Berezin and Berezin-Toeplitz Quantizations for General Function Spaces

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ABSTRACT

The standard Berezin and Berezin-Toeplitz quantizations on a Kähler manifold are based on operator symbols and on Toeplitz operators, respectively, on weighted L^2-spaces of holomorphic functions (weighted Bergman spaces). In both cases, the construction basically uses only the fact that these spaces have a reproducing kernel. We explore the possibilities of using other function spaces with reproducing kernels instead, such as L^2-spaces of harmonic functions, Sobolev spaces, Sobolev spaces of holomorphic functions, and so on. Both positive and negative results are obtained.

Key words: Berezin quantization, Berezin-Toeplitz quantization, star product, harmonic Bergman space, Sobolev-Bergman space, reproducing kernel.

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Introduction

Let (Ω, ω) be a symplectic manifold, $C^\infty(\Omega)$ the usual space of smooth complex valued functions on Ω, and $C^\infty(\Omega)[[h]]$ the ring of all power series with $C^\infty(\Omega)$ coefficients in a formal parameter h. Recall that a *star-product* on Ω is a $\mathbf{C}[[h]]$-bilinear mapping $* : C^\infty(\Omega)[[h]] \times C^\infty(\Omega)[[h]] \to C^\infty(\Omega)[[h]]$ such that:

(i) $*$ is associative;

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(ii) there exist bilinear operators $C_j : C^\infty(\Omega) \times C^\infty(\Omega) \to C^\infty(\Omega)$ ($j = 0, 1, \dots$) such that $\forall f, g \in C^\infty(\Omega)$,

$$f * g = \sum_{j=0}^{\infty} C_j(f,g) h^j; \tag{1}$$

(iii) the operators C_j satisfy

$$C_0(f,g) = fg, \tag{2}$$

$$C_1(f,g) - C_1(g,f) = \frac{i}{2\pi}\{f,g\} \tag{3}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket with respect to ω, and

$$C_j(\mathbf{1},\cdot) = C_j(\cdot,\mathbf{1}) = 0 \qquad \forall j \geq 1. \tag{4}$$

Note that the last requirement means precisely that $\mathbf{1}$ is the identity element for $*$.

A star-product is called *differential* if C_j are bidifferential operators, i.e., in terms of local coordinates can be expressed as finite sums

$$C_j(f,g) = \sum_{\alpha,\beta \text{ multiindices}} c_{j\alpha\beta} \cdot (D^\alpha f) \cdot (D^\beta g) \qquad \Big(D^\alpha := \frac{\partial^{\alpha_1+\dots+\alpha_m}}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_m}}\Big), \tag{5}$$

with some coefficient functions $c_{j\alpha\beta}$ (which must then belong to $C^\infty(\Omega)$). Two star-products $*$, $*'$ are called *equivalent* if there exists a formal power series of operators $M = I + M_1 h + M_2 h^2 + \cdots$ such that $Mf *' Mg = M(f * g)$. Finally, if Ω has complex structure, a star-product is said to have the property of *separation of variables* (or to be of *anti-Wick type*) if $f * g = fg$ at every point in a neighborhood of which either f or $\bar g$ is holomorphic; and to have the property of the separation of variables *in the reverse order* (or to be of *Wick type*) if $f * g = fg$ at every point in a neighborhood of which either $\bar f$ or g is holomorphic. In other words, a star product is of Wick type if the operators (5) contain only holomorphic derivatives of f and anti-holomorphic derivatives of g, and of anti-Wick type if they contain only anti-holomorphic derivatives of f and holomorphic derivatives of g.

Due to a role which they play in Hochschild cohomology, the bilinear operators C_j are sometimes called *cochains*; we will occasionally use this terminology too.

Construction of star-products is the subject of *deformation quantization*. In contrast to the conventional schemes, in which one assigns to real-valued functions $f \in C^\infty(\Omega)$ (the classical observables) self-adjoint operators Q_f on some fixed Hilbert space (the quantum observables) in some appropriate way, in deformation quantization one instead defines the spectrum of an element $f \in C^\infty(\Omega)[[h]]$ using the so-called star-exponential, and then uses it as a substitute for the spectrum of the operators quantum Q_f; see, e.g., the recent survey [1] for more information.

On the algebraic (i.e., formal power series) level, the question of existence of star products and their classification has been resolved completely: they exist on any symplectic (even Poisson) manifold, and their classes up to equivalence are classified by formal power series with coefficients in the second de Rham cohomology group $H^2(\Omega, \mathbf{R})$. See, e.g., the recent survey [19] for further details. Classification of star products with separation of variables is also available [20].

However, it has also turned out that equivalent (hence, mathematically isomorphic) star-products may lead to different spectra for the same observable. For this reason, there has been increased interest in various particular star-products which are "more canonical" than others, in the sense that they are in some way related to some analytic or geometric objects on Ω. For Kähler manifolds, such examples are furnished by the *Berezin* and the *Berezin-Toeplitz* star-products.

Let us briefly recall the definition of these. (More details can be found in section 1 below). Let Φ be a real-valued Kähler potential for ω (so that $\omega = \frac{i}{2}\partial\bar\partial\Phi$). Assume, for simplicity, that Φ exists globally, and for each $h > 0$ let $L^2_{h,\mathrm{hol}}$ be the *weighted Bergman space* of all holomorphic functions on Ω with respect to the measure $d\mu_h(x) := e^{-\Phi(x)/h}\, d\mu(x)$, where $\mu = \bigwedge^d \omega$ ($d = \dim_{\mathbf{C}} \Omega$) is the Riemannian volume element on Ω. Denote by $K^{(h)}(x,y)$ the reproducing kernel of $L^2_{h,\mathrm{hol}}$, and for each bounded linear operator T on $L^2_{h,\mathrm{hol}}$ define its *symbol* $\widetilde{T}$ by

$$\widetilde{T}(x) := \frac{\langle TK^{(h)}(\cdot,x), K^{(h)}(\cdot,x)\rangle}{K^{(h)}(x,x)}.$$

It can be shown that the correspondence $T \leftrightarrow \widetilde{T}$ is one-to-one; thus one can transfer the operator multiplication into an associative product $*_h$ on functions by declaring that

$$\widetilde{T} *_h \widetilde{S} = \widetilde{TS}, \qquad \forall\, T, S \in \mathcal{B}(L^2_{h,\mathrm{hol}}).$$

Consider now functions f on $\Omega \times \mathbf{R}_+$ such that $f(\cdot,h) = \widetilde{T_h}$ for some $T_h \in \mathcal{B}(L^2_{h,\mathrm{hol}})$, for each h; and define the multiplication of two such functions by $(f * g)(\cdot,h) := f(\cdot,h) *_h g(\cdot,h)$. If it happens that the product $*$ so defined can actually be extended to all of $C^\infty(\Omega)[[h]]$ and satisfies (2)–(4), then it defines a differential star-product with separation of variables. That's the *Berezin* star-product.

The definition of the *Berezin-Toeplitz* star-product runs as follows. Keeping the above notation, one defines for each $f \in L^\infty(\Omega)$ the associated *Toeplitz operator* T^h_f on $L^2_{h,\mathrm{hol}}$ by the formula

$$T^h_f \phi := P_h(f\phi), \qquad \phi \in L^2_{h,\mathrm{hol}},$$

where P_h is the orthogonal projection in $L^2(\Omega, d\mu_h)$ onto its holomorphic subspace $L^2_{h,\mathrm{hol}}$. If it happens that

$$\lim_{h\to 0} \widetilde{T^h_f}(x) = 0 \ \forall x \quad \text{implies} \quad f = 0,$$

and that the product of two Toeplitz operators admits an asymptotic expansion

$$T_f^h T_g^h = \sum_{j=0}^{\infty} h^j T^h_{C_j(f,g)},$$

in the sense that

$$\Big\| T_f^h T_g^h - \sum_{j=0}^{N} h^j T^h_{C_j(f,g)} \Big\| = O(h^{N+1}) \qquad \text{as } h \to 0,\ \forall N = 0,1,2,\dots, \tag{6}$$

for all $f, g \in C^\infty(\Omega)$ with compact support, with some bidifferential operators C_j (independent of f and g) satisfying (2)–(4), then these operators C_j define, via the recipe (1), a differential star-product with separation of variables in the reverse order — the *Berezin-Toeplitz* star-product.

We see that both these star-products originate from the Bergman spaces — i.e., spaces of square-integrable holomorphic functions. It seems therefore very natural to ask if other function spaces with reproducing kernels can be used instead. In particular, it would be very nice to have such spaces which, unlike the spaces of holomorphic functions, do not require that Ω have complex structure, so that the quantization procedures are applicable not only to Kähler manifolds but to an arbitrary symplectic manifold.

For instance, instead of spaces of holomorphic functions, one might try spaces of square-integrable *harmonic* functions, i.e., the subspaces L^2_{harm} of all harmonic functions in L^2. More generally, one can even consider the subspace L^2_A of all functions annihilated by a given (fixed) hypoelliptic partial differential operator A. [Recall that a linear partial differential operator A is called *hypoelliptic* if every distribution annihilated by A is automatically C^∞; examples of hypoelliptic operators are $\bar\partial$ (holomorphic functions), the Laplace operator Δ (harmonic functions), $\frac{\partial}{\partial t} - \Delta$ (caloric functions), or elliptic operators.] Then the value of a function $f \in L^2_A$ at a point makes sense, and usually depends continuously on f, so there exists a reproducing kernel $K_A(x,y)$. Hence one can define operator symbols, as well as the Toeplitz operators, and ask about their usability for quantization.

Another example are the Sobolev spaces. These are defined, for a nonnegative integer order s, by

$$H^s(\Omega) := \{\, f : D^\alpha f \in L^2(\Omega)\ \forall |\alpha| \le s \,\},$$

for negative integers s by duality, and for $s \in \mathbf{R} \setminus \mathbf{Z}$ by interpolation [25]. By the Sobolev theorem, $H^s(\Omega)$ has a reproducing kernel if $s > \frac{n}{2}$. Consequently, in this case we can still define the operator symbols as before. (However, it is no longer possible to define Toeplitz operators.)

One can also combine the ideas in the last two paragraphs, and consider the subspaces H^s_A of functions in the Sobolev spaces H^s annihilated by a given (hypo)elliptic linear partial differential operator A (for instance, Sobolev spaces of holomorphic or harmonic functions). These spaces have reproducing kernels even for all real s [24]. This time we can again define the operator symbols without problems, and the Toeplitz operators T_f can be defined at least for those f which are multipliers of the spaces H^s_A into the corresponding spaces H^s (the role of the Bergman projections P_h being taken over by the orthogonal projections of H^s onto H^s_A).

Can one make either the Berezin or the Berezin-Toeplitz quantization still work in any of the above-mentioned settings?

The aim of the present paper is to give answers to this question. After reviewing some preliminaries in section 1, we begin with the negative ones by showing in section 2 that the Berezin quantization based on the harmonic Bergman spaces L^2_{harm} does not work. (Hence it cannot work for a general L^2_A either, unless the hypoelliptic operator A is subject to some conditions.) Similarly, the Berezin quantization based on the Sobolev spaces $H^s(\Omega)$ breaks down. Negative answers continue in section 3, where we prove that also the Berezin-Toeplitz quantization based on the harmonic Bergman spaces does not work, even for the simplest case of the harmonic analogues of the standard weighted Bergman spaces on the unit disc. This also means that the corresponding quantizations based on the harmonic Sobolev spaces H^s_{harm} cannot work (since they do not work already for $s = 0$), and probably indicates that it is not very reasonable to expect them to work for the spaces L^2_A with other hypoelliptic operators A more general than $\bar\partial$.

Finally, results in the positive direction begin to appear in section 4, where it is shown that the Berezin quantization works on holomorphic Sobolev spaces, and even on certain more general Hilbert spaces of holomorphic functions (see there for the precise statements). Our method is based on representing the inner product in these spaces as a deformation of the ordinary Bergman space product, and then deducing the results from those for the Bergman spaces. Some concrete examples are also given.

The remaining case of the Berezin-Toeplitz quantization on holomorphic Sobolev spaces is discussed in section 5, where it is shown to work for the Sobolev analogues of the Fock space on the complex plane. Whether it works also in more general situations is a question which remains open as of this writing.

The final section 6 contains a small table conveniently summarizing our findings, and a few concluding comments.

Some of the results of the present paper were announced on the 3rd Conference on Contemporary Problems in Mathematical Physics (COPROMAPH3) in Cotonou, Benin, in November 2003 [13].

Notation. Throughout the paper, we use the usual notational conventions concerning multiindices, e.g., $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ or $\alpha! = \alpha_1! \cdots \alpha_d!$ if $\alpha = (\alpha_1, \dots, \alpha_d)$. The symbol ∂ will denote the operator of holomorphic differentiation, that is, $\partial^\alpha = \partial^{|\alpha|}/\partial z_1^{\alpha_1} \cdots \partial z_d^{\alpha_d}$ for $z \in \mathbf{C}^d$; similarly for $\bar\partial$. Abusing notation slightly, we will sometimes also use ∂ for the holomorphic part of the exterior derivative on differential forms. Similarly, dz will denote the Lebesgue measure, but in a few instances will also stand for the differential 1-form. The symbol D^α will denote partial derivative with respect to real variables.

Finally, we will use a somewhat nonstandard definition of the Laplace operator on $\mathbf{C}^d$,

$$\Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$$

which differs from the usual one by a factor of 4.

1. Hilbert spaces with reproducing kernels

We begin by reviewing in more detail the constructions of the Berezin and the Berezin-Toeplitz quantizations, sketched above, so as to make it clearer which properties of the Bergman spaces are really needed to make things work.

1.1. Reproducing kernels

Consider, quite generally, a Hilbert space H whose elements are functions on some set $\mathfrak{S}$. Assume that

$$\forall y \in \mathfrak{S}, \text{ the evaluation functional } f \mapsto f(y) \text{ is continuous on } H. \tag{7}$$

Then, by the Riesz-Fischer theorem, there exist $K_y \in H$ such that

$$f(y) = \langle f, K_y \rangle_H \qquad \forall f \in H.$$

The function $K(x, y) := \langle K_y, K_x \rangle_H = K_y(x)$ is called the *reproducing kernel* of H. It satisfies $K(y, x) = \overline{K(x, y)}$, and $K(x, x) = \|K_x\|^2 \geq 0$ (with equality occurring if and only if $f(x) = 0 \; \forall f \in H$).

Note that we have not made any assumptions whatsoever on the set $\mathfrak{S}$ or on the functions in H. However, if it happens that $\mathfrak{S}$ is not only a set but a (smooth, complex, ...) manifold and the elements of H are continuous (C^∞, holomorphic, ...), then the inclusion $K_y \in H$ and the equality $K(y, x) = \overline{K(x, y)}$ imply that $K(x, \bar{y})$ is also continuous (C^∞, holomorphic) in each variable. In particular, if $\mathfrak{S}$ is a complex manifold and the elements of H are holomorphic functions, then $K(x, y)$ will be holomorphic in x and $\bar{y}$.

Let $\mathcal{B}(H)$ denote the algebra of all (bounded linear) operators on H. Then for any $T \in \mathcal{B}(H)$, we can write

$$Tf(x) = \langle Tf, K_x \rangle = \langle f, T^* K_x \rangle.$$

Thus T is uniquely determined by the function $T^* K_x(y)$ on $\mathfrak{S} \times \mathfrak{S}$; hence, also by the function

$$T(x, y) := \overline{T^* K_x(y)} / \overline{K_x(y)} = \frac{\langle T K_y, K_x \rangle}{\langle K_y, K_x \rangle} = \frac{T K_y(x)}{K(x, y)}.$$

defined at all points where $K(x,y) \neq 0$.

It follows from the definition that the mapping $T \mapsto T(x,y)$ is linear, $T^*(x,y) = \overline{T(y,x)}$, and $I(\cdot,\cdot) = \mathbf{1}$, where I is the identity operator and $\mathbf{1}$ the function constant one. Also, if the elements of H are continuous (smooth, holomorphic), then so is $T(x,\bar{y})$ in each variable, at all points where $K(x,\bar{y}) \neq 0$.

A function $g : \mathfrak{S} \to \mathbf{C}$ is called a *multiplier* of H if the operator $M_g : f \mapsto gf$ of multiplication by g maps H into itself. By the Closed Graph Theorem, M_g is then a bounded linear operator on H. The set of all multipliers of H is an algebra, which will be denoted by $\mathcal{M}_H$. Clearly, for each $g \in \mathcal{M}_H$,

$$M_g(x,y) = g(x) \qquad \forall x,y \in \mathfrak{S}. \tag{8}$$

For any $g \in \mathcal{M}_H$, we further have the important relation

$$M_g^* K_x = \overline{g(x)} K_x \qquad \forall x \in \mathfrak{S} \tag{9}$$

(i.e., each K_x is an eigenvector of M_g^*). Indeed, by the reproducing property of K_x,

$$\langle f, M_g^* K_x\rangle = \langle M_g f, K_x\rangle = \langle gf, K_x\rangle = g(x)f(x) = g(x)\langle f, K_x\rangle$$

for any $f \in H$, and the assertion follows.

1.2. Operator symbols

Assume now that, in fact,

$$\text{the functions } T(x,y) \text{ are uniquely determined by their restrictions } T(x,x) \text{ to the diagonal.} \tag{10}$$

This is the case, for instance, whenever the functions $T(x,y)$ are holomorphic in x and $\bar{y}$, by a well-known theorem in complex analysis. Then also each operator $T \in \mathcal{B}(H)$ is uniquely determined by the said restriction, which we denote by

$$\widetilde{T}(x) := T(x,x) = \frac{\langle TK_x, K_x\rangle}{K(x,x)} = \frac{TK_x(x)}{K(x,x)}. \tag{11}$$

One calls $\widetilde{T}$ the *symbol* of the operator T. The mapping $T \mapsto \widetilde{T}$ is linear, $\widetilde{T^*} = \overline{\widetilde{T}}$ and $\tilde{I} = \mathbf{1}$. Further, by the Schwarz inequality we have $|\widetilde{T}(x)| \leq \|T\|$, hence $\widetilde{T}$ is actually a bounded function on $\mathfrak{S}$. Finally, if the elements of H are holomorphic functions, then $\widetilde{T}$ is C^ω (:=real-analytic).

Of course, for (11) to be defined it is necessary that

$$K(x,x) > 0 \quad \forall x, \qquad \text{i.e.,} \quad K_x \neq 0 \quad \forall x, \tag{12}$$

which we will *assume to be fulfilled* from now on.

Now let
$$\mathcal{A}_H := \{\widetilde{T} : \ T \in \mathcal{B}(H)\}$$
be the vector space of symbols of all bounded linear operators on H (which is a space of bounded functions on $\mathfrak{S}$ by the last remark), and let us define a multiplication $*_H$ on $\mathcal{A}_H$ by
$$\widetilde{T} *_H \widetilde{S} := \widetilde{TS}.$$
Note that this makes sense since the correspondence $T \leftrightarrow \widetilde{T}$ is 1-to-1 by assumption. Then $(\mathcal{A}_H, *_H)$ is a (noncommutative, associative) algebra of bounded functions on Ω. (It is isomorphic to the algebra $\mathcal{B}(H)$ of all bounded linear operators on H.)

Observe that if g is a multiplier of H, then by (8) $\widetilde{M_g} = g$. Thus for any $T \in \mathcal{B}(H)$
$$\begin{aligned}(g *_H \widetilde{T})(x) &= (\widetilde{M_g} *_H \widetilde{T})(x) = \widetilde{M_g T}(x) \\ &= K(x,x)^{-1}(M_g T K_x)(x) \\ &= K(x,x)^{-1} g(x)(TK_x)(x) \\ &= g(x)\widetilde{T}(x). \end{aligned} \tag{13}$$
Consequently, left $*_H$-multiplication by g coincides with the pointwise multiplication, for any $g \in \mathcal{M}_H$.

Similarly, by (9), $\widetilde{M_g^*} = \bar{g}$, and thus
$$\begin{aligned}(\widetilde{T} *_H \bar{g})(x) &= (\widetilde{T} *_H \widetilde{M_g^*})(x) = \widetilde{TM_g^*}(x) \\ &= K(x,x)^{-1}\langle TM_g^* K_x, K_x\rangle \\ &= K(x,x)^{-1}\langle T\overline{g(x)} K_x, K_x\rangle \\ &= \overline{g(x)}\widetilde{T}(x), \end{aligned} \tag{14}$$
so the right $*_H$-multiplication by $\bar{g}$ coincides with pointwise multiplication.

Note that (14) can also be obtained from (13) — or vice versa — by means of the following symmetry property of the $*_H$-multiplication,
$$g *_H f = \overline{\bar{f} *_H \bar{g}} \qquad \forall f, g \in \mathcal{A}_H, \tag{15}$$
which is a consequence of the fact that $ST = (T^* S^*)^*$.

1.3. Berezin quantization

Let us now return to our Kähler manifold Ω from the Introduction. As before let Φ be a real-valued potential for the Kähler form ω; let us assume, for simplicity, that Φ exists globally (otherwise one just has to replace functions by sections of certain line bundles), and let us apply the construction from §1.2 to the weighted Bergman space $H = L^2_{h,\mathrm{hol}} = L^2_{\mathrm{hol}}(\Omega, d\mu_h)$. The mean-value theorem for holomorphic functions implies that (7) is fulfilled, hence the space has a reproducing kernel $K^{(h)}(x,y)$. Since the elements of $L^2_{h,\mathrm{hol}}$ are holomorphic functions, (10) is also satisfied in view of the remarks above, and thus we can introduce the algebras $\mathcal{A}_H =: \mathcal{A}_h$ with the products $*_H =: *_h$. For each h, the elements of $\mathcal{A}_h$ are bounded real-analytic functions on Ω. Further, by (13) and (14), the product $f *_h g$ reduces to the pointwise product fg whenever either f or $\bar{g}$ is a bounded holomorphic function.

The idea of the *Berezin quantization* is now to glue together the noncommutative products $*_h$ in such a way as to obtain a star-product.

More precisely, consider the direct sum $(\mathcal{A}, *)$ of all the algebras $(\mathcal{A}_h, *_h)$, $h > 0$.

Definition. We will say that an element $f = \{f_h(x)\}_h \in \mathcal{A}$, or more generally any family of functions $f_h \in C^\infty(\Omega)$ indexed by $h > 0$, *admits an asymptotic expansion in nonnegative powers of* h *as* $h \searrow 0$ (or just *admits an asymptotic expansion as* $h \searrow 0$ for short) if

$$f_h(x) = \sum_{j=0}^{\infty} h^j f_j(x) \qquad \text{as } h \to 0 \tag{16}$$

with some $f_j \in C^\infty(\Omega)$.

Definition. We will say that a linear subset $\mathcal{A}_0 \subset \mathcal{A}$ is *total* if for any $m > 0$, $x \in \Omega$ and complex numbers $c_{j\alpha}$, there exists $f \in \mathcal{A}_0$ having the asymptotic expansion (16) and such that $D^\alpha f_j(x) = c_{j\alpha}$ for all multiindices $|\alpha| \le m$ and $j = 0, 1, \dots, m$.

Suppose now that we can find a linear subset $\mathcal{A}_0$ of $\mathcal{A}$ such that

- each $f \in \mathcal{A}_0$ admits an asymptotic expansion as $h \searrow 0$;
- $\mathcal{A}_0$ is total;
- for any $f, g \in \mathcal{A}_0$ and $x \in \Omega$, the product $f * g$ admits an asymptotic expansion as $h \searrow 0$

$$(f * g)_h(x) = \sum_{i,j,k \ge 0} h^{i+j+k} C_k(f_i, g_j)(x) \qquad \text{as } h \to 0, \tag{17}$$

  where $C_k : C^\infty(\Omega) \times C^\infty(\Omega) \to C^\infty(\Omega)$ are bidifferential operators satisfying (2) and (3).

Then C_k define, via the recipe (1), a differential star-product on Ω. This is the *Berezin star product* (Berezin [3], Engliš [11]).

Remark. Note that owing to the second condition, the bidifferential operators C_j in (17) are uniquely determined. (This was precisely the motivation behind the definition of a total set.)

Finally, since $f * g = fg$ whenever either f or $\bar{g}$ is bounded holomorphic, this star-product usually has the property of separation of variables. Namely, suppose that Ω is such that

$$\begin{aligned}&\text{for any } m \geq 0,\, x_0 \in \Omega \text{ and complex numbers } c_\alpha,\, |\alpha| \leq m, \text{ there exists}\\&\text{a bounded holomorphic function } f \in H^\infty(\Omega) \text{ such that } \partial^\alpha f(x_0) = c_\alpha\\&\text{for all multiindices } \alpha \text{ with } |\alpha| \leq m.\end{aligned} \tag{18}$$

Since we know that $C_k(f,g) = 0\ \forall k \geq 1$ whenever $f \in H^\infty$, and since each C_k, being a differential operator, involves only finitely many derivatives of f, it then follows that $C_k(f,g) = 0\ \forall k \geq 1$ even for any holomorphic f (not necessarily bounded). Similarly for $C_k(f,g)$ with $\bar{g}$ holomorphic. Thus the star-product has the property of separation of variables.

Observe that the condition (18) is obviously fulfilled, e.g., whenever Ω is a bounded domain in $\mathbf{C}^d$, since then the polynomials are contained in $H^\infty(\Omega)$.

Motivated by the condition (18), we make one more definition.

Definition. We will say that a family H_h of reproducing kernel Hilbert spaces on Ω, indexed by $h > 0$, has *sufficiently many holomorphic multipliers* if for each $x_0 \in \Omega$, $m \geq 0$ and complex numbers c_α, $|\alpha| \geq m$, there exists a holomorphic function f such that $\partial^\alpha f(x_0) = c_\alpha$ for all multiindices α with $|\alpha| \leq m$, and f is a multiplier of H_h for all $h > 0$.

As the argument above shows, a star product arising from any such family H_h will automatically have the property of separation of variables.

In situations when there are not sufficiently many holomorphic multipliers, one has to check the property of separation of variables directly on a case-by-case basis.

Remark. Note that in view of (16), the Berezin star-product inherits the symmetry property

$$g * f = \overline{\bar{f} * \bar{g}},$$

i.e., $C_j(g,f) = \overline{C_j(\bar{f},\bar{g})}$ for any j.

1.4. Toeplitz operators

Let us now come back to the end of §1.1, and continue by introducing instead of (10) the assumption

$$\begin{aligned}&\text{the space } H \text{ is a closed subspace of a larger Hilbert space } K,\\&\text{whose elements are also functions on } \mathfrak{S}.\end{aligned} \tag{19}$$

Let $\mathcal{M}(H,K) \equiv \mathcal{M}$ be the set of all *multipliers* of H into K, i.e., functions m on $\mathfrak{S}$ such that $mf \in K\ \forall f \in H$. No matter what H and K are, $\mathcal{M}(H,K)$ is always a vector space containing the constant functions, as well as the subspaces $\mathcal{M}(H,H) \equiv \mathcal{M}_H$ and $\mathcal{M}(K,K) \equiv \mathcal{M}_K$ of all self-multipliers of H and K, respectively. (In fact, $\mathcal{M}_H$

and $\mathcal{M}_K$ are algebras, under pointwise multiplication, and $\mathcal{M}(H,K)$ is a module over both $\mathcal{M}_H$ and $\mathcal{M}_K$.) For $m \in \mathcal{M}$, the associated *Toeplitz operator* T_m on H is defined by

$$T_m f = P(mf),$$

where $P : K \to H$ is the orthogonal projection. Explicitly,

$$T_m f(x) = \langle mf, K_x \rangle_K.$$

Clearly, T_m depends linearly on m, and $T_g = M_g$ if g is a multiplier of H; in particular, $T_{\mathbf{1}} = I$ and

$$T_m T_g = T_{mg} \qquad \text{whenever } g \in \mathcal{M}_H.$$

(However, $T_{m_1} T_{m_2} \neq T_{m_1 m_2}$ and $\neq T_{m_1} T_{m_2}$ for general $m_1, m_2 \in \mathcal{M}$.) Also, if the multiplier m is bounded as operator from H into K (with the norm denoted by $\|m\|_{\mathcal{M}}$), then $\|T_m\| \leq \|m\|_{\mathcal{M}}$.

Finally, if in addition

$$K = L^2(\mathfrak{S}, d\nu) \tag{20}$$

for some measure ν on $\mathfrak{S}$, then $m \in \mathcal{M}$ implies $\bar{m} \in \mathcal{M}$ and

$$(T_m)^* = T_{\bar{m}}; \tag{21}$$

hence also

$$T_{\bar{g}} T_m = T_{\bar{g}m} \qquad \text{whenever } g \in \mathcal{M}_H. \tag{22}$$

Note also that in this case $\mathcal{M}_K = L^\infty(\mathfrak{S}, d\nu)$ (hence also $\mathcal{M} \supset L^\infty(\mathfrak{S}, d\nu)$) and $\|T_m\| \leq \|m\|_\infty$.

Remark. In principle, one can take for K the space H itself. Then the Toeplitz operators will be just the multipliers of H and, in particular, will satisfy $T_f T_g = T_g T_f = T_{fg}$ $\forall f, g \in \mathcal{M}$. Unfortunately, this last property makes these Toeplitz operators rather uninteresting from our point of view.

1.5. Berezin-Toeplitz quantization

Let us now again apply the preceding paragraph to the particular case of the weighted Bergman spaces $L^2_{h,\mathrm{hol}}$. Clearly, (19) and (20) are then satisfied for $K = L^2(\Omega, d\mu_h)$. The corresponding Toeplitz operators $T_m =: T^h_m$ are therefore defined for any $m \in \mathcal{M}_K = L^\infty(\Omega)$ (for the most part, though, we will use them only for m smooth and with compact support), and

$$T^h_m T^h_g = T^h_{mg}, \qquad T^h_{\bar{g}} T^h_m = T^h_{\bar{g}m}, \tag{23}$$

whenever g is a bounded holomorphic function.

Definition. Let us call a linear subset $\mathcal{V} \subset C^\infty(\Omega)$ *total* if for any $m \geq 0$, $x \in \Omega$, and complex numbers c_α there exists $f \in \mathcal{V}$ such that $D^\alpha f(x) = c_\alpha$ for all multiindices α with $|\alpha| \leq m$.

This definition should not cause confusion with the previous definition of total set, as one of them pertains to subsets of $C^\infty(\Omega)$ while the other to subsets the algebra $\mathcal{A}$ consisting of functions of x and h.

Suppose now that we can show that for any f in some total set $\mathcal{V} \subset C^\infty(\Omega)$,

$$\lim_{h\searrow 0} \widetilde{T_f^h}(x) = 0 \ \forall x \quad \text{implies} \quad f = 0, \tag{24}$$

and that for any $f, g \in \mathcal{V}$ there exists an asymptotic expansion

$$T_f^h T_g^h = \sum_{j=0}^\infty h^j T_{C_j(f,g)}^h \qquad \text{as } h \searrow 0 \tag{25}$$

for some bidifferential operators C_j (independent of f and g) satisfying (2)–(4). Then the operators C_j define, via the recipe (1), a differential star-product on Ω. This is the *Berezin-Toeplitz star-product.* Here (25) is to be understood in the sense of operator norms, i.e., as in (6).

Proof. As the last claim is not so standard, we indicate the proof here. Thus let $*$ be defined by (1) and let us, for instance, check the associativity. In terms of the cochains C_j, this amounts to

$$\sum_{j+k=N} C_j(C_k(e,f),g) = \sum_{j+k=N} C_j(e, C_k(f,g)) \qquad \forall N = 0,1,2,\dots,$$

for any $e, f, g \in \mathcal{V}$. However, denoting, for a moment, the left-hand side by l_N and the right-hand side by r_N, it follows from (6) that

$$(T_e^h T_f^h) T_g^h = \sum_{j=0}^\infty T_{l_j}^h h^j, \qquad T_e^h (T_f^h T_g^h) = \sum_{j=0}^\infty T_{r_j}^h h^j,$$

asymptotically as $h \searrow 0$ (as always, in the sense of operator norms). Since multiplication of operators in $\mathcal{B}(L^2_{h,\mathrm{hol}})$ is associative, it is thus enough to check that

$$\left\| \sum_{j=0}^N T_{l_j - r_j}^h h^j \right\| = O(h^{N+1}) \qquad \forall N$$

can only hold if $l_j = r_j \ \forall j$. However, as $|\widetilde{T}(x)| \leq \|T\|$ for any operator T and any $x \in \Omega$, the last equality implies that

$$\sum_{j=0}^N \widetilde{T_{l_j - r_j}^h} h^j = O(h^{N+1}) \qquad \forall N,$$

and the claim follows by (24) and a straightforward induction on N.

The other items in the definition of star-product can be checked similarly. □

Remark. The last proof is implicit, e.g., in [28], where, however, the assumption

$$\lim_{h\searrow 0}\|T_f^h\| = \|f\|_\infty$$

was employed instead of (24).

Comparing the property (23) of the Toeplitz operators with (6), we also see that the Berezin-Toeplitz star product satisfies $f * g = fg$ whenever either $\bar f$ or g belong to $\mathcal{M}_H$, i.e., are bounded holomorphic. That is, if the spaces $L^2_{h,\mathrm{hol}}$ have sufficiently many holomorphic multipliers, then the Berezin-Toeplitz star-product has the property of separation of variables in the reverse order.

Remark. Note that in view of (21), the Berezin-Toeplitz star-product also has the symmetry property (15).

1.6. The Berezin transform

Assume now that both the assumption (10) of §1.2 and the assumption (19) of §1.4 are satisfied. Thus we have both the operator symbols and the Toeplitz operators. Consequently, we can form for any $m \in \mathcal{M}$ the symbol of the Toeplitz operator T_m:

$$B_H m(x) := \widetilde{T_m}(x) = \frac{\langle mK_x, K_x\rangle}{K(x,x)}.$$

The operator B_H is called the *Berezin transform*. Clearly, B_H is linear, $\|B_H m\|_\infty \le \|m\|_{\mathcal{M}}$, and, by the reproducing property of K_x, $B_H g = g$ whenever $g \in \mathcal{M}_H$. (In particular, $B_H \mathbf{1} = \mathbf{1}$.)

If also (20) holds, then further $B_H\bar m = \overline{B_H m}$ for any $m \in \mathcal{M}$, and $B_H\bar g = \bar g$ for all $g \in \mathcal{M}_H$.

1.7. Karabegov's alternative approach

The Berezin transform is actually related to the Berezin star product in a rather nice way, first noted by Karabegov [20]. Namely, consider the setup from §§1.3 and 1.5, i.e., the weighted Bergman spaces $L^2_{h,\mathrm{hol}}$, the associated products $*_h$, and the Toeplitz operators T^h_f; and further denote, for each $h > 0$, by B_h the corresponding Berezin transform. Note that the property of separation of variables of the Berezin star-product,

$$f * g = fg \qquad \text{if } f \text{ or } \bar g \text{ is holomorphic,}$$

means precisely that the coordinate expressions (5) for the corresponding bidifferential operators C_j involve only holomorphic derivatives of g and anti-holomorphic derivatives of f. Consequently, the C_j are uniquely determined by their values $C_j(\bar f, g)$ for holomorphic f and g. Since in that case

$$\bar{f} *_h g = \widetilde{(T^h_f)^*} *_h \widetilde{T^h_g} = \widetilde{T^h_{\bar{f}}} *_h \widetilde{T^h_g} = \widetilde{T^h_{\bar{f}} T^h_g} = \widetilde{T^h_{\bar{f}g}} = B_h(\bar{f}g), \tag{26}$$

we see that we can recover the coefficients C_j of the Berezin star-product by the following simple recipe.

Assume that we can establish the asymptotic expansion

$$B_h = I + Q_1 h + Q_2 h^2 + \cdots \qquad \text{as } h \searrow 0, \tag{27}$$

for the Berezin transforms B_h, where Q_j are some differential operators. Denote by $c_{j\alpha\beta}$ the coefficients of Q_j:

$$Q_j f = \sum_{\alpha,\beta \text{ multiindices}} c_{j\alpha\beta}\, \partial^\alpha \bar{\partial}^\beta f.$$

Then the coefficients of the Berezin star-product are given by

$$C_j(f,g) := \sum_{\alpha,\beta} c_{j\alpha\beta}\, (\bar{\partial}^\beta f)(\partial^\alpha g). \tag{28}$$

Here the asymptotic expansion (27) is meant pointwise, i.e.,

$$B_h f(x) = \sum_{j=0}^{N} Q_j f(x) + O(h^{N+1}), \quad Q_0 = I, \quad \text{as } h \searrow 0, \quad \forall N = 0,1,\dots, \tag{29}$$

for all $x \in \Omega$; and in order that the cochains C_j be uniquely determined, we should require this to hold for all f in some total subset of $C^\infty(\Omega)$.

Remark. It is another idea from [20] that the above formula

$$\bar{f} *_H g = B_H(\bar{f}g), \qquad \forall f, g \in \mathcal{M}_H, \tag{30}$$

can also be used in the other direction, i.e., to *define* the Berezin transform B_H in situations when the ordinary definition does not apply. We will have more to say about this in section 4.

1.8. Applicability

The situations when (17), (27), and (6) are known to hold (for the Bergman spaces $L^2_{h,\mathrm{hol}}$) are $\Omega = \mathbf{C}^d$ with the Euclidean Kähler structure [9], Ω a bounded symmetric domain with the invariant Kähler form [6], compact Kähler manifolds Ω [4] (here one has to use sections of line bundles instead of functions, because the potential Φ does not exist globally; and Ω has to satisfy a certain cohomology integrality condition — see below) and pseudoconvex domains $\Omega \subset \mathbf{C}^d$ satisfying certain assumptions [11]. Since not all details have appeared in the literature, and also for later reference, we now review these examples in more detail.

1.8.1. The Fock space

Let $\Omega = \mathbf{C}^d \cong \mathbf{R}^{2d}$ with the standard Euclidean Kähler structure, i.e., $\omega = \frac{i}{2}\sum_{j=1}^d dz_j \wedge d\bar{z}_j$. One can take $\Phi(z) = \|z\|^2$ as the Kähler potential, and the volume element $\mu = \bigwedge^d \omega$ coincides with the Lebesgue measure dz (up to normalization). Thus the spaces $L^2_{h,\mathrm{hol}}$ are

$$L^2_{h,\mathrm{hol}} = L^2_{\mathrm{hol}}(\mathbf{C}^d, e^{-\|z\|^2/h}(\pi h)^{-d} dz) \tag{31}$$

where, for convenience, we have introduced the normalization factor $(\pi h)^{-d}$ ensuring that $\|\mathbf{1}\|_h = 1$ for all h. The monomials $\{z^\alpha\}$, with α a multiindex, form an orthogonal basis of (31), with norms

$$\|z^\alpha\|_h^2 = \alpha! h^\alpha.$$

Consequently, the reproducing kernel is given by

$$K^{(h)}(x,y) = e^{\langle x,y\rangle/h},$$

and the Berezin transform becomes

$$\begin{aligned} B_h f(x) &= \int_{\mathbf{C}^d} f(y)\, \frac{|K^{(h)}(x,y)|^2}{K^{(h)}(x,x)}\, d\mu_h(y) \\ &= (\pi h)^{-d} \int_{\mathbf{C}^d} f(y)\, e^{-\|x-y\|^2/h}\, dx, \end{aligned} \tag{32}$$

that is, just the well-known heat semigroup, i.e., $B_h = e^{h\Delta}$. Thus formally

$$B_h = I + h\Delta + \frac{h^2}{2!}\Delta^2 + \cdots . \tag{33}$$

The last formula can be made rigorous using the familiar stationary phase (WJKB) method. Recall that the latter tells us that if S is a complex-valued C^∞ function on a domain in $\mathbf{C}^d$ having a unique critical point x_0 (i.e., $S'(x_0) = 0$) which is nondegenerate (i.e., $\det S''(x_0) \neq 0$) and is a global maximum for $\operatorname{Re} S$, then for any smooth function ϕ with compact support, the integral

$$(\pi h)^{-d} e^{-S(x_0)/h} \int \phi(x)\, e^{S(x)/h}\, dx \tag{34}$$

has the asymptotic expansion

$$\sum_{j=0}^\infty h^j\, L_j \phi(x_0) \qquad \text{as } h \searrow 0.$$

(In particular, if $x_0 \notin \operatorname{supp}\phi$ then the integral decays faster than any power of h.) Here L_j are certain differential operators (independent of ϕ) whose coefficients involve only the phase function S and its derivatives; the formula for L_j is quite complicated, but simplifies if the phase function S is quadratic: namely, if $S(x) = -\langle Q(x-x_0), x-x_0\rangle_{\mathbf{C}^d}$ for some matrix Q with positive real part, then x_0 is a (unique) critical point of S and

$$L_j = \frac{1}{j!}\mathcal{Q}^j, \tag{35}$$

where $\mathcal{Q} = \langle Q^{-1}\partial, \partial\rangle$.

In particular, for the integral (32), we thus obtain

$$B_h f(x) = \sum_{j=0}^{\infty} \frac{h^j}{j!}\,\Delta^j f(x) \qquad \text{as } h \searrow 0, \tag{36}$$

for any $f \in C^\infty(\mathbf{C}^d)$ with compact support.

Let us now attend to each of the quantization methods discussed in the preceding sections.

For the *Berezin quantization*, we claim that we can use the total set

$$\begin{aligned} \mathcal{A}_0 &= \Big\{ \sum_{j\in\text{finite}} h^{n_j} \widetilde{T^h_{f_j}} \;:\; n_j \geq 0,\ f_j \in \mathcal{D}(\mathbf{C}^d) \Big\} \\ &\equiv \Big\{ \sum_{j\in\text{finite}} h^{n_j} B_h f_j \;:\; n_j \geq 0,\ f_j \in \mathcal{D}(\mathbf{C}^d) \Big\}. \end{aligned} \tag{37}$$

(Here $\sum_{j\in\text{finite}}$ means that the summation is over j in some finite set.) Indeed, let us verify that the three items in §1.3 hold:

- By virtue of (36), for any $f \in \mathcal{D}(\mathbf{C}^d)$ and $n \geq 0$,

  $$h^n \widetilde{T^h_f} = \sum_{r\geq 0}^{\infty} h^{r+n} \frac{1}{r!}\Delta^r f;$$

  thus indeed any function in $\mathcal{A}_0$ has an asymptotic expansion as $h \searrow 0$.

- Fix an integer $m \geq 0$, multiindices α, β, and function $\psi \in C^\infty(\mathbf{R})$ such that $\psi(t) = 1$ for $|t| \leq 1$ and $\psi(t) = 0$ for $|t| \geq 2$, and let $f(z) = \frac{1}{\alpha!\beta!}(z-z_0)^\alpha (\bar z - \bar z_0)^\beta \psi(\|z-z_0\|)$. Then $h^m B_h f \in \mathcal{A}_0$ and, by (36), $B_h f(z) = \frac{1}{\alpha!\beta!}(z-z_0)^\alpha (\bar z - \bar z_0)^\beta + O(h)$ for $\|z-z_0\| < 1$, so $\partial^\gamma \bar\partial^\eta (h^m B_h f)(z_0) = \delta_{\alpha\gamma}\delta_{\beta\eta} h^m + O(h^{m+1})$. By an easy induction argument (attending first to coefficients at h^0, then at h^1, and so on), it therefore follows that the set $\mathcal{A}_0$ is total.

- Finally, for any $f, g \in \mathcal{D}(\mathbf{C}^d)$,

$$
\begin{aligned}
(\widetilde{T^h_f} *_h \widetilde{T^h g})(z) &= \widetilde{T^h_f T^h_g}(z) \\
&= \int_{\mathbf{C}^d}\int_{\mathbf{C}^d} f(x)g(y)\,\frac{K^{(h)}(z,x)K^{(h)}(x,y)K^{(h)}(y,z)}{K^{(h)}(z,z)}\,d\mu_h(x)\,d\mu_h(y) \\
&= (\pi h)^{-2d}\int_{\mathbf{C}^d}\int_{\mathbf{C}^d} f(x)g(y)\,e^{(\langle z,x\rangle+\langle x,y\rangle+\langle y,z\rangle-\|x\|^2-\|y\|^2-\|z\|^2)/h}\,dx\,dy.
\end{aligned}
$$

  The stationary phase method again shows that this has the asymptotic expansion

$$\sum_{r=0}^{\infty} h^r\,M_r(f,g)(z) \qquad \text{as } h \searrow 0,$$

  where

$$M_r(f,g)(z) = \frac{1}{r!}\Big[\Delta_x + \Delta_y + \sum_{j=1}^{d}\frac{\partial^2}{\partial\bar{x}_j\partial y_j}\Big]^r f(x)g(y)\Big|_{x=y=z}.$$

  In particular,

$$M_0(f,g) = fg, \qquad M_1(f,g) = g\Delta f + f\Delta g + \sum_{j=1}^{d}\frac{\partial f}{\partial\bar{z}_j}\,\frac{\partial g}{\partial z_j},$$

$$M_1(f,g) - M_1(g,f) = \frac{i}{2\pi}\{f,g\}.$$

  Thus $u * v$ has an asymptotic expansion as $h \searrow 0$ for any $u, v \in \mathcal{A}_0$, and (2), (3), and (4) are fulfilled.

Furthermore, if f is holomorphic on some open set $\mathcal{U} \subset \Omega$, then by (36) $B_h f = f + O(h^\infty)$ on $\mathcal{U}$, while the formula for M_r shows that $M_r(f,g) = f \cdot \frac{1}{r!}\Delta^r g$ on $\mathcal{U}$; thus, modulo $O(h^\infty)$, $f *_h B_h g = B_h f *_h B_h g = \sum_{r=0}^{\infty} h^r f\,\frac{1}{r!}\Delta^r g = f \cdot B_h g$ on $\mathcal{U}$; that is, $f * v = fv$ on $\mathcal{U}$ for all $v \in \mathcal{A}_0$. Similarly (or by (15)) for $v * g$ with $\bar{g}$ holomorphic. Thus we see that the Berezin star-product has the separation of variables.

Remark. We had to establish the separation of variables by a direct argument, since the spaces $L^2_{\mathrm{hol}}(\mathbf{C}^d, e^{-\|z\|^2/h}(\pi h)^{-d}dz)$ do not have sufficiently many holomorphic multipliers in the sense of §1.3 — in fact, the only holomorphic multipliers turn out to be the constant functions.

The validity of the *Berezin-Toeplitz* quantization on the Fock space has been settled by Coburn [9] and also Borthwick [5]; it turns out that (24) and (25) hold for $\mathcal{V} = BC^\infty(\mathbf{C}^d)$, the space of all smooth functions whose partial derivatives of all orders are bounded.

Finally, the *asymptotics of the Berezin transform* are given by (33) or (36), for any $f \in BC^\infty(\mathbf{C}^d)$. In particular, (28) tells us that the cochains C_j in (17) are given by

$$C_j(f,g) = \sum_{|\alpha|=j} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial \bar{z}^\alpha} \frac{\partial^\alpha g}{\partial z^\alpha}.$$

The cochains for the Berezin-Toeplitz star-product turn out to be given by the similar formula

$$C_j(f,g) = \sum_{|\alpha|=j} \frac{(-1)^{|\alpha|}}{\alpha!} \frac{\partial^\alpha f}{\partial z^\alpha} \frac{\partial^\alpha g}{\partial \bar{z}^\alpha}. \tag{38}$$

Remark. Actually, (6) was proved in [5,9] only for $N \leq 1$, but their argument in fact yields the conclusion in full generality.

Remark. An alternative to using the set $\mathcal{A}_0$ in (37) above is to extend Berezin's procedure from §1.3 by admitting unbounded operators. Namely, let $\mathcal{O}_h$, for each $h > 0$, be the set of all linear combinations of operators $T^h_{z^\alpha} T^h_{\bar{z}^\beta}$, where α, β range over all multiindices. Note that $T^h_{z^\alpha}$ are just the operators of multiplication by z^α, while $T^h_{\bar{z}^\beta}$ coincide with the differentiation $h^{|\beta|}\partial^\beta$; in particular, all operators in $\mathcal{O}_h$ have a common dense domain consisting of all functions of the form

$$\sum_{j\in\text{finite}} p_j(x)\, e^{\langle x, a_j\rangle}, \qquad p_j \text{ polynomials, } a_j \in \mathbf{C}^d; \tag{39}$$

and we have the commutation relations $[T^h_{\bar{z}_k}, T^h_{z_j}] = h\delta_{jk} I$. It follows that each $\mathcal{O}_h$ is an associative algebra, with an involution given by $T^h_{z^\alpha} T^h_{\bar{z}^\beta} \mapsto T^h_{z^\beta} T^h_{\bar{z}^\alpha}$; further, since the reproducing kernels belong to (39), the operator symbols $\widetilde{T}$ are defined for all $T \in \mathcal{O}_h$, and, by a short computation, $\widetilde{T^h_{z^\alpha} T^h_{\bar{z}^\beta}} = z^\alpha \bar{z}^\beta$. Thus we may use $\mathcal{O}_h$ instead of $\mathcal{A}_h$ in §1.3 to define the Berezin star-product, and the advantage is that instead of $\mathcal{A}_0$ we can take simply $\mathcal{O}_0 := \{\sum_{j\in\text{finite}} h^{n_j} \widetilde{T^h_{p_j} T^h_{\overline{q_j}}}\} = \{\sum_{j\in\text{finite}} h^{n_j} p_j \overline{q_j}\}$, with $n_j \geq 0$ and p_j, q_j polynomials, as the total set.

We remark that in principle it is possible to define the operator symbols (11) even for any unbounded operator whose domain contains K_x for all $x \in \Omega$; furthermore, if we in addition require that the operators be closed then the mapping $T \mapsto \widetilde{T}$ will still be one-to-one. However, there are difficulties when one attempts to define the $*_H$-multiplication of these symbols by the formula $\widetilde{S} *_H \widetilde{T} = \widetilde{ST}$, since the domain of the composition ST need not contain the K_x any longer and thus $\widetilde{ST}$ need not be defined in general.

1.8.2. The unit disc

Let $\Omega = \mathbf{D} = \{ z \in \mathbf{C} : |z| < 1 \}$ with the Poincaré metric, i.e., $\omega = \frac{i}{2}(1-|z|^2)^{-2} dz \wedge d\bar{z}$, corresponding to the Kähler potential $\Phi(z) = -\log(1-|z|^2)$. The volume element

μ is the invariant measure $d\mu(z) = (1-|z|^2)^{-2}dz$. Thus, we arrive at the spaces

$$L^2_{h,\mathrm{hol}} = L^2_{\mathrm{hol}}(\mathbf{D}, \tfrac{\nu-1}{\pi}(1-|z|^2)^{\nu-2}dz) \tag{40}$$

where we have again introduced the normalization factor $\frac{\nu-1}{\pi}$ ensuring that $\|\mathbf{1}\|_\nu = 1$, and denoted

$$\nu = \frac{1}{h}$$

for typographic reasons. Again, the monomials z^k, $k = 0, 1, 2, \dots$, form an orthogonal basis of (40), with

$$\|z^k\|^2_\nu = \frac{k!}{(\nu)_k},$$

where

$$(\nu)_k := \nu(\nu+1)\cdots(\nu+k-1) \tag{41}$$

is the Pochhammer symbol. Consequently, the reproducing kernel and the Berezin transform are given by

$$\begin{aligned} K_\nu(x,y) &= (1-x\overline{y})^{-\nu}, \\ B_\nu f(x) &= \frac{\nu-1}{\pi}\int_{\mathbf{D}} f(y)\,\frac{(1-|x|^2)^\nu}{|1-x\overline{y}|^{2\nu}}\,(1-|y|^2)^{\nu-2}\,dy \\ &= \frac{\nu-1}{\pi}\int_{\mathbf{D}} f\Big(\frac{x-y}{1-\bar{x}y}\Big)\,(1-|y|^2)^{\nu-2}\,dy. \end{aligned}$$

An application of the stationary phase again implies that B_ν has an asymptotic expansion of the required form as $\nu \to +\infty$. This time the phase function $S(y) = \log(1-|y|^2)$ is not quadratic, so some effort is needed to compute the coefficients explicitly; the first two terms of the expansion are

$$B_\nu f(z) = f(z) + \nu^{-1}\,(1-|z|^2)^2\Delta f(z) + O(\nu^{-2}). \tag{42}$$

Let us now again attend, in turn, to our three quantization methods.

For the *Berezin quantization*, we claim that (16) and (17) are satisfied for the total set

$$\mathcal{A}_0 = \Big\{\sum_{j\in\text{finite}} h^{n_j}\widetilde{T^h_{p_j}T^h_{\overline{q_j}}} \;:\; n_j \geq 0,\; p_j, q_j \text{ polynomials}\Big\}. \tag{43}$$

Indeed, as $\widetilde{T^h_{p_j}T^h_{\overline{q_j}}} = p_j\overline{q_j}$ by (13) or (14), (16) trivially holds. Further, the function $f(z) = h^n\frac{1}{j!k!}(z-z_0)^j(\bar{z}-\bar{z}_0)^k \in \mathcal{A}_0$ satisfies $\partial^l\bar{\partial}^m f(z_0) = \delta_{jl}\delta_{km}h^n$, implying that $\mathcal{A}_0$ is total. Finally, since $\mathbf{D}$ is a bounded domain and thus polynomials are automatically bounded on $\mathbf{D}$, the validity of (17) follows from Berezin [3, Theorem 2.2] (only the coefficients at h^0 and h^1) and [11, Theorem 12]. For the same reason, the spaces (40) have sufficiently many holomorphic multipliers and thus the Berezin star-product has the property of separation of variables.

The validity of the *Berezin-Toeplitz quantization* has been established by Klimek and Lesniewski [22]; this time, with the total set $\mathcal{V} = C^\infty(\overline{\mathbf{D}})$. (Again, they only proved (6) for $N \leq 1$, but their argument easily extends to arbitrary N.)

Finally, the *asymptotic expansion* of the Berezin transform is given by (42), for any $f \in L^\infty \cap C^\infty(\mathbf{D})$. The higher order terms turn out to be of the form $Q_j = p_j(\widetilde{\Delta})$, where $\widetilde{\Delta} = (1-|z|^2)^2\Delta$ is the invariant Laplacian on $\mathbf{D}$, and p_j are polynomials involving Bernoulli numbers etc.; see [10]. Note that, in view of (28), this implies that the coefficients of the cochains C_j of the Berezin star-product are *polynomials*; and using (50) below, it follows that, likewise,

the coefficients of the cochains C_j of the Berezin-Toeplitz star-product are polynomials. (44)

Remark. In (43), one might even allow $\mathcal{A}_0$ to contain some infinite sums, namely,

$$\mathcal{A}_0 = \Big\{ \sum_j h^{n_j} p_j \overline{q_j} \;:\; \sum h^{n_j} \|p_j\|_\infty \|q_j\|_\infty < \infty \text{ for } h \text{ sufficiently small} \Big\}.$$

The proofs will work without change.

Remark. Alternatively, one can also use for $\mathcal{A}_0$ the same set as for the Fock space:

$$\mathcal{A}_0 = \Big\{ \sum_{j\in\text{finite}} h^{n_j} \widetilde{T^h_{f_j}} \;:\; n_j \geq 0,\; f_j \in \mathcal{D}(\mathbf{D}) \Big\}; \tag{45}$$

the required assertions then follow again by application of the stationary phase method, see Theorem 1.2 below. However, for bounded domains it seems simpler to use (43).

This remark applies also to §§1.8.3 and 1.8.4 below. □

1.8.3. Bounded symmetric domains

The preceding example generalizes to arbitrary bounded symmetric domains $\Omega \subset \mathbf{C}^d$ with the invariant metric, i.e., given by the potential $\Phi(z) = \log K(z,z)$, where $K(x,y)$ is the unweighted Bergman kernel of Ω. The volume element turns out to be $K(z,z)\,dz$ (up to a constant factor), and

$$L^2_{h,\text{hol}} = L^2_{\text{hol}}(\Omega, K(z,z)^{1-\alpha} dz), \qquad \alpha = \frac{1}{h}.$$

The reproducing kernels are

$$K_\alpha(x,y) = c_\alpha K(x,y)^\alpha$$

with some constants c_α, and

$$B_\alpha f(x) = c_\alpha \int_\Omega f(y) \left[\frac{|K(x,y)|^2}{K(x,x)K(y,y)} \right]^\alpha K(y,y)\, dy.$$

An application of the stationary phase again implies that the Berezin transform has an asymptotic expansion $I + \alpha^{-1}\widetilde{\Delta} + \cdots$ as $h \searrow 0$, where $\widetilde{\Delta}$ is the invariant Laplacian on Ω; the higher-order coefficients have been computed explicitly by Unterberger and Upmeier [30]. For the Berezin quantization, the same total set (43) can be used as in the preceding example (with the same proof). The validity of the Berezin-Toeplitz quantization is due to Borthwick, Lesniewski, and Upmeier [6], for $f, g \in \mathcal{D}(\Omega)$. (Once more, (6) was proved in [6] only for $N \leq 1$, but their argument in fact yields the conclusion in full generality; see [14], where an extension to a class of functions $f, g \in C^\infty(\Omega)$ not necessarily with compact support is also derived.) The asymptotic expansion of the Berezin transform is again valid for any $f \in L^\infty \cap C^\infty(\Omega)$.

1.8.4. Pseudoconvex domains

Our last example concerns bounded strictly pseudoconvex domains $\Omega \subset \mathbf{C}^d$, $d \geq 1$, with smooth boundary, and a smooth strictly plurisubharmonic function Φ on Ω, so that $\omega = \frac{i}{2}\partial\bar\partial\Phi$ makes Ω into a Kähler manifold. In this case, there is no explicit formula for the reproducing kernels $K^{(h)}(x,y)$ of $L^2_{h,\mathrm{hol}} = L^2_{\mathrm{hol}}(\Omega, e^{-\Phi/h} d\mu)$; however, it has been shown in [11] that there is an asymptotic expansion

$$K^{(h)}(x,y) = e^{\Phi(x,y)/h} h^{-d} \sum_{j=0}^\infty h^j \beta_j(x,y) \qquad \text{as } h \searrow 0, \tag{46}$$

for (x,y) near the diagonal, where β_j are some coefficient functions and $\Phi(x,y)$ is a kind of "sesquianalytic extension" of $\Phi(x) = \Phi(x,x)$ to a neighborhood of the diagonal. (In particular, (12) holds as soon as h is sufficiently small.) If $\Phi(x)$ behaves reasonably at the boundary, then one can estimate $K^{(h)}(x,y)$ also away from the diagonal, and apply the stationary phase method as before to get asymptotics of the Berezin transform and to establish the validity of the Berezin and the Berezin-Toeplitz quantizations. (For the former, one can take again either (43) or (45) as the total set; and for the latter, one can again allow any $f, g \in C^\infty(\overline{\Omega})$. The asymptotics of the Berezin transform again hold for any $f \in C^\infty(\Omega) \cap L^\infty(\Omega)$. See [11] for the details.

1.8.5. Manifolds

The previous example can be generalized from domains Ω with the plurisubharmonic function Φ to Kähler manifolds (Ω, ω) for which there exists a holomorphic Hermitian line bundle $\mathcal{L}$ with compatible connection whose curvature form coincides with ω. For this line bundle to exist, it is necessary and sufficient that ω satisfy the *integrality condition*, well known from geometric quantization. One can then consider, for each

$m = 1, 2, 3, \dots$, the space $L^2_{h,\mathrm{hol}} = L^2_{\mathrm{hol}}(\Omega, \mathcal{L}^m)$ of all square-integrable holomorphic sections of the m-th tensor power $\mathcal{L}^m$ of $\mathcal{L}$, where $h = \frac{1}{m}$. Reproducing kernels and operator symbols can be defined more or less in the same way as before, as well as the Toeplitz operators and the Berezin transform (see Peetre [27]); and everything what has been said in §§1.3, 1.5, and 1.7 extends also to this setting. For compact manifolds, the asymptotics (46) of the reproducing kernels were established by Catlin [7], and the Berezin-Toeplitz quantization by Bordemann, Meinrenken, and Schlichenmaier [4]; the asymptotics of the Berezin transform, as well as a detailed discussion of the separation of variables, appear in [21].

Since the extension from domains to manifolds involves only routine technicalities (most of the work is always done in local charts), we will not say anything more about them and restrict attention to domains $\Omega \subset \mathbf{C}^d$ with globally defined Kähler potential Φ for the rest of this paper.

1.9. Possible generalizations

Let us review our findings so far about the two quantization procedures:

(I) the Berezin quantization, described in §1.3, uses only the symbols of operators, cf. (17), and needs the assumptions (10) and (12);

(II) the Berezin-Toeplitz quantization, described in §1.5, uses only the Toeplitz operators, cf. (6), and needs the assumption (19).

Moreover, the Berezin star-product usually has the property of separation of variables (for instance, if there are sufficiently many holomorphic multipliers), and so has the Berezin-Toeplitz (in the reverse order) provided that in addition (20) holds. Furthermore,

(III) if the Berezin quantization has the property of separation of variables, then we can recover the coefficients C_j using solely the Berezin transform, as described in §1.7, cf. (27); for this we need both operator symbols and Toeplitz operators — hence, both the assumptions (10), (12) and the assumption (19) — and, further, for the latter the ambient space must be L^2 — i.e., (20) must hold. ((20) is needed to ensure that $T_f^* = T_{\bar{f}}$ in (26).)

Of course, all the time we are also tacitly assuming (7), which guarantees the existence of the reproducing kernel in the first place.

Summarizing, we thus see that what is really needed for these quantization procedures is just a family L_h of Hilbert spaces, indexed by a small positive parameter $h > 0$, such that

I. for the **Berezin quantization**,

  (a) each L_h is a reproducing kernel space of functions on Ω satisfying (12);

(b) for each h, the correspondence $T \mapsto \widetilde{T}$ is one-to-one; hence one can transfer the operator multiplication into the noncommutative products $*_h$ in the algebras $\mathcal{A}_h$ of operator symbols;

(c) there exists a total subset $\mathcal{A}_0$ of the direct sum $\mathcal{A} = \bigoplus_{h>0} \mathcal{A}_h$ such that (16) and (17) hold.

Further, the spaces L_h have sufficiently many holomorphic multipliers, then the resulting Berezin star-product has the property of separation of variables.

II. for the **Berezin-Toeplitz** quantization,

(a) each L_h is a reproducing kernel space and a subspace of another Hilbert space $\mathcal{K}_h$ of functions on Ω; hence we can define, for each h, Toeplitz operators on L_h by $T^h_g f := P_h(gf)$, where $P_h : \mathcal{K}_h \to L_h$ is the orthogonal projection;

(b) there exists a total subset $\mathcal{V}$ of $C^\infty(\Omega)$ such that for any $f, g \in \mathcal{V}$, the Toeplitz operators T^h_f and T^h_g satisfy (24) and (6), where the bidifferential operators C_j satisfy (2)–(4).

Further, if the spaces L_h have sufficiently many holomorphic multipliers and $\mathcal{K}_h = L^2(\Omega, d\nu_h)$ for some measures ν_h, then the resulting Berezin-Toeplitz star-product has the property of separation of variables (in the reverse order).

Moreover,

III. for **Karabegov's approach**,

(a) the items I(a)–I(c) and II(a) above should be satisfied, the last with $\mathcal{K}_h = L^2(\Omega, d\nu_h)$ for some measures ν_h; thus we can define, for each h, the associated Berezin transform B_h;

(b) the spaces L_h have sufficiently many holomorphic multipliers; thus the Berezin star-product has the property of separation of variables;

(c) there exists a total subset $\mathcal{V} \subset C^\infty(\Omega)$ such that for any $f \in \mathcal{V}$, the asymptotic expansion (29) holds.

Then the coefficients of the Berezin star-product can be recovered from the recipe (28).

Clearly, the last item (III) requires much stronger hypotheses than I or II.

Let us now analyze which of the above items are satisfied when instead of the weighted Bergman spaces $L^2_{h,\mathrm{hol}}$ one tries for L_h the various alternatives mentioned at the end of the Introduction. (And if they are satisfied, whether they lead to interesting star-products.)

At the moment, we have fairly complete results for I, and also for II when the spaces $\mathcal{K}_h$ are taken to be $L^2(\Omega, d\nu_h)$ for some measures ν_h. No results at all will be offered for III, and only a single one for II with other spaces $\mathcal{K}_h$ (namely, for $\mathcal{K}_h$ the Sobolev spaces H^s, and L_h the subspaces H^s_{hol} of all holomorphic functions therein); as has already been remarked above, all this has to wait for future work.

In the rest of this paper, we will call the quantizations based on the spaces $L^2_{h,\mathrm{hol}}$ from §§1.3 and 1.5 the *traditional* Berezin and Berezin-Toeplitz quantization, respectively, in order to distinguish them from their generalizations based on the spaces L_h as in I–III above.

1.10. Some more technicalities

Before continuing with the main line of exposition, we pause to collect here various additional material; some of it will be needed in §4. We start with a simple observation concerning the relationship between the Berezin and the Berezin-Toeplitz quantization.

Proposition 1.1. *Assume that for some spaces L_h as in §1.9,*

- *the Berezin-Toeplitz quantization works on Ω, with total set $\mathcal{V}$;*
- *the Berezin transform $B_h f$ has the asymptotic expansion* (29) *for any $f \in \mathcal{V}$; and*
- *the correspondence between operators and their symbols is one-to-one.*

(That is, the items I(a), I(b), II(a), II(b), and III(c) from §1.9 hold.)

Then the Berezin quantization also works on Ω for the total set

$$\mathcal{A}_0 = \Big\{ \sum_{j\in\text{finite}} h^{n_j} \widetilde{T^h_{f_j}} \ :\ n_j \geq 0,\ f_j \in \mathcal{V} \Big\}.$$

Proof. We need to check I(c), i.e., (16), (17), and the totality of the set $\mathcal{A}_0$. Since $\widetilde{T^h_f} = B_h f$, the validity of (16) is evident from (29). Likewise, the totality of $\mathcal{A}_0$ follows from the fact that $B_h f = f + O(h)$ and the totality of $\mathcal{V}$. Finally, since $|\widetilde{T}(x)| \leq \|T\|$ for any point x and any bounded linear operator T, taking operator symbols in (6) we get

$$\begin{aligned}\widetilde{T^h_f} *_h \widetilde{T^h_g} &= \sum_{j=0}^{\infty} h^j \widetilde{T^h_{C_j(f,g)}} \\ &= \sum_{j=0}^{\infty} h^j B_h C_j(f,g) = \sum_{j,k=0}^{\infty} h^{j+k} Q_k C_j(f,g) \end{aligned} \tag{47}$$

for all $f, g \in \mathcal{V}$; this settles (17). Further, expanding the left-hand side $B_h f * B_h g$ of (47) and comparing the coefficients at h^0 and h^1 with the right-hand side shows that, as formal power series in h,

$$f * g + h[Q_1 f * g + f * Q_1 g] + O(h^2) = fg + h[C_1(f,g) + Q_1(fg)] + O(h^2),$$

that is (writing temporarily C_j^B and C_j^{BT} to distinguish the cochains C_j for the Berezin and Berezin-Toeplitz star-products, respectively),

$$C_0^B(f,g) + hC_1^B(f,g) + h[C_0^B(Q_1 f, g) + C_0^B(f, Q_1 g)] = fg + hC_1^{BT}(f,g) + hQ_1(fg).$$

Thus

$$C_0^B(f,g) = fg \tag{48}$$

and, upon interchanging f and g and subtracting,

$$C_1^B(f,g) - C_1^B(g,f) = C_1^{BT}(f,g) - C_1^{BT}(g,f), \tag{49}$$

proving (2) and (3). Finally, since $B_h\mathbf{1} = \mathbf{1}$, (47) implies that $B_h f * \mathbf{1} = B_h f$ and $\mathbf{1} * B_h g = B_h g$ for all f and g, thus proving (4). This completes the proof. □

Remark. Viewing B_h as a formal series of operators (27) on $C^\infty(\Omega)$, (47) implies that

$$Bf * Bg = B(f * g) \tag{50}$$

where the $*$ on the left is the Berezin star-product, and the one on the right is the Berezin-Toeplitz star-product; thus these two star-products are equivalent and the Berezin transform mediates the equivalence (cf. [20], or [11, page 239]).

We continue by a definition. We will say that Ω admits Berezin quantization *in the strong form* (based on the family of spaces L_h) if in addition to the items (16) and (17), we even have the following strengthening of (17):

for any $k \geq 1$ and $f_1, f_2, \dots, f_k \in \mathcal{A}_0$, the star product $f_1 * f_2 * \cdots * f_k$ also admits an asymptotic expansion as $h \searrow 0$ (which for $k = 2$ specializes to (17)). (51)

It follows from the above proof (upon applying (47) repeatedly) that if

$$C_j^{BT}(f,g) \in \mathcal{V} \quad \forall f, g \in \mathcal{V} \tag{52}$$

then Ω even admits Berezin quantization in the strong form. In particular, in view of (38) and (44), the Fock spaces from §1.8.1 as well as the unit disc from §1.8.2 admit Berezin quantization in the strong form. Unfortunately, (52) is not always fulfilled in practice, so we establish this directly.

Theorem 1.2. *Consider the spaces $L^2_{h,\mathrm{hol}}$ on the domains Ω from §§1.8.1–1.8.3. Then Ω admits the traditional Berezin quantization in the strong form with total set*

$$\mathcal{A}_0 = \Big\{ \sum_{j \in \text{finite}} h^{n_j} \widetilde{T^h_{f_j}} \; : \; n_j \geq 0,\ f_j \in C^\infty(\Omega) \cap L^\infty(\Omega) \Big\}.$$

Proof. We present the proof for the Fock spaces from §1.8.1; the other cases are similar. It suffices to show that for any $k \geq 1$ and $f_1, \dots, f_k \in L^\infty \cap C^\infty$,

$$\widetilde{T^h_{f_1}} *_h \widetilde{T^h_{f_2}} *_h \cdots *_h \widetilde{T^h_{f_k}} = [T^h_{f_1} T^h_{f_2} \cdots T^h_{f_k}]^\sim$$

has an asymptotic expansion as $h \searrow 0$, with leading term $h^0 \cdot f_1 f_2 \dots f_k$. Indeed, taking $k = 1$ this implies first of all that (16) holds, with $\widetilde{T^h_f} = f + O(h)$; from the latter it also follows (as in §1.8.1) that $\mathcal{A}_0$ is total. From §1.8.1 we also know that for $k = 2$ and $f_1, f_2 \in \mathcal{D}(\Omega)$, the coefficients C_j have the desired properties (2)–(4); since differential operators are uniquely determined by their restrictions to $\mathcal{D}(\Omega)$, (2)–(4) remain in force also for any $f, g \in C^\infty(\Omega) \cap L^\infty(\Omega)$. Finally, for $k \geq 2$ the existence of the asymptotic expansion is precisely what is required by (51), and the proof will be complete.

By the definition of the Toeplitz operator,

$$T^h_f \phi(x) = \langle \phi, K^{(h)}_x \rangle = \int_\Omega \phi(y) f(y) K^{(h)}(x, y)\, d\mu_h(y),$$

i.e., T^h_f is an integral operator with kernel $f(y)K^{(h)}(x, y)$. Consequently,

$$\begin{aligned}
[T^h_{f_1} \cdots T^h_{f_k}]^\sim(z) &= K^{(h)}(z, z)^{-1} \langle T^h_{f_1} \dots T^h_{f_k} {K^{(h)}}_z, {K^{(h)}}_z \rangle \\
&= \int_\Omega \int_\Omega \cdots \int_\Omega f_1(y_1) \cdots f_k(y_k) \\
&\quad \times \frac{K^{(h)}(z, y_1) K^{(h)}(y_1, y_2) K^{(h)}(y_2, y_3) \cdots K^{(h)}(y_k, z)}{K^{(h)}(z, z)} \\
&\quad \times d\mu_h(y_1) \cdots d\mu_h(y_k) \\
&= \int_{\mathbf{C}^d} \int_{\mathbf{C}^d} \cdots \int_{\mathbf{C}^d} f_1(y_1) \cdots f_k(y_k)\, e^{(\langle z, y_1\rangle + \langle y_1, y_2\rangle + \cdots + \langle y_k, z\rangle)/h} \\
&\quad \times e^{-(\|y_1\|^2 + \cdots + \|y_k\|^2 + \|z\|^2)/h} \frac{dy_1 \cdots dy_k}{(\pi h)^{kd}}.
\end{aligned} \tag{53}$$

Now recall that in the stationary phase method, concerning the asymptotics of the integral (34), the hypothesis of the compact support of ϕ can be relaxed: it is enough that ϕ be such that the integral exists for some $h = h_0$, and that the maximum of $\operatorname{Re} S$ at the critical point x_0 dominate also the values of $\operatorname{Re} S$ at the boundary or at infinity, in the sense that $\operatorname{Re} S(x) \to \operatorname{Re} S(x_0) \implies x \to x_0$. The latter condition is obviously fulfilled whenever S is a quadratic form, hence also for the integral (53); and the former condition is likewise satisfied in our case since the existence of the integral (53) for all h follows from the boundedness of $T^h_{f_j}$ for any $f_j \in L^\infty(\Omega)$. Thus the stationary phase applies, and the existence of the asymptotic expansion as $h \searrow 0$ follows, with leading term equal to $h^0 \cdot f_1(z) \cdots f_k(z)$ by virtue of (35). This settles the claim. □

Remark. Under some extra assumptions, the last proof works without change also for the pseudoconvex domains from §1.8.4. We have stated the theorem only for situations in which we will need it later in §4.3.

We conclude this section by elaborating on one more of the ideas above. Namely, assume that (12) holds and that the Berezin-Toeplitz quantization works on Ω, i.e., the items I(a), II(a), and II(b) from §1.9 are fulfilled. Upon taking operator symbols on both sides of (6) and recalling that $\|\widetilde{T}\|_\infty \leq \|T\|$, we obtain the equality (47), viz.,

$$B_h f *_h B_h g = \sum_{j=0}^{\infty} h^j B_h C_j(f,g) \qquad \forall f, g \in \mathcal{A}_0. \tag{54}$$

We might call this a *weak Berezin-Toeplitz quantization.* It turns out that this is often something that we are already familiar with.

Proposition 1.3. *Assume that the items* I(a)*,* I(b)*,* I(c)*, and* III(c) *from* §*1.9 are fulfilled. Then the weak Berezin-Toeplitz quantization* (54) *holds for all* f*,* g *in a total set* $\mathcal{V} \subset C^\infty(\Omega)$ *if and only if the Berezin quantization holds with the total set* $\mathcal{A}_0 = \{\sum_{j\in\text{finite}} h^{n_j} B_h f_j \;:\; n_j \geq 0,\, f_j \in \mathcal{V}\}$.

Proof. Immediate upon comparing (29) with (16), and (54) with (17); the fact that $\mathcal{A}_0$ is total if and only if $\mathcal{V}$ is follows again from the fact that the leading term in (29) is the identity operator. □

2. Berezin quantization via spaces of nonholomorphic functions

Returning to the main line of exposition, let us now analyze if the Berezin quantization can be accomplished using other spaces than the traditional $L^2_{h,\text{hol}}$. We start with the case of the Berezin quantization based on the subspaces L^2_A of functions in some L^2 annihilated by a given hypoelliptic operator A, or more generally on the analogous subspaces H^s_A of the Sobolev space H^s with arbitrary $s \in \mathbf{R}$ (of course, the former are just the special case $s = 0$ of the latter), as well as on the Sobolev spaces H^s themselves.

We thus need to check the three items I(a)–I(c) in §1.9: namely, that (a) the point evaluations $f \mapsto f(y)$ are continuous; (b) the correspondence $T \mapsto \widetilde{T}$ is one-to-one; and, finally, (c) if the space is made to vary with the Planck parameter h in a suitable way, we can find a total subset such that (17) holds.

Consider first the simplest case among the above (apart from the holomorphic functions, corresponding to $A = \bar\partial$), namely, the spaces L^2_{harm} of harmonic functions, obtained for the choice $A = \Delta$. From the point of view of practical applications, these spaces would have the obvious advantage over L^2_{hol} in that they do not require Ω to have complex structure.

Thus let $L^2_{\text{harm}}(\Omega, d\nu)$ be the subspace of all harmonic functions in $L^2(\Omega, d\nu)$ for some measure ν on Ω. The first item (a) presents no problem: from the mean value property of harmonic functions, it is immediate that point evaluations are continuous on $L^2_{\rm harm}$, for instance, whenever $d\nu$ has a continuous positive density with respect to the Lebesgue measure. The associated reproducing kernel, the (weighted) *harmonic Bergman kernel* $H(x,y)$, is then a function harmonic in both x and y, and in distinction to the usual Bergman kernel is *real-valued*. Indeed, for any $f \in L^2_{\rm harm}$,

$$\overline{\langle f, H_x\rangle} = \overline{f(x)} = \langle \bar f, H_x\rangle = \overline{\langle f, \overline{H_x}\rangle},$$

implying that $H_x = \overline{H_x}$. Thus $H(x,y) = H(y,x) = \overline{H(x,y)}$.

Example. For $L^2_{\rm harm}(\mathbf{C}, e^{-|z|^2/h}(\pi h)^{-1}dz)$, we have $H(x,y) = 2\,\mathrm{Re}\, e^{x\bar y} - 1$; and for $L^2_{\rm harm}(\mathbf{D}, dz)$, $H(x,y) = 2\,\mathrm{Re}(1-x\bar y)^{-2} - 1$. For the analogous space on the unit ball in $\mathbf{R}^n$, the kernel is already much more complicated; see [2].

Unfortunately, however, the item (b) breaks down completely. Indeed, consider the operator

$$T = \langle \cdot, \bar f\rangle g - \langle \cdot, \bar g\rangle f$$

for some linearly independent $f, g \in L^2_{\rm harm}$. Then by the reproducing property of H_x,

$$\langle TH_x, H_x\rangle = \langle H_x, \bar f\rangle\langle g, H_x\rangle - \langle H_x, \bar g\rangle\langle f, H_x\rangle = f(x)g(x) - g(x)f(x) = 0.$$

Thus $\widetilde T = 0$, yet $T \neq 0$.

More generally, defining a "complex conjugate" of an operator by

$$\overline{T} f := \overline{T\bar f},$$

the same argument shows that

$$(\overline{T} - T^*)^\sim = 0, \quad \text{for any } T. \tag{55}$$

Thus there is no hope of performing the Berezin quantization in this case.

Remark. A possible solution to this problem might be to find a (noncommutative) subalgebra $\mathcal{X}$ of $\mathcal{B}(L^2_{\rm harm})$ such that $T \mapsto \widetilde T$ is injective on $\mathcal{X}$, yet $\mathcal{X}$ is "total" in some sense. Currently, this is an open problem.

Remark. It is possible to define Toeplitz operators T_f on $L^2_{\rm harm}$ as in §1.4 (taking $K = L^2$, and $f \in L^\infty$). One can then show that in several standard situations (like $\Omega = \mathbf{C}$ with the Gaussian measure, or $\Omega = \mathbf{D}$ with the measures as in (40)), the mapping $T \mapsto \widetilde T$ on $\mathcal{B}(L^2_{\rm harm})$ is, nonetheless, one-to-one when restricted to the Toeplitz operators; i.e., $\widetilde{T_f} = 0$ implies $f = 0$. (The same is likewise true for the analogous spaces $L^2_{\rm ph}(\Omega, d\nu)$ of *pluriharmonic* functions for any bounded rotation-invariant domain Ω and finite rotation-invariant measure ν; see [15] for the proofs of all these facts.) This seems quite remarkable, since already for commutators $[T_f, T_g]$ of two Toeplitz operators this is no longer true: indeed, since $(T_fT_g)^* = T_{\bar g}T_{\bar f}$, while $\overline{T_fT_g} = T_{\bar f}T_{\bar g}$, applying (55) to $T = T_{\bar f}T_{\bar g}$, we see that

$$(T_fT_g - T_gT_f)^\sim = 0, \qquad \forall f, g \in L^\infty. \tag{56}$$

On the other hand, it is easy to choose f, g so that $T_f T_g - T_g T_f \neq 0$. For instance, if Ω admits a nonconstant bounded holomorphic function f, then

$$\langle T_f T_{\bar f} f, f\rangle = \|T_{\bar f} f\|^2 = \|P|f|^2\|^2,$$

while

$$\langle T_{\bar f} T_f f, f\rangle = \|T_f f\|^2 = \|f^2\|^2 = \|\,|f|^2\,\|^2.$$

Since $\|P|f|^2\| = \|\,|f|^2\,\| \iff |f|^2$ is harmonic $\iff f \equiv \text{const.}$, we see that $T_f T_{\bar f} - T_{\bar f} T_f \neq 0$.

Currently, very little seems to be known about these Toeplitz operators [8, 18, 23, 26, 29], and even less about the corresponding Berezin transforms [15].

The same argument also shows that the Berezin quantization must break down if we try to use, instead of L^2_{harm}, the Sobolev spaces H^s, their harmonic subspaces H^s_{harm}, etc. — more generally, any function space which contains the complex conjugate $\bar f$ whenever it contains f and on which the complex conjugation $f \mapsto \bar f$ is an isometry.

Remark. In the holomorphic case, the injectivity of the map $T \mapsto \widetilde{T}$ stemmed from the fact that any function $f(x, y)$ holomorphic in x and conjugate-holomorphic in y is uniquely determined by its values on the diagonal $x = y$. Similarly, for the spaces L^2_A and H^s_A defined by a hypoelliptic operator A, the symbol map $T \mapsto \widetilde{T}$ will be injective as soon as A has the following property: any function $f(x, y)$ satisfying $A_x f(x, \bar y) = A_y f(x, \bar y) = 0\ \forall x, y$ and $f(x, x) = 0\ \forall x$, vanishes identically. The characterization of such operators A, however, remains rather elusive as of this writing.

3. Berezin-Toeplitz quantization via harmonic Bergman spaces

Let us now turn to the Berezin-Toeplitz quantization based on the spaces L^2_A. The Toeplitz operators, of course, are defined using $K = L^2$, so the assumption (19) is fulfilled, as well as (20). From the item II at the end of §1.9 we know that we need to verify the property (24) and the asymptotic expansion (6) for the product of two such Toeplitz operators. We are going to show that, unfortunately, this again fails already in what one would, apparently, expect to be the simplest generalization of the classical case, namely, for Toeplitz operators on the harmonic Bergman spaces

$$L^2_{\text{harm}}(\mathbf{D}, \tfrac{\nu-1}{\pi}(1-|z|^2)^{\nu-2}\,dz)$$

on the unit disc $\mathbf{D}$, where we again use the notation $\nu := \frac{1}{h}$. These are the harmonic counterparts of the Bergman spaces occurring in the traditional Berezin-Toeplitz quantization on $\mathbf{D}$ from §1.8.2.

These spaces have an orthogonal basis $\{z^{[m]}\}_{m\in\mathbf{Z}}$, where

$$z^{[m]} = \begin{cases} z^m & m \geq 0, \\ \bar z^{-m} & m \leq 0. \end{cases}$$

The norm of $z^{[m]}$ is easily computed to be $|m|!/(\nu)_{|m|}$. It follows that $e^{(h)}{}_m := \sqrt{(\nu)_{|m|}/|m|!}\, z^{[m]}$ is an orthonormal basis. Since

$$\begin{aligned}
\langle T^h_{z^k\bar z^l} z^{[m]}, z^{[n]}\rangle &= \frac{\nu-1}{\pi}\int_{\mathbf{D}} z^k \bar z^l z^{[m]} z^{[-n]} (1-|z|^2)^{\nu-2}\, dz \\
&= \frac{\nu-1}{\pi}\int_0^1\int_0^{2\pi} r^{k+l+|m|+|n|} e^{(k-l+m-n)i\theta}(1-r^2)^{\nu-2}\, d\theta\, r\, dr \\
&= \delta_{k+m,l+n}\,(\nu-1)\int_0^1 t^{(k+l+|m|+|n|)/2}(1-t)^{\nu-2}\, dt \\
&= \delta_{k+m,l+n}\frac{\left(\frac{k+l+|m|+|n|}{2}\right)!}{(\nu)_{\frac{k+l+|m|+|n|}{2}}},
\end{aligned}$$

we have

$$\langle T^h_{z^k\bar z^l} e^{(h)}{}_m, e^{(h)}{}_n\rangle = \sqrt{\frac{(\nu)_{|m|}(\nu)_{|n|}}{|m|!|n|!}}\,\delta_{k+m,l+n}\frac{\left(\frac{k+l+|m|+|n|}{2}\right)!}{(\nu)_{\frac{k+l+|m|+|n|}{2}}}.$$

Consequently,

$$T^h_{z^k\bar z^l} e^{(h)}{}_m = \sqrt{\frac{(\nu)_{|m|}(\nu)_{|m+k-l|}}{|m|!|m+k-l|!}}\,\frac{\left(\frac{k+l+|m|+|m+k-l|}{2}\right)!}{(\nu)_{\frac{k+l+|m|+|m+k-l|}{2}}}\, e^{(h)}{}_{m+k-l}. \tag{57}$$

Now if (6) were true, we would have asymptotically,

$$\|T^h_f T^h_g - T^h_{fg} - hT^h_{C_1(f,g)}\| = O(h^2)$$

as $h \searrow 0$. Applying this to $f = z$, $g = \bar z$, then vice versa, and subtracting and using (3), we get

$$\|\nu(T^h_z T^h_{\bar z} - T^h_{\bar z} T^h_z) - T^h_\phi\| \to 0 \tag{58}$$

as $\nu = 1/h \to +\infty$, where $\phi = \frac{i}{2\pi}\{z,\bar z\} = -(1-|z|^2)^2$. Now from (57) it follows that the operator $\nu(T^h_z T^h_{\bar z} - T^h_{\bar z} T^h_z)$ is diagonal with respect to the basis $\{e^{(h)}{}_m\}$, with eigenvalues

$$\begin{aligned}
c_m(\nu) :&= \nu\cdot(\|T^h_{\bar z} e^{(h)}{}_m\|^2 - \|T^h_z e^{(h)}{}_m\|^2) \\
&= \nu\left[\frac{(\nu)_{|m|}(\nu)_{|m-1|}}{|m|!|m-1|!}\,\frac{\left(\frac{|m|+1+|m-1|}{2}\right)!^2}{(\nu)^2_{\frac{|m|+1+|m-1|}{2}}} - \frac{(\nu)_{|m|}(\nu)_{|m+1|}}{|m|!|m+1|!}\,\frac{\left(\frac{|m|+1+|m+1|}{2}\right)!^2}{(\nu)^2_{\frac{|m|+1+|m+1|}{2}}}\right],
\end{aligned}$$

which are easily seen to satisfy (using Stirling's formula)

$$\lim_{\nu\to+\infty} c_m(\nu) = \begin{cases} -1 & m \ge 1 \\ 0 & m = 0 \\ +1 & m \le -1 \end{cases}.$$

Observe that the right-hand side is an odd (and nonzero) function of m. On the other hand, T^h_ϕ is also diagonal with respect to the same basis, but with eigenvalues

$$d_m(\nu) := \langle T^h_\phi e^{(h)}{}_m, e^{(h)}{}_m\rangle = \frac{(\nu)_{|m|}}{|m|!}\frac{\nu-1}{\pi}\int_{\mathbf{D}} \phi(z)|z|^{2|m|}(1-|z|^2)^{\nu-2}\,dz$$

which depend only on $|m|$. Thus (58) cannot hold. (Actually, it is not difficult to show that $d_m(\nu) = -(\nu-1)_{|m|+1}/(\nu+1)_{|m|+1}$, whence $\lim_{\nu\to+\infty} d_m(\nu) = -1\ \forall m$, so $\liminf_{h\to 0}\|\frac{1}{h}[T^h_z, T^h_{\bar z}] - T^h_\phi\| \geq 2$.)

Another way of seeing that either (24) or (6) must fail is by using (56): namely, (6) and (3) imply that

$$T^h_f T^h_g - T^h_g T^h_f = \sum_{j=1}^\infty h^j\, T^h_{C_j(f,g)-C_j(g,f)} = \frac{ih}{2\pi}\, T^h\{f,g\} + O(h^2)$$

(in operator norm). Taking operator symbols and dividing by h, we thus get

$$\frac{1}{h}(T^h_f T^h_g - T^h_g T^h_f)^\sim = \frac{i}{2\pi}\widetilde{T^h_{\{f,g\}}} + O(h) \qquad \text{as } h\searrow 0.$$

However, by (56), $(T^h_f T^h_g - T^h_g T^h_f)^\sim = 0$, so

$$\widetilde{T^h_{\{f,g\}}} \to 0 \quad \text{as } h\searrow 0.$$

Thus (24) implies that $\{f,g\} = 0\ \forall f,g\in\mathcal{V}$, which is absurd.

Again, the last argument also works for any other subspace of L^2 in the place of L^2_{harm} as long as it is preserved by complex conjugation (i.e., contains $\bar f$ whenever it contains f) and the complex conjugation is an isometry on it.

4. Berezin quantization on general Hilbert spaces of holomorphic functions

Here by "general" we mean that the scalar product need not come from plain integration; i.e., (20) need not hold. (They need not be the holomorphic subspaces of some L^2.) Thus we can still define operator symbols, and the correspondence between the operators and their symbols is still one-to-one, but in general there are no Toeplitz operators, and no Berezin transform. (Sometimes the Toeplitz operators and the Berezin transform can also be defined, but some care is needed; see section 5.)

The simplest example of such spaces are the Sobolev spaces of holomorphic functions H^s_{hol}. Another important example is the analytic continuation of the weighted Bergman spaces on a bounded symmetric domain $\Omega\subset\mathbf{C}^d$ (i.e., Hermitian symmetric space $\Omega\cong G/K$ of non-compact type) from §1.8.3. Namely, let $K(x,y)$ stand for the ordinary Bergman kernel of Ω with respect to the Lebesgue measure dx. Then the weighted Bergman spaces

$$L^2_{\mathrm{hol}}(\Omega, K(x,x)^{-\alpha}\,dx)$$

are well-defined for $\alpha > -1/p$ and have reproducing kernels $K_\alpha(x,y) = \mathrm{const}\cdot K(x,y)^{\alpha+1}$, where p is an integer called the *genus* of Ω. (These spaces are known in representation theory as the *holomorphic discrete series* of G.) It was shown by Rossi and Vergne [31] that these spaces admit an *analytic continuation* to smaller values of ν; and for these values, with a few exceptions, they are *not* subspaces of any L^2 space.

For instance, for the unit disc $\mathbf{D}$, the analytic continuation of the weighted Bergman spaces

$$\mathcal{L}_\nu := L^2_{\mathrm{hol}}(\mathbf{D}, \tfrac{\nu-1}{\pi}(1-|z|^2)^{\nu-2}\,dz), \qquad \nu > 1,$$

is given by $\mathcal{L}_\nu = \{f \text{ holomorphic on } \mathbf{D}\colon \|f\|_\nu < \infty\}$, where

$$\|f\|^2_\nu = \sum_{j=0}^\infty \frac{j!}{(\nu)_j}|f_j|^2 \qquad \text{if } f(z) = \sum_{j=0}^\infty f_j z^j,$$

for any $\nu > 0$; here $(\nu)_j$ is the Pochhammer symbol (41). For $\nu = 1$, this is the Hardy space on the unit circle $\mathbf{T}$; for $0 < \nu < 1$, these spaces are not subspaces of any L^2 space.

4.1. New scalar product

Our idea will be to get such spaces from "known" ones by a deformation of the scalar product.

Consider, quite generally, a Hilbert space $\mathcal{H}$ of functions on Ω, and let M be any self-adjoint operator on $\mathcal{H}$ (usually unbounded) which is *positive* in the sense that

$$\langle Mx, x\rangle > 0 \qquad \forall x \in \operatorname{dom} M,\ x \neq 0. \tag{59}$$

Define a new scalar product on $\operatorname{dom} M$ by

$$\langle f, g\rangle_M := \langle Mf, g\rangle$$

and let $\mathcal{H}_M$ be the completion of $\operatorname{dom} M$ with respect to the corresponding norm. Note that if M is bounded (so that $\operatorname{dom} M = \mathcal{H}$), then $\mathcal{H}_M \supset \mathcal{H}$; on the other hand, if M^{-1} is bounded, then $\mathcal{H}_M$ is contained in $\mathcal{H}$ and coincides (as a set) with the domain of $M^{1/2}$. (In our applications, either M or M^{-1} will always be bounded.)

Examples. For $\mathcal{H} = L^2(\mathbf{R}^n)$ and $M = (I-\Delta)^s$, with Δ the Laplace operator, we get the familiar Sobolev spaces $\mathcal{H}_M = H^s(\mathbf{R}^n)$, $s \in \mathbf{R}$.

For $\mathcal{H} = L^2_{\mathrm{hol}}(\Omega)$ and $M = (I + D^*D)^s$, where $D : f \mapsto (\frac{\partial f}{\partial z_j})_{j=1}^d$ is the operator of holomorphic differentiation from L^2_{hol} into $\bigoplus^d L^2_{\mathrm{hol}}$, we get the holomorphic Sobolev spaces $\mathcal{H}_M = H^s_{\mathrm{hol}}(\Omega)$.

Now assume that $\mathcal{H}$ has a reproducing kernel $K(x,y)$. Then we have

$$f(x) = \langle f, K_x\rangle = \langle Mf, M^{-1}K_x\rangle = \langle f, M^{-1}K_x\rangle_M.$$

Thus if $K_y \in \operatorname{dom} M^{-1}$ for all y, then we see that $\mathcal{H}_M$ has also a reproducing kernel, $L(x,y)$, given by

$$L_y = M^{-1}K_y, \quad \text{or} \quad L(x,y) = \langle M^{-1}K_y, K_x\rangle.$$

(This also shows, in particular, that $\mathcal{H}_M$ is again a space of functions on Ω.)

Remark. The condition $K_y \in \operatorname{dom} M^{-1}$ can in fact be relaxed to

$$K_y \in \operatorname{dom} M^{-1/2}. \tag{60}$$

See [16, section 4] for the details.

Clearly, if $\mathcal{H}$ is a space of *holomorphic* functions, then so will be $\mathcal{H}_M$ (since L_y depends anti-holomorphically on y if K_y does). Further, the condition (12) will be satisfied for $\mathcal{H}_M$ if it is satisfied for $\mathcal{H}$.

The operator symbols on $\mathcal{H}_M$ can be expressed in terms of those on $\mathcal{H}$:

$$\begin{aligned}\widetilde{T}^M(x) &= \frac{\langle TL_x, L_x\rangle_M}{\langle L_x, L_x\rangle_M} = \\ &= \frac{\langle MTM^{-1}K_x, M^{-1}K_x\rangle}{\langle MM^{-1}K_x, M^{-1}K_x\rangle} \\ &= \frac{\widetilde{TM^{-1}}(x)}{\widetilde{M^{-1}}(x)}.\end{aligned} \tag{61}$$

(In particular, T is uniquely determined by $\widetilde{T}^M$ whenever it is uniquely determined by $\widetilde{T}$.) This implies also the relation between the associated products $*_{\mathcal{H}} =: *$ and $*_{\mathcal{H}_M} =: *_M$: by definition, $\widetilde{T}^M *_M \widetilde{S}^M = \widetilde{TS}^M$, which translates into

$$f *_M g = \frac{qf *_{\mathcal{H}} \widetilde{M} *_{\mathcal{H}} qg}{q}, \tag{62}$$

where we have denoted, for the sake of brevity,

$$q := \widetilde{M^{-1}}.$$

An operator T on $\mathcal{H}$ is bounded on $\mathcal{H}_M$ if and only if $\langle MTx, Tx\rangle \le c\langle Mx, x\rangle$ $\forall x \in \operatorname{dom} M$, for some finite c; that is, if and only if $M^{1/2}TM^{-1/2}$ is bounded on $\mathcal{H}$. The mapping $T \mapsto M^{-1/2}TM^{1/2}$ is thus an isomorphism (except that it does not

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preserve involution) from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H}_M)$, and the algebra $\mathcal{A}_{\mathcal{H}_M}$ of all symbols of bounded operators on $\mathcal{H}_M$ is related to $\mathcal{A}_{\mathcal{H}}$ by

$$\mathcal{A}_{\mathcal{H}_M} = \{(\widetilde{M^{-1/2}} *_{\mathcal{H}} f *_{\mathcal{H}} \widetilde{M^{-1/2}})/q;\ f \in \mathcal{A}_{\mathcal{H}}\}. \tag{63}$$

Finally, if $*_{\mathcal{H}}$ has the property of separation of variables, then so has $*_M$.

In particular, if $\mathcal{H}$ is one of our weighted holomorphic Bergman spaces $L^2_{h,\mathrm{hol}}$, *which we will assume from now on*, and M any positive selfadjoint operator thereon such that (60) holds, then $\mathcal{H}_M$ will always be a reproducing kernel Hilbert space of holomorphic functions and $*_M$ will have the separation of variables.

Observe that for f, g holomorphic we may by (30) also define the M-Berezin transform by

$$B^M(\bar f g) =: \bar f *_M g = \frac{q\bar f *_{\mathcal{H}} \widetilde{M} *_{\mathcal{H}} qg}{q}, \tag{64}$$

and M-Toeplitz operators by

$$(T^M_{\bar f g})^{\sim M} = B^M(\bar f g) \qquad \text{(if such operator exists).} \tag{65}$$

Example 4.1. Take $\mathcal{H} = L^2_{\mathrm{hol}}(\mathbf{D}, \frac{1}{\pi}dz)$ and

$$M : z^n \mapsto (n+1)z^n,$$

i.e., $Mf = (zf)'$. Then $\langle z^m, z^n\rangle_M = \delta_{nm} = \langle z^m, z^n\rangle_{H^2}$, where H^2 is the Hardy space on the unit circle $\mathbf{T}$. Hence $\mathcal{H}_M = H^2$, $L(x,y) = 1/(1-\bar y x)$ (the Szegö kernel). The Berezin transform can be computed to be

$$B^M(z^n \bar z^m) = z^{[n-m]} = \begin{cases} z^{n-m} & n \geq m, \\ \bar z^{m-n} & n \leq m; \end{cases}$$

that is,

$$B^M f = \text{the Poisson extension of } f|_{\mathbf{T}}$$

for any polynomial f in $z, \bar z$. Similarly, for such f, T^M_f turns out to be the usual (Hardy space) Toeplitz operator with symbol $f|_{\mathbf{T}}$. (For more general functions f, $B^M f$ and T^M_f need not be defined.)

Example 4.2. Take again the same $\mathcal{H}$ but now with

$$M : z^n \mapsto \frac{(n+1)!}{(\nu)_n}\, z^n, \qquad \nu > 0.$$

Then $\mathcal{H}_M$ are the analytic continuation of the weighted Bergman spaces $\mathcal{L}_\nu \equiv L^2_{\mathrm{hol}}(\mathbf{D}, \frac{\nu-1}{\pi}(1-|z|^2)^{\nu-2}\,dz)$, $\nu > 1$, mentioned above.

Example 4.3. Take the same $\mathcal{H}$ but with

$$M : z^n \mapsto \frac{(n+1)! \cdot [1 + n(n+\nu-1)]}{(\nu)_n}\, z^n.$$

Then $\langle f, g\rangle_M = \langle f, g\rangle_\nu + \langle f', g'\rangle_\nu$, so $\mathcal{H}_M$ coincides with the holomorphic Sobolev space $H^1_{\mathrm{hol}}(\mathbf{D}, \frac{\nu-1}{\pi}(1-|z|^2)^{\nu-2}\,dz) := \{\, f \in \mathcal{L}_\nu \ :\ f' \in \mathcal{L}_\nu \,\}$.

4.2. Applications to quantization

Fix now a weighted Bergman space $\mathcal{H}$, take a family of operators M_h on it, and let $\mathcal{H}_h := \mathcal{H}_{M_h}$, $*_h$, B_h, etc., be the corresponding M-objects. Can we find M_h so that as $h \to 0$, the spaces $\mathcal{H}_h$ satisfy the conditions I(a)–I(c) from §1.9, and thus can be used to define a Berezin star-product?

Example 4.4. Consider the weighted Bergman spaces $L^2_{h,\mathrm{hol}} = L^2_{\mathrm{hol}}(\Omega, \mu_h)$ from the traditional Berezin quantization, and let $\mathcal{H} := L^2_{h_0,\mathrm{hol}}$ for some chosen (fixed) value of h_0. Let $\iota_h : \mathcal{H} \to L^2_{h,\mathrm{hol}}$ stand for the identity mapping $f \mapsto f$, considered as a mapping from $\mathcal{H}$ into $L^2_{h,\mathrm{hol}}$. Assume that ι_h is densely defined and has dense range (that is, $\mathcal{H} \cap L^2_{h,\mathrm{hol}}$ is dense in both $\mathcal{H}$ and $L^2_{h,\mathrm{hol}}$; except for the compact Kähler manifolds, this is satisfied in all situations where the traditional Berezin quantization is known to work, as soon as h_0 is so small that $\mathcal{H}$ does not reduce to the constant zero). Then

$$M_h := \iota_h^* \iota_h$$

is a densely defined, positive selfadjoint operator, so we can apply to it our construction. Since $\langle M_h f, f\rangle_{\mathcal{H}} = \langle f, f\rangle_{L^2_{h,\mathrm{hol}}}$, we have $\mathcal{H}_h = L^2_{h,\mathrm{hol}}$. Thus we see that the spaces $L^2_{h,\mathrm{hol}}$ from the traditional Berezin quantization can be recovered by the above construction.

Example 4.5 (More concrete). Take $\Omega = \mathbf{C}$, $\mathcal{H} = L^2_{\mathrm{hol}}(\mathbf{C}, e^{-|z|^2}\pi^{-1}\,dz)$, and

$$M_h : z^j \mapsto h^j z^j \qquad \text{(i.e., } M_h f(z) = f(hz)\text{)}.$$

Then we have

$$\begin{aligned}
\mathcal{H}_h &= L^2_{\mathrm{hol}}(\mathbf{C}, e^{-|z|^2/h}(\pi h)^{-1}\,dz) = L^2_{h,\mathrm{hol}},\\
\widetilde{M}_h(z) &= e^{(h-1)|z|^2},\\
\widetilde{M_h^{-1}}(z) &= e^{(h^{-1}-1)|z|^2},
\end{aligned}$$

and

$$f *_h g = e^{(1-h^{-1})|z|^2}\big[e^{(h^{-1}-1)|z|^2} f *_{\mathcal{H}} e^{(h-1)|z|^2} *_{\mathcal{H}} e^{(h^{-1}-1)|z|^2} g\big].$$

(Note that we had to go outside the realm of formal power series in h!)

Taking, for instance, $f = z^k \bar{z}^l$, $g = z^m \bar{z}^n$, $m > l$, an explicit computation shows that

$$\begin{aligned}
z^k\bar z^l *_h z^m\bar z^n &= z^k\bar z^n e^{(1-h^{-1})|z|^2}\big[M_h^{-1}T^*_{z^l}M_hT_{z^m}M_h^{-1}\big]^\sim \\
&= z^k\bar z^n e^{(1-h^{-1})|z|^2} z^{m-l}h^l\frac{m!}{(m-l)!}e^{-|z|^2}\ {}_1F_1(m+1;m-l+1;h^{-1}|z|^2) \\
&= z^{m+k}\bar z^{n+l}\frac{m!}{(m-l)!}\Big(\frac{h}{|z|^2}\Big)^l e^{-h^{-1}|z|^2}\ {}_1F_1(m+1;m-l+1;h^{-1}|z|^2) \\
&= z^{m+k}\bar z^{n+l}\frac{m!}{(m-l)!}\Big(\frac{h}{|z|^2}\Big)^l\ {}_1F_1(-l;m-l+1;-h^{-1}|z|^2) \qquad \text{by [17, 6.3 (7)]} \\
&= z^{m+k}\bar z^{n+l}l!\Big(\frac{h}{|z|^2}\Big)^l L_l^{m-l}(-h^{-1}|z|^2) \qquad \text{by [17, 6.9 (36)]} \\
&= z^{m+k}\bar z^{n+l}\sum_{j=0}^{l}\frac{l!m!}{j!(l-j)!(m-j)!}\frac{h^j}{|z|^{2j}},
\end{aligned}$$

where ${}_1F_1$ is the confluent hypergeometric function and L_n^α are the Laguerre polynomials. Thus

$$z^k\bar z^l *_h z^m\bar z^n = \sum_{j=0}^\infty \frac{h^j}{j!}\frac{\partial^j}{\partial\bar z^j}(z^k\bar z^l)\cdot\frac{\partial^j}{\partial z^j}(z^m\bar z^n).$$

Consequently, taking for $\mathcal{A}_0$ the set of all polynomials $f\equiv f_h(z)$ in h, z, and $\bar z$, we see that $\forall f,g\in\mathcal{A}_0$

$$f *_h g = \sum_{j=0}^\infty h^j\,C_j(f,g), \qquad \text{with} \quad C_j(f,g)=\frac{1}{j!}\frac{\partial^j f}{\partial\bar z^j}\frac{\partial^j g}{\partial z^j}.$$

Thus (17) holds. Since (16) and the totality of $\mathcal{A}_0$ are obvious, we thus obtain a differential star-product on $\mathbf{C}$ (identical with the traditional Berezin star-product, which in this case is equivalent to the classical Moyal product on $\mathbf{C}$).

In general, proving the validity of (17) for a given family M_h is quite difficult. (Even in the special case of the preceding two examples, it is tantamount to proving the correct semiclassical behavior of the traditional Berezin quantization!) It is therefore of advantage to take another route: namely, to try to establish that if we have one family M_h which works, then we can get another one by modifying the former in a suitable way. Since for the former we can take the operators M_h corresponding to the traditional Berezin quantization (Example 4.4), this will solve our problem.

Thus let as usual Ω be a domain with a Kähler metric admitting a global potential and assume that the traditional Berezin quantization based on the weighted Bergman spaces $L^2_{h,\mathrm{hol}}=L^2_{\mathrm{hol}}(\Omega,\mu_h)$ works on Ω. Assume further that for each h, U_h is a positive self-adjoint operator on $L^2_{h,\mathrm{hol}}$ such that

$$\widetilde{U_h}=\sum_{j=0}^\infty h^j u_j \qquad \text{as } h\searrow 0$$

for some $u_j \in C^\infty(\Omega)$, $u_0 > 0$. Consider the spaces

$$L_h := (L^2_{h,\mathrm{hol}})_{U_h}. \tag{66}$$

(Thus, in contrast to Examples 4.4 and 4.5, now both the operator M and the space $\mathcal{H}$ vary with h.) Let us now see whether these spaces can be used instead of $L^2_{h,\mathrm{hol}}$ for performing the Berezin quantization. In view of the results in §4.1, the items I(a) and I(b) from §1.9 are always satisfied; thus we only need to check if we can find a total set $\mathcal{A}_0$ for which (16) and (17) hold.

To prevent confusion, we keep the notations $*_h, *, \mathcal{A}_h$ and $\mathcal{A}_0$ for the objects associated to $L^2_{h,\mathrm{hol}}$, and denote by $*'_h, *', \mathcal{A}'_h$ and $\mathcal{A}_0{}'$ the ones pertaining to the spaces (66). We also keep the notation C_j for the bidifferential operators C_j from the traditional Berezin quantization, denoting those corresponding to (66) (provided we establish their existence) by C'_j.

We claim, first of all, that once a suitable total set $\mathcal{A}_0{}'$ has been found so that (16) and (17) hold, the resulting operators C'_j will automatically satisfy (2)–(4). Indeed, let us denote, for brevity,

$$u = \sum_{j=0}^{\infty} h^j u_j \in C^\infty(\Omega)[[h]].$$

Let q be the inverse of u in $C^\infty(\Omega)[[h]]$ with respect to $*$; the latter exists since $u_0 > 0$ does not vanish by hypothesis, and thus is invertible as an element of $C^\infty(\Omega)$. Moreover, $q_0 = 1/u_0$. Let further r be the formal square root of u in $C^\infty(\Omega)[[h]]$, i.e., $r * r = u$; again, r exists since $u_0 > 0$ is an element of $C^\infty(\Omega)$ having a smooth square root, and $r_0 = \sqrt{u_0}$. Now from $r * r = u$ and $q * u = \mathbf{1}$, together with (62) and (63), it follows that the star-products $*$ and $*'$ on $C^\infty(\Omega)[[h]]$ are equivalent

$$V(f *' g) = Vf * Vg \tag{67}$$

via the operator

$$Vf := r * qf * r = f + hV_1 f + h^2 V_2 f + \cdots .$$

By the same argument as in (48) and (49), the desired claim follows.

Thus to show that the spaces (66) can be used for Berezin quantization, it is enough to find a total subset $\mathcal{A}_0{}' \subset \oplus_h \mathcal{A}'_h$ such that each $f \in \mathcal{A}_0{}'$, as well as $f *' g$ for each $f, g \in \mathcal{A}_0{}'$, have an asymptotic expansion in nonnegative powers of h as $h \searrow 0$.

This is accomplished by the following theorem. We formulate it only for the Fock space and the disc, though it probably remains valid in more general situations.

Recall that a function on $\mathbf{C}^d$ or $\mathbf{D}$ is called *radial* if it depends only on $|z_1|, \dots, |z_d|$ or $|z|$, respectively. For any radial function w, the Toeplitz operators T^h_w are diagonal with respect to the standard basis of monomials; we will call the function w *temperate* if the eigenvalues c_α of T^h_w on z^α satisfy

$$\sup_\alpha \frac{c_\alpha}{c_{\alpha+\beta}} < \infty, \qquad \sup_\alpha \frac{c_{\alpha+\beta}}{c_\alpha} < \infty \tag{68}$$

for any h and any multiindex β. (Here and below, for the unit disc "multiindex" just means "nonnegative integer.")

Theorem 4.6. *Consider either the Fock spaces on $\Omega = \mathbf{C}^d$ from §1.8.1, or the spaces $L^2_{h,\mathrm{hol}}$ on $\Omega = \mathbf{D}$ from §1.8.2. Let $w > 0$ be a positive temperate radial function in $L^\infty(\Omega) \cap C^\infty(\Omega)$, and for each $h > 0$ set*

$$U_h = (T^h_w)^{-1}.$$

Denote by ${\mathcal{A}_0}'$ the set of all linear combinations of functions of the form $h^n \widetilde{T^h_{z^l \bar{z}^m \phi(z)}}^{U_h}$, where $n \geq 0$, l, m are multiindices (for $\mathbf{C}^d$) or $l, m \geq 0$ (for $\mathbf{D}$), and $\phi \in \mathcal{D}(\Omega)$ is radial. Then the spaces (66) *with the total set ${\mathcal{A}_0}'$ define a Berezin quantization on Ω in the strong form.*

Remark. Of course, the last theorem is a complete triviality on the level of formal power series, in view of the equivalence (67) — one could even forget about the complicated definitions of ${\mathcal{A}_0}'$ and special forms of U_h. The point is to make things work even on the rigorous (not only formal power series) level.

Proof. First of all, note that $w > 0$ implies that T^h_w is positive:

$$\langle T^h_w f, f\rangle = \int_\Omega w|f|^2 \, d\mu_h \geq 0 \quad \forall f \in L^2_{h,\mathrm{hol}}, \qquad \text{with equality iff } f = 0. \tag{69}$$

Hence T^h_w is injective and its inverse $(T^h_w)^{-1} = U_h$ is also positive. Thus U_h satisfies the condition (59) and the definition (66) makes sense. Also, as U_h^{-1} is bounded, the condition (60) is trivially fulfilled.

Second, we claim that the operators $T^h_{z^l \bar{z}^m \phi(z)}$ as above are bounded on the spaces L_h (that was the sole reason behind such a complicated definition of ${\mathcal{A}_0}'$). Indeed, we know from §4.1 that this is equivalent to $U_h^{1/2} T^h_{z^l \bar{z}^m \phi(z)} U_h^{-1/2}$ being bounded on $L^2_{h,\mathrm{hol}}$. However, the latter is clear since

$$\langle U_h^{1/2} T^h_{z^l \bar{z}^m \phi(z)} U_h^{-1/2} z^\alpha, z^\beta \rangle = \sqrt{\frac{c_\alpha}{c_\beta}} \, \langle T^h_\phi z^{l+\alpha}, z^{m+\beta}\rangle,$$

and the second factor on the right is nonzero only if $\alpha - \beta = m - l$; thus the boundedness follows from the boundedness of $T^h_{z^l \bar{z}^m \phi}$ and (68).

Hence ${\mathcal{A}_0}' \subset \mathcal{A}'$.

Now let $k \geq 1$ and let $f_1, \dots, f_k \in L^\infty \cap C^\infty$ be any functions such that $T^h_{f_j}$ are bounded on L_h. Then by the definition of $*'_h$ and (61),

$$\begin{aligned} \widetilde{T^h_{f_1}}^{U_h} *'_h \widetilde{T^h_{f_2}}^{U_h} *'_h \cdots *'_h \widetilde{T^h_{f_k}}^{U_h} &= [T^h_{f_1} \cdots T^h_{f_k}]^{\sim U_h} \\ &= \frac{[T^h_{f_1} \cdots T^h_{f_k} T^h_w]^\sim}{\widetilde{T^h_w}}. \end{aligned}$$

(Note that $\widetilde{T^h_w} > 0$ by (69).) Since both the numerator and the denominator of the last formula have asymptotic expansions as $h \searrow 0$, by Theorem 1.2, so does the left-hand side. This establishes (51), (16) (upon taking $k = 1$), and, in view of the discussion preceding the theorem, also (17) (upon taking $k = 2$).

Finally, as, in particular,

$$\widetilde{T^h_f}^{U_h} = \frac{\widetilde{T^h_f} * \widetilde{T^h_w}}{\widetilde{T^h_w}} = \frac{fw + O(h)}{w + O(h)} = f + O(h),$$

the totality of the set $\mathcal{A}_0{}'$ follows from the (evident) totality of the set of all linear combinations of the functions $h^n z^l \bar{z}^m \phi(z)$.

The proof is complete. □

Remark. For the Fock space and $w \in BC^\infty(\mathbf{C}^d)$, an alternative proof of Theorem 4.6 may be given based on the fact that bidifferential operators C_j from the traditional Berezin-Toeplitz quantization (38) obviously preserve BC^∞ (i.e., (52) holds for $\mathcal{V} = BC^\infty$) and the observation before Theorem 1.2. Similarly for the unit disc and $w \in C^\infty(\overline{\mathbf{D}})$, by (44).

4.3. Some examples

As our first application of Theorem 4.6, consider the Fock space $L^2_{h,\mathrm{hol}} = L^2_{\mathrm{hol}}(\mathbf{C}, e^{-|z|^2/h}(\pi h)^{-1} dz)$ on $\mathbf{C}$ and take

$$w(z) = (1 + |z|^2)^{-s}$$

with some $s > 0$. Then T^h_w is a diagonal operator with respect to the standard monomial orthogonal basis $\{z^k\}$, with eigenvalues

$$\begin{aligned} c_k = \frac{\langle T^h_w z^k, z^k\rangle}{\langle z^k, z^k\rangle} &= \frac{1}{k! h^k} \int_0^\infty (1+t)^{-s} t^k e^{-t/h} h^{-1}\, dt \\ &= \frac{1}{k!} \int_0^\infty (1+yh)^{-s} y^k e^{-y}\, dy. \end{aligned}$$

It follows that, for each fixed h, $c_k \sim \text{const} \cdot k^{-s}$ as $k \to \infty$. Hence, in particular, $\frac{c_k}{c_{k+1}}$ is positive and tends to one, so w is temperate. Thus Theorem 4.6 applies, and we obtain a Berezin quantization of $\mathbf{C}$ based on the spaces L_h from (66). Note that for s an integer and $k \geq s$,

$$\begin{aligned} \langle z^k, z^k\rangle_{L_h} &= \langle U_h z^k, z^k\rangle_{L^2} = \langle c_k^{-1} z^k, z^k\rangle_{L^2} \approx (k+1)^s \langle z^k, z^k\rangle_{L^2} \\ &= (k+1)^s k! h^k \approx \sum_{j=0}^s k^j\, k! h^k \approx \sum_{j=0}^s \frac{k!^2}{(k-j)!^2} (k-j)! h^{k-j} \\ &= \sum_{j=0}^s \langle (z^k)^{(j)}, (z^k)^{(j)}\rangle_{L^2} = \langle z^k, z^k\rangle_{H^s}, \end{aligned}$$

for each fixed $h > 0$; hence L_h coincides as a set with the holomorphic Sobolev space $H^s_{\mathrm{hol}}(\mathbf{C}, e^{-|z|^2/h}dz)$, with equivalent norms. We have thus arrived at a Berezin star product based on Sobolev spaces of holomorphic functions.

The same construction works also for the Fock spaces on $\mathbf{C}^d$ with $d > 1$, using the function $w(z) = \prod_{j=1}^d (1 + |z_j|^2)^{-s}$.

For a second example, consider the disc $\mathbf{D}$ with the usual spaces $L^2_{h,\mathrm{hol}} = L^2_{\mathrm{hol}}(\mathbf{D}, \frac{\nu-1}{\pi}(1-|z|^2)^{\nu-2}dz)$, $\nu = \frac{1}{h}$, and take

$$w(z) = (1-|z|^2)^{2s}, \qquad s > 0.$$

Again, T^h_w is a diagonal operator with respect to the standard basis, with eigenvalues

$$c_k = \frac{(\nu)_k}{k!}\,(\nu-1)\int_0^1 t^k(1-t)^{2s+\nu-2}\,dt = \frac{(\nu-1)_{k+1}}{(\nu+2s-1)_{k+1}} \sim (k+1)^{-2s}.$$

As before, it follows that w is temperate, and thus the last theorem applies and yields a Berezin quantization of $\mathbf{D}$ based on the corresponding spaces (66). Note that again, for s an integer and $k \geq s$,

$$\begin{aligned}\langle z^k, z^k\rangle_{L_h} &= c_k^{-1}\langle z^k, z^k\rangle_{L^2} \approx (k+1)^{2s}\,\frac{k!}{(\nu)_k} \approx \sum_{j=0}^s k^{2j}\,\frac{k!}{(\nu)_k} \approx \sum_{j=0}^s k^{2j}\,k^{1-\nu} \\ &\approx \sum_{j=0}^s k^{2j}\,(k-j)^{1-\nu} \approx \sum_{j=0}^s \frac{k!^2}{(k-j)!^2}\,\langle z^{k-j}, z^{k-j}\rangle_{L^2} = \langle z^k, z^k\rangle_{H^s},\end{aligned}$$

for each fixed ν, so that L_h again coincides with the holomorphic Sobolev space $H^s_{\mathrm{hol}}(\mathbf{D}, (1-|z|^2)^{\nu-2}dz)$, with equivalent norms.

4.4. Concluding remarks

Remark. We have seen in Example 4.4 that for $\Omega = \mathbf{D}$, the unit disc, and $\mathcal{H} = L^2_{\mathrm{hol}}(\mathbf{D}, \pi^{-1}\,dz)$, the ordinary (unweighted) Bergman space, the operators M_h corresponding to the passage from $\mathcal{H}$ to $L^2_{h,\mathrm{hol}} = L^2_{\mathrm{hol}}(\mathbf{D}, \frac{h^{-1}-1}{\pi}(1-|z|^2)^{h^{-1}-2}\,dz)$ of the traditional Berezin quantization are given by $M_h z^j = \frac{(j+1)!}{(1/h)_j} z^j$. Similarly, we have seen in §4.3 that the operator M corresponding to the passage from $\mathcal{H}$ to the holomorphic s-th order Sobolev space $H^s_{\mathrm{hol}}(\mathbf{D}, \pi^{-1}\,dz)$ is given by $Mz^j = m_j z^j$, where

$$\begin{aligned}m_j = \frac{\|z^j\|^2_{H^s_{\mathrm{hol}}}}{\|z^j\|^2_{\mathcal{H}}} &= \frac{1}{\|z^j\|^2_{\mathcal{H}}}\sum_{k=0}^s \frac{j!^2}{(j-k)!^2}\,\|z^{j-k}\|^2_{\mathcal{H}} \\ &= \sum_{j=0}^s \frac{j!(j+1)!}{(j-k)!(j-k+1)!} \asymp (j+1)^{2s}\end{aligned}$$

(at least for s a nonnegative integer). Since $\frac{(j+1)!}{(\nu)_j} \asymp (j+1)^{2-\nu}$, we see that $L^2_{h,\mathrm{hol}}$ coincides with H^s_{hol} for $s = 1 - \frac{1}{2h}$, with equivalent norms. The Berezin quantization $h \to 0$ can thus be equivalently viewed as a quantization using Sobolev spaces H^s_{hol} with $s \to -\infty$ (suitably renormalized). Unfortunately, for the Fock space this analogy breaks down.

Remark. We finish by commenting on a natural question, namely: which reproducing kernel spaces of holomorphic functions, on a given domain Ω, can be obtained by the "deformation" construction from the beginning of this section?

The answer is: essentially, *all of them.* Indeed, take $\mathcal{H} := L^2_{\mathrm{hol}}(\mathbf{C}^d, e^{-|z|^2}\,dz)$; this space contains all polynomials and they are dense in it. Thus if $\mathcal{K}$ is any other reproducing kernel Hilbert space of holomorphic functions on a domain $\Omega \subset \mathbf{C}^d$, such that the polynomials and contained and dense in it, then $\mathcal{H} \cap \mathcal{K}$ is dense in both $\mathcal{H}$ and $\mathcal{K}$. Thus the argument from Example 4.4 can be applied to show that $\mathcal{K} = (\mathcal{H})_M$ for $M = \iota^*\iota$, with ι the restriction map from $\mathcal{H}$ into $\mathcal{K}$.

Remark. Related question: which star-products can be obtained from the Berezin star-product using the machinery from this section?

Certainly not all, since in view of (67) they must be equivalent to the star product we have started with, and also have separation of variables (by §4.1). Currently, the answer to this question is unclear.

It should be noted that for a somewhat analogous problem for the Berezin-Toeplitz quantization — namely, which star-products can be obtained from the recipe

$$Q^{(h)}_f Q^{(h)}_g = \sum_{j=0}^{\infty} h^j\, Q^{(h)}_{C_j(f,g)} \qquad \text{as } h \to 0$$

(in the sense of operator norms) where $f \mapsto Q_f$ is a mapping from $\mathcal{D}(\Omega)$ to (bounded linear) operators on $L^2_{h,\mathrm{hol}}$ — the answer is known in the case of symmetric spaces Ω and invariant star-products: then *any* G-equivalent star product can be obtained in this way. See [12, 14]. It is a conjecture that the assertion remains in force even in general (i.e., for non-symmetric spaces).

5. Berezin-Toeplitz quantization on holomorphic Sobolev spaces

We proceed to discuss the remaining case of the Berezin-Toeplitz quantization on general spaces of holomorphic functions, and as before we start with the simplest ones among them, the holomorphic Sobolev spaces H^s_{hol}. Of course, for the larger space K in (19) we take the whole Sobolev spaces H^s, so that the Toeplitz operators are defined by $T_f\phi = P(f\phi)$ where P is the projection in H^s onto its holomorphic subspace H^s_{hol}. Now one cannot take an arbitrary L^∞ functions for the symbol f, but only those for which the multiplication by f maps H^s_{hol} into H^s; thus, for instance, any f such that all its derivatives of orders $\le s$ exist and are bounded will be fine. Note that since (20) is violated, these Toeplitz operators do not usually have properties as nice as before; for instance, $T_f^* \ne T_{\bar f}$ in general.

We actually offer only a single result, concerning the simplest case of the holomorphic Sobolev spaces $H^s_{h,\mathrm{hol}}$ on the complex plane $\mathbf{C}$. We will also again assume that the order s is an integer and $s \ge 0$. That is, we consider the spaces

$$L_h = H^s_{h,\mathrm{hol}} := H^s_{\mathrm{hol}}(\mathbf{C}, e^{-|z|^2/h}(\pi h)^{-1} dz),$$

equipped with the norm

$$\|f\|_s^2 := \sum_{j=0}^{s} \|\partial^j f\|^2, \tag{70}$$

where the norms on the right are taken in $L^2(\mathbf{C}, e^{-|z|^2/h}\frac{dz}{\pi h})$. The following result was quite a surprise for the author.

Theorem 5.1. *For any $f \in BC^\infty(\mathbf{C})$ and $h > 0$, the Toeplitz operator T_f^h on $H^s_{h,\mathrm{hol}}$ is just the restriction to $H^s_{h,\mathrm{hol}} \subset L^2_{h,\mathrm{hol}}$ of the Toeplitz operator T_f^h on $L^2_{h,\mathrm{hol}}$.*

Proof. To avoid confusion, let us temporarily denote the Toeplitz operators on $H^s_{h,\mathrm{hol}}$ by $V^{(h)}{}_f$, while keeping the notation T_f^h for the Toeplitz operators on $L^2_{h,\mathrm{hol}}$. Thus we want to show that $V^{(h)}{}_f = T_f^h|_{H^s_{h,\mathrm{hol}}}$. Since the monomials form an orthogonal basis both in $L^2_{h,\mathrm{hol}}$ and in $H^s_{h,\mathrm{hol}}$, we have

$$V^{(h)}{}_f z^k = \sum_{j=0}^{\infty} \frac{\langle f z^k, z^m\rangle_s}{\langle z^m, z^m\rangle_s}\, z^m, \qquad T_f^h z^k = \sum_{j=0}^{\infty} \frac{\langle f z^k, z^m\rangle_0}{\langle z^m, z^m\rangle_0}\, z^m,$$

so it suffices to show that

$$\frac{\langle f z^k, z^m\rangle_s}{\langle z^m, z^m\rangle_s} = \frac{\langle f z^k, z^m\rangle_0}{\langle z^m, z^m\rangle_0}, \qquad \forall k, m \ge 0. \tag{71}$$

Indeed, then $V^{(h)}{}_f z^k = T_f^h z^k$ for all k, and the assertion will follow in view of the continuity of $V^{(h)}{}_f$ and T_f^h on their corresponding spaces.

Now

$$\langle f z^k, z^m\rangle_0 = \int_{\mathbf{C}} f(z)\, z^k\, \bar z^m\, e^{-|z|^2/h}\, \frac{dz}{\pi h}.$$

Let us integrate by parts on the right-hand side. Owing to the factor $e^{-|z|^2/h}$, the boundary term vanishes in view of the boundedness of f (it would be enough that $f = O(e^{|z|^{2-\delta}})$, $\delta > 0$). We therefore obtain

$$\begin{aligned}\langle f z^k, z^m\rangle_0 &= \int_{\mathbf{C}} \partial(z^k f)\, \bar z^m\, \frac{h}{\bar z} e^{-|z|^2/h}\, \frac{dz}{\pi h} \\ &= \frac{h}{m} \int_{\mathbf{C}} \partial(z^k f)\, \overline{\partial z^m}\, e^{-|z|^2/h}\, \frac{dz}{\pi h} = \frac{h}{m} \langle \partial(z^k f), \partial z^m\rangle_0.\end{aligned}$$

Repeating the partial integrations, we thus see that

$$\langle fz^k, z^m\rangle_0 = \frac{(m-j)!h^j}{m!}\langle \partial^j(z^k f), \partial^j z^m\rangle_0$$

for any $f \in BC^\infty$ and any $j \le m$. Consequently,

$$\begin{aligned}\langle fz^k, z^m\rangle_s &= \sum_{j=0}^{\min(m,s)} \langle \partial^j(fz^k), \partial^j z^m\rangle_0 \\ &= \sum_{j=0}^{\min(m,s)} \frac{m!}{(m-j)!h^j}\,\langle fz^k, z^m\rangle_0 \\ &\equiv c_{m,s,h}\,\langle fz^k, z^m\rangle_0.\end{aligned}$$

Setting $f = \mathbf{1}$, $k = m$, we get, in particular,

$$\langle z^m, z^m\rangle_s = c_{m,s,h}\,\langle z^m, z^m\rangle_0.$$

Since the constants $c_{m,s,h}$ do not depend on f and k, (71) follows, and the proof is complete. □

Corollary. *For any integer order $s > 0$, the Berezin-Toeplitz quantization works for the spaces $L_h = H^s_{\mathrm{hol}}(\mathbf{C}, e^{-|z|^2/h}\frac{dz}{\pi h})$ on $\mathbf{C}$, with the same total set as for the traditional Berezin-Toeplitz quantization in §1.8.1; moreover, the resulting star-product is identical to the one from §1.8.1 (coming from the spaces $L^2_{h,\mathrm{hol}}$).*

The situation described in the last theorem is probably highly untypical, and it certainly does not prevail even for the unit disc. Namely, if we consider the spaces

$$L_h = H^s_{h,\mathrm{hol}} \equiv H^s_{\mathrm{hol}}(\mathbf{D}, \tfrac{\nu-1}{\pi}(1-|z|^2)^{\nu-2}dz), \tag{72}$$

with the norm (70) (where, as before, $\nu = \frac{1}{h}$ and $s \ge 0$ is an integer), then a short computation reveals that

$$T^h_{\bar z^l} z^j = \begin{cases} 0 & j < l, \\ q_{lj}(s) z^{j-l} & j \ge l, \end{cases} \tag{73}$$

where

$$q_{jl}(s) = \frac{j!}{(j-l)!}\,\frac{\sum_{k=0}^s \frac{1}{(j-l-k)!(\nu)_{j-k}}}{\sum_{j=0}^s \frac{1}{(j-l-k)!(\nu)_{j-l-k}}}$$

varies with s. (For the Fock space, the analogous expression turns out to be $\frac{j!}{(j-l)!}h^l$, which is independent of s.)

Remark. The formula (73) also implies that $T_{\bar z}T_{\bar z} \neq T_{\bar z^2}$ (even though $T_zT_z = T_{z^2}$) and, in particular, that $T_z^* \neq T_{\bar z}$ on $H^s_{h,\mathrm{hol}}$; thus we see that the validity of (21) and (22) is indeed restricted to subspaces of L^2 in general.

Currently, we do not know whether the Berezin-Toeplitz quantization works also on the holomorphic Sobolev spaces (72) on **D**, as it does on the "Fock-Sobolev" spaces from Theorem 5.1. Computations using the formula (73) seem to indicate that this might be the case. Needless to say, the case of holomorphic Sobolev spaces on other domains, or even of more general spaces of holomorphic functions that are not subspaces of an L^2 space, is likewise open.

6. Other vistas?

The following table briefly summarizes our findings in this paper.

	Berezin quantization	Berezin-Toeplitz quantization
L^2_{hol}	OK	OK
$L^2_{\mathrm{harm}}, L^2_A$	no	no
H^s	no	—
H^s_{hol}	OK	OK?
$H^s_{\mathrm{harm}}, H^s_A$	no	no

We should, finally, note that there might exist still other possible approaches: for instance, in principle, one could also use the Toeplitz operators T^M and the Berezin transform B^M, defined by (65) and (64) above, to carry out the Berezin-Toeplitz quantization or Karabegov's Berezin quantization along the lines of §§1.5 and 1.7, respectively. So far, this possibility remains unexplored.

It should be noted at this point that these Toeplitz operators and Berezin transforms are in general *different* from those defined in §1.4 and §1.6, respectively, in situations when both definitions make sense (for instance, for the spaces H^s_{hol} from Example 4.3). Indeed, the Toeplitz operators from (65) always satisfy $T_{\bar f}T_g = T_{\bar f g}$ if f or g is holomorphic, while those defined in §1.4 need not (cf. the Remark at the end of §5). Similarly, the Berezin transform from (64) always satisfies $B\bar f = \bar f$ if f is holomorphic, while for the one from §1.7 $B\bar f = \widetilde{T_{\bar f}}$, which may differ from $\bar f = \widetilde{T_f^*}$ as $T_f^* \neq T_{\bar f}$ in general (by the same Remark).

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