{R}^2)) = \mathbb{Z}$ ,  $\pi_2(B^{0,+}(S^1, \mathbb{R}^2)) = \mathbb{Z}$ , and  $\pi_k(B^{0,+}(S^1, \mathbb{R}^2)) = 0$  for  $k > 2$ . Factoring out the larger group  $\text{Diff}(S^1)$  gives a two-sheeted covering and the final result.

### 1. A simple proof that $\mathbb{Z} \subseteq \pi_1(B^0(S^1, \mathbb{R}^2))$

**Proposition 1.1.**  $\text{Imm}^0(S^1, \mathbb{R}^2)/\text{Diff}^+(S^1)$  is not contractible.

*Proof.* We shall view a curve  $c \in \text{Imm}(S^1, \mathbb{R}^2)$  as a  $2\pi$ -periodic plane valued function. A smooth function  $a = a(c, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is called an argument of a curve  $c$  if

$$\frac{c'(\theta)}{|c'(\theta)|} = \exp(i a(\theta));$$

it is unique up to addition of an integer multiple of  $2\pi$ . If the curve  $c$  has degree  $k$  then  $a(\theta + 2\pi) - a(\theta) = 2k\pi$ . Thus, a curve  $c$  is in  $\text{Imm}^0(S^1, \mathbb{R}^2)$  if and only if some (any) argument of  $c$  is  $2\pi$ -periodic. For a curve  $c \in \text{Imm}^0(S^1, \mathbb{R}^2)$ , we define the average argument  $\alpha(c) \in S^1$  by

$$\alpha(c) = \exp\left(\frac{i}{l(c)} \int_0^{2\pi} a(c, \theta) |c'(\theta)| d\theta\right),$$

which does not depend on the choice of  $a(c, \cdot)$  and defines a well-defined smooth mapping  $\alpha : \text{Imm}^0(S^1, \mathbb{R}^2) \rightarrow S^1$ . Also, since any argument  $a$  of a degree 0 curve is  $2\pi$ -periodic,  $\alpha(c)$  is invariant under the action of  $\text{Diff}^+(S^1)$ . So we can view  $\alpha$  as a map

$$\alpha : B^{0,+}(S^1, \mathbb{R}^2) = \text{Imm}^0(S^1, \mathbb{R}^2)/\text{Diff}^+(S^1) \rightarrow S^1.$$

For  $\varphi \in S^1 \subset \mathbb{C} = \mathbb{R}^2$ , the rotation map  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  act on  $B^{0,+}(S^1, \mathbb{R}^2)$  and obviously

$$\alpha(\varphi \cdot c) = \varphi \cdot \alpha(c).$$

So choosing a free orbit  $S^1 \cdot C$  for the rotation action of  $S^1$  on  $B^{0,+}(S^1, \mathbb{R}^2)$ , the composition

$$S^1 \cdot C \hookrightarrow \text{Imm}^0(S^1, \mathbb{R}^2)/\text{Diff}^+(S^1) \xrightarrow{\alpha} S^1$$

equals the identity on  $S^1$ , thus  $\pi_1(S^1) = \mathbb{Z} \subset \pi_1(B^{0,+}(S^1, \mathbb{R}^2))$ .

Moreover,  $\alpha(c(-)) = -\alpha(c)$  implies that  $\alpha$  factors as follows, where the vertical arrows are 2-sheeted coverings:

$$\begin{CD} B^{0,+}(S^1, \mathbb{R}^2) @= \text{Imm}^0(S^1, \mathbb{R}^2)/\text{Diff}^+(S^1) @>\alpha>> S^1 \\ @. @V2VV @VV2V \\ B^0(S^1, \mathbb{R}^2) @= \text{Imm}^0(S^1, \mathbb{R}^2)/\text{Diff}(S^1) @>\bar{\alpha}>> S^1 \end{CD}$$

Thus we also get in a similar way  $\pi_0(S^1) = \mathbb{Z} \subset \pi_0(B^0(S^1, \mathbb{R}^2))$ . □

### 2. The homotopy type of $\text{Imm}^0(S^1, \mathbb{R}^2)$

**Proposition 2.1.** *The space  $\text{Imm}^0(S^1, \mathbb{R}^2)$  of degree 0 immersions in the plane is homotopy equivalent to  $S^1$ .*

*Proof.* This will follow from 2.2–2.5 below. □

**2.2.** Let  $\text{Imm}^{0,*}(S^1, \mathbb{R}^2) := \{c \in \text{Imm}^0(S^1, \mathbb{R}^2) : c(0) = 0\}$ . Clearly we have  $\text{Imm}^0(S^1, \mathbb{R}^2) \cong \text{Imm}^{0,*}(S^1, \mathbb{R}^2) \times \mathbb{R}^2$  and  $\text{Imm}^0(S^1, \mathbb{R}^2) \sim \text{Imm}^{0,*}(S^1, \mathbb{R}^2)$ , where  $\cong$  denotes homeomorphic and  $\sim$  homotopy equivalent. Let us define a map

$$\begin{aligned} \Phi: \text{Imm}^{0,*} &\rightarrow C^\infty(S^1, \mathbb{R}_+) \times C^\infty(S^1, S^1) \\ \Phi(c)(\theta) &= \left( |c_\theta(\theta)|, \frac{c_\theta(\theta)}{|c_\theta(\theta)|} \right) =: (v(\theta), e(\theta)). \end{aligned}$$

The map  $\Phi$  is injective. For  $(v, e) = \Phi(c)$ , the winding number of  $e$  equals the degree 0 of  $c$  and thus  $\int_0^{2\pi} v \cdot e \, d\theta = 0$ .

**Lemma.** *The length of the image of  $e$  is greater than  $\pi$ .*

*Proof.* If not, there exists a number  $r \in \mathbb{R}$  such that

$$\exp(ir) \in \text{Im}(e) \subset \exp(i[r - \pi/2, r + \pi/2]).$$

Then,  $\langle \exp(ir), e(\theta) \rangle$  is nonnegative for any  $\theta$  and strictly positive for some  $\theta$ . Therefore  $\int_0^{2\pi} \langle \exp(ir), v \cdot e \rangle \, d\theta > 0$ . This contradicts

$$\int_0^{2\pi} \langle \exp(ir), v \cdot e \rangle \, d\theta = \left\langle \exp(ir), \int_0^{2\pi} v \cdot e \, d\theta \right\rangle = \langle \exp(ir), 0 \rangle = 0. \quad \square$$

**2.3.** Let us define the set

$$C_{>\pi}^{\infty,0}(S^1, S^1) = \{e \in C^\infty(S^1, S^1) : \text{deg}(e) = 0, \text{length}(\text{Im}(e)) > \pi\}$$

and consider the map

$$\text{pr}_2 \circ \Phi: \text{Imm}^{0,*}(S^1, S^1) \rightarrow C_{>\pi}^{\infty,0}(S^1, S^1),$$

where  $\text{pr}_2$  denotes the second projection.

**Lemma.** *The map  $\text{pr}_2 \circ \Phi: \text{Imm}^{0,*}(S^1, S^1) \rightarrow C_{>\pi}^{\infty,0}(S^1, S^1)$ , is surjective, has contractible fibers, admits a global smooth section, and is a homotopy equivalence.*

*Proof.* For a map  $e \in C_{>\pi}^{\infty,0}(S^1, S^1)$ , there exist points  $\theta_1, \theta_2, \theta_3$  such that  $0 \in \text{int}([e(\theta_1), e(\theta_2), e(\theta_3)])$ , where  $[\cdot, \cdot, \cdot]$  denotes the convex hull of three points. Let  $v_1 \in C^\infty(S^1, \mathbb{R}_{>0})$  be a map such that  $\int_0^{2\pi} v_1 d\theta = 1$  and  $v_1(\theta)$  is close to 0 if  $\theta$  is not close to  $\theta_1$ . Then  $\int_0^{2\pi} v_1 \cdot e d\theta$  is close to  $e(\theta_1)$ . We also define  $v_2$  and  $v_3$  similarly, so that

$$0 \in \text{int} \left( \left[ \int_0^{2\pi} v_1 \cdot e d\theta, \int_0^{2\pi} v_2 \cdot e d\theta, \int_0^{2\pi} v_3 \cdot e d\theta \right] \right).$$

Therefore there exist positive numbers  $a_1, a_2, a_3$  with

$$a_1 \int_0^{2\pi} v_1 \cdot e d\theta + a_2 \int_0^{2\pi} v_2 \cdot e d\theta + a_3 \int_0^{2\pi} v_3 \cdot e d\theta = 0.$$

Define  $c$  by

$$c(\theta) = \int_0^\theta (a_1 v_1(u) + a_2 v_2(u) + a_3 v_3(u)) e(u) du.$$

Then  $c$  is in  $\text{Imm}^{0,*}$  and  $(\text{pr}_2 \circ \Phi)(c) = e$ , which means that  $\text{pr}_2 \circ \Phi$  is surjective.

We next show that for any  $e \in C_{>\pi}^{\infty,0}(S^1, S^1)$ , the inverse image  $(\text{pr}_2 \circ \Phi)^{-1}(e)$  is contractible. Namely, let  $V(e) \subset C^\infty(S^1, \mathbb{R}_+)$  be given by

$$V(e) = \left\{ v \in C^\infty(S^1, \mathbb{R}_+) : \int_0^{2\pi} v \cdot e d\theta = 0 \right\},$$

an open convex subset of the linear subspace  $\{v \in C^\infty(S^1, \mathbb{R}) : \int_0^{2\pi} v \cdot e d\theta = 0\} \subset C^\infty(S^1, \mathbb{R})$ . Thus  $V(e)$  is contractible for each  $e$ . Moreover,  $V(e)$  is homeomorphic to  $(\text{pr}_2 \circ \Phi)^{-1}(e)$  by the map  $\text{pr}_1 \circ \Phi: (\text{pr}_2 \circ \Phi)^{-1}(e) \rightarrow V(e)$ .

For fixed  $\theta_1, \theta_2, \theta_3$  the construction above works for each  $e \in C_{>\pi}^{\infty,0}(S^1, S^1)$  for which 0 is contained in the interior of the convex hull of  $e(\theta_1), e(\theta_2), e(\theta_3)$ ; these  $e$  form an open set in  $C_{>\pi}^{\infty,0}(S^1, S^1)$  on which we get a continuous (even smooth) section of  $\text{pr}_2 \circ \Phi$ . Open sets like that cover  $C_{>\pi}^{\infty,0}(S^1, S^1)$ . So we get smooth local sections whose domains cover the base. Since the base is open in a nuclear Fréchet space, it is smoothly paracompact (see [3, 16.10]) we can use convexity of all fibers and a smooth partition of unity on the base  $C_{>\pi}^{\infty,0}(S^1, S^1)$  to construct a global smooth section  $s$ .

Finally, since all fibers are convex, there is a smooth strong fiber preserving deformation retraction of  $\text{Imm}^{0,*}(S^1, S^1)$  onto the image the global section  $s$ .  $\square$

**2.4.** To study the topology of  $C_{>\pi}^{\infty,0}(S^1, S^1)$ , we introduce the set of  $2\pi$ -periodic functions

$$C^{\infty,p}(\mathbb{R}, \mathbb{R}) = \{c \in C^\infty(\mathbb{R}, \mathbb{R}) : c(\theta + 2\pi) = c(\theta)\}.$$

For  $c \in C^{\infty,p}(\mathbb{R}, \mathbb{R})$  let  $\text{Var}(c) = \max c - \min c$  and let  $\text{Ave}(c) = \frac{1}{2\pi} \int_0^{2\pi} c \, d\theta$ . For  $k \geq 0$ ,  $C^{\infty,p}_{>k}(\mathbb{R}, \mathbb{R}) = \{c \in C^{\infty,p}(\mathbb{R}, \mathbb{R}) : \text{Var}(c) > k\}$ . Define a diffeomorphism  $g : C^{\infty,p}_{>0}(\mathbb{R}, \mathbb{R}) \rightarrow C^{\infty,p}_{>\pi}(\mathbb{R}, \mathbb{R})$  by

$$g(c) = \frac{\text{Var}(c) + \pi}{\text{Var}(c)}(c - \text{Ave}(c)) + \text{Ave}(c).$$

The diffeomorphism  $g$  satisfies  $g(c(\theta + 2n\pi)) = g(c)(\theta + 2n\pi)$ , thus induces the diffeomorphism

$$\tilde{g} : C^{\infty,0}_{>0}(S^1, S^1) \rightarrow C^{\infty,0}_{>\pi}(S^1, S^1),$$

where  $C^{\infty,0}_{>0}(S^1, S^1)$  denotes the set of nonconstant smooth maps of degree 0 in  $C^\infty(S^1, S^1)$ .

**2.5.** We consider now the evaluation  $\text{ev}_1$  at  $1 \in S^1$  whose fiber at  $1 \in S^1$  is the smooth manifold of based smooth loops of degree 0 in  $S^1$ , with the constant loop 1 deleted:

$$C^{\infty,0}((S^1, 1), (S^1, 1)) \setminus \{1\} \hookrightarrow C^{\infty,0}_{>0}(S^1, S^1) \xrightarrow{\text{ev}_1} S^1$$

**Lemma.** *The map  $\text{ev}_1 : C^{\infty,0}_{>0}(S^1, S^1) \rightarrow S^1$  is a smooth trivial fibration with a global section and smoothly contractible fibers. Moreover, it is a homotopy equivalence.*

*Proof.* A smooth section  $s : S^1 \rightarrow C^{\infty,0}_{>0}(S^1, S^1)$  of  $\text{ev}_1$  is given by  $s(\varphi)(\theta) = \varphi \cdot \exp(i \text{Im}(\theta))$ . The fiber of  $\text{ev}_1$  over  $\varphi$  is the space  $C^{\infty,0}((S^1, 1), (S^1, \varphi)) \setminus \{\varphi\}$  consisting of all non-constant smooth loops of degree 0 mapping 1 to  $\varphi$ , which is diffeomorphic to the fiber  $C^{\infty,0}((S^1, 1), (S^1, 1)) \setminus \{1\}$  via multiplication by  $\varphi$ .

It remains to show that the fiber  $C^{\infty,0}((S^1, 1), (S^1, 1)) \setminus \{1\}$  is contractible. Via lifting to the universal cover,  $C^{\infty,0}((S^1, 1), (S^1, 1))$  is diffeomorphic to the space  $\{f \in C^{\infty,p}(\mathbb{R}, \mathbb{R}) : f(0) = 0\}$  of periodic functions mapping 0 to 0. Via Fourier expansion  $f(t) = \sum_{n \in \mathbb{Z}} a_n \exp(int)$  this is isomorphic to the space of all rapidly decreasing complex sequences  $(a_k)_{k \in \mathbb{Z}}$  with  $\bar{a}_k = a_{-k}$  and  $\sum_k a_k = 0$ . This space is isomorphic to the space  $\mathfrak{s}$  of rapidly decreasing sequences  $(b_n)_{n \geq 1}$  by  $a_n = b_n$  for  $n \geq 1$ ,  $a_{-n} = \bar{b}_n$ , and  $a_0 = 2 \text{Re}(\sum_{n \geq 1} b_n)$ .

Now we have to show that this is still contractible if we remove the constant sequence 0. Then it is homotopy equivalent to its intersection with the sphere in  $\ell^2$ , i.e., to the space  $S := \{b \in \mathfrak{s} : \sum_{n \geq 1} b_n^2 = 1\}$ . But this is contractible by a standard argument which is explained on page 513 of [3] for the space of finite sequences. Namely, consider the homotopy  $A : \mathfrak{s} \times [0, 1] \rightarrow \mathfrak{s}$  through isometries which is given by  $A_0 = \text{Id}$  and by

$$A_t(b_1, b_2, b_3, \dots) = (b_1, \dots, b_{n-2}, b_{n-1} \cos \theta_n(t), b_{n-1} \sin \theta_n(t), \\ b_n \cos \theta_n(t), b_n \sin \theta_n(t), b_{n+1} \cos \theta_n(t), b_{n+1} \sin \theta_n(t), \dots)$$

for  $\frac{1}{n+1} \leq t \leq \frac{1}{n}$ , where  $\theta_n(t) = \varphi(n((n+1)t - 1)) \frac{\pi}{2}$  for a fixed smooth function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  which is 0 on  $(-\infty, 0]$ , grows monotonely to 1 in  $[0, 1]$ , and equals 1 on

$[1, \infty)$ . The mapping  $A$  is Lipschitz continuous for each seminorm  $\|b\|_k = \sup\{|b_n|n^k : n \geq 1\}$  of  $\mathfrak{s}$  with constant  $2^k$ , and is isometric for  $\ell^2$ . Then  $A_{1/2}(b_1, b_2, \dots) = (b_1, 0, b_2, 0, \dots)$  is in  $\mathfrak{s}_{\text{odd}}$ , and on the other hand  $A_1(b_1, b_2, \dots) = (0, b_1, 0, b_2, 0, \dots)$  is in  $\mathfrak{s}_{\text{even}}$ . This is a variant of a homotopy constructed by [6]. Now  $A_t|_S$  for  $0 \leq t \leq 1/2$  is a homotopy on  $S$  between the identity and  $A_{1/2}(S) \subset \mathfrak{s}_{\text{odd}}$ . The latter set is contractible, for example in a stereographic chart.  $\square$

**2.6.** If we put together all mappings constructed above we get the following commutative diagram where we indicate isomorphism  $\cong$ , homotopy equivalence  $\sim$ , or 2-sheeted covering  $\twoheadrightarrow$ , and a free orbit  $S^1 \cdot c$  for the rotation action on  $\text{Imm}^0$ :

$$\begin{array}{ccccccc}
 & & S^1 & \xrightarrow{=} & S^1 & \xrightarrow{2} & S^1 \\
 & \nearrow \cong & \uparrow \alpha & & \uparrow \alpha & & \uparrow \bar{\alpha} \\
 S^1 \cdot c & \xrightarrow{c} & \text{Imm}^0 & \twoheadrightarrow & B^{0,+} & \xrightarrow{2} & B^0 \\
 \uparrow = & & \downarrow \sim \tilde{g}_1 \circ \text{pr}_2 \circ \Phi & & & & \\
 S^1 & \xleftarrow{\sim} & C_{>0}^{\infty,0} & & & & 
 \end{array}$$

### 3. The homotopy type of $B^0(S^1, \mathbb{R}^2)$

**Proposition 3.1.** *The mapping  $\text{Imm}^0(S^1, \mathbb{R}^2) \rightarrow B^{0,+}(S^1, \mathbb{R}^2)$  is a (Serre) fibration.*

*Proof.* First we replace  $\text{Imm}^0(S^1, \mathbb{R}^2)$  by the subset  $\text{Imm}_a^0(S^1, \mathbb{R}^2)$  consisting of all immersions which are parametrized by scaled arc-length which is a strong deformation retract, see [4, 2.6]. The normalizer of the  $\text{Diff}^+(S^1)$ -action on it is just the action of  $S^1$  which shifts the initial point. We have to show that for any compactly generated space  $P$  and a homotopy  $h : [0, 1] \times P \rightarrow B^{0,+}$  whose initial value  $h(0, \cdot)$  admits a continuous lift there exists a continuous lift of the whole homotopy:

$$\begin{array}{ccc}
 \{0\} \times P & \xrightarrow{H(0, \cdot)} & \text{Imm}_a^0(S^1, \mathbb{R}^2) \\
 \downarrow c & \nearrow H & \downarrow \\
 [0, 1] \times P & \xrightarrow{h} & B^{0,+}(S^1, \mathbb{R}^2)
 \end{array}$$

To get the lift  $H$  we just have to specify the initial point coherently from  $H(0, p)(1)$  over  $[0, 1] \ni t \mapsto h(t, p)$ .

For that we need a description of the elements in  $B^{0,+}(S^1, \mathbb{R}^2)$ . A point  $C$  in it can be described by the following data:

For some  $n$  and  $i = 1, \dots, n$ , there are open sets  $U_i = U_i(C) \subseteq \mathbb{R}^2$ , smooth functions  $f_i = f_i(C) : U_i \rightarrow \mathbb{R}$  such that  $f_i^{-1}(0) =: C_i$  is a component  $C_i$  of  $C$  with

$\text{grad}(f_i)$  is a unit vector field with flow lines unit speed straight lines passing orthogonally through  $C_i$  in such a way that for  $x \in C_i$  the frame consisting of  $\text{grad}(f_i)(x)$  and the unit tangent to  $C_i$  at  $x$  is positively oriented. The unparameterized smooth oriented 1-manifolds  $C_1, C_2, \dots, C_n$  (in that order) describe  $C$ . Note that there is a choice for the  $U_i$  and their cyclic order, but then the  $f_i$  are unique.

For every  $p \in P$  the initial point  $H(0, p)(1)$  lies in some component  $h(0, p)_i$  of  $h(0, p)$ , and we may move it orthogonally along  $\text{grad}(f_i(h(t, p)))$  to get a coherent choice of initial points. This takes care of the lift  $H$ .  $\square$

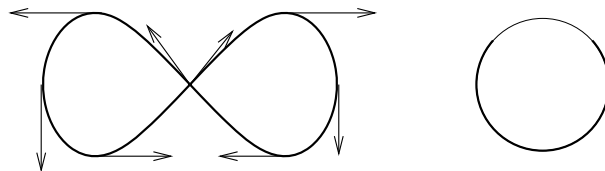
**Lemma 3.2.** *The fiber  $\text{Diff}^+(S^1)$  maps homotopically trivial into the fibration*

$$\text{Imm}^0(S^1, \mathbb{R}^2) \rightarrow B^{0,+}(S^1, \mathbb{R}^2).$$

*Proof.* As in the proof of 3.1 we consider the space  $\text{Imm}_a^0(S^1, \mathbb{R}^2)$  of degree 0 immersions with constant speed parameterizations. Let  $c$  be the unit speed parameterized horizontal figure eight, and consider the diagram where  $c^*(f) = c \circ f$ :

$$\begin{array}{ccccc}
 S^1 & \xrightarrow{c_*} & \text{Imm}_a^0 & & \\
 \sim \downarrow c & & \sim \downarrow c & \searrow & \\
 \text{Diff}^+(S^1) & \xrightarrow{c_*} & \text{Imm}^0 & \twoheadrightarrow & B^{0,+} \\
 & & \sim \downarrow \tilde{g}_{1 \circ \text{pr}_2} \circ \Phi & & \\
 S^1 & \xleftarrow{\sim \text{ev}_1} & C_{>0}^{\infty,0} & & 
 \end{array}$$

We have to show that the mapping from the upper left  $S^1$  to the lower left  $S^1$  is nullhomotopic. It is essentially (suppressing  $\tilde{g}^{-1}$ ) given by  $\beta \mapsto \frac{c'(\beta)}{|c'(\beta)|}$ . From the figure



we see that this mapping covers everything below the northern polar region twice and avoids the northern polar region, so it is nullhomotopic.  $\square$

**Corollary 3.3.** *We have the following homotopy groups:*

$$\begin{array}{ll}
 \pi_1(B^{0,+}(S^1, \mathbb{R}^2)) = \mathbb{Z}, & \pi_1(B^0(S^1, \mathbb{R}^2)) = \mathbb{Z}, \\
 \pi_2(B^{0,+}(S^1, \mathbb{R}^2)) = \mathbb{Z}, & \pi_2(B^0(S^1, \mathbb{R}^2)) = \mathbb{Z}, \\
 \pi_k(B^{0,+}(S^1, \mathbb{R}^2)) = 0, & \pi_k(B^0(S^1, \mathbb{R}^2)) = 0 \quad \text{for } k > 2.
 \end{array}$$

*Proof.* By 3.1 we have the long exact homotopy sequence

$$\cdots \rightarrow \pi_k(S^1) \xrightarrow{0} \pi_k(\text{Imm}_a^0) \rightarrow \pi_k(B^{0,+}) \rightarrow \pi_{k-1}(S^1) \rightarrow \cdots$$

and by section 2 the space  $\text{Imm}_a^0$  is homotopy equivalent to  $S^1$ . This gives the homotopy groups of  $B^{0,+}(S^1, \mathbb{R}^2)$ . Since  $B^{0,+}(S^1, \mathbb{R}^2) \rightarrow B^0(S^1, \mathbb{R}^2)$  is a two-sheeted covering, we can also read off the homotopy groups of  $B^{0,+}(S^1, \mathbb{R}^2)$ .  $\square$

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## References

- [1] W. C. Graustein, *A new form of the four-vertex theorem*, Mh. Math. Physics **43** (1936), 381–384.
- [2] ———, *Extensions of the four-vertex theorem*, Trans. Amer. Math. Soc. **41** (1937), no. 1, 9–23.
- [3] A. Kriegl and P. W. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence, RI, 1997.
- [4] P. W. Michor and D. Mumford, *Riemannian geometries on spaces of plane curves*, J. Eur. Math. Soc. **8** (2006), 1–48.
- [5] ———, *Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms*, Doc. Math. **10** (2005), 217–245 (electronic), available at [arXiv:math.DG/0409303](https://arxiv.org/abs/math/0409303).
- [6] T. R. Ramadas, *On the space of maps inducing isomorphic connections*, Ann. Inst. Fourier (Grenoble) **32** (1982), no. 1, viii, 263–276.
- [7] G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York, 1978.
- [8] H. Whitney, *Regular families of curves*. I, Proc. Nat. Acad. Sci USA **18** (1932), 275–278; II, Proc. Nat. Acad. Sci USA **18** (1932), 340–342.