On the complicatedness
of the pair (g, K)

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In memoriam Atilio Bauchiero

ABSTRACT. Let $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{p}$ be the complexification of a Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}_\mathbb{R}$ and let K be the analytic subgroup of the adjoint group of $\mathfrak{g}$ with Lie algebra $\mathrm{ad}_\mathfrak{g}(\mathfrak{f})$. Let L be an algebraic connected linear reductive complex group acting on a finite dimensional vector space V. In the study of the orbits of this sort of actions, there are some criteria of «non complicatedness»: e.g., «cofreeness» (the ring of all polynomial functions on V is a free module over the ring of all L-invariants), etc. From this viewpoint, we show that the pair $(\mathfrak{g}, K)$ is complicated, at least when $\mathfrak{g}_\mathbb{R}$ is not a product of copies of $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$.

1. INTRODUCTION

Let $\mathfrak{g}_\mathbb{R} = \mathfrak{f}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$ be a Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{p}$ be the corresponding complexification. Let θ be the associated Cartan involution. Also let $\mathfrak{a}_\mathbb{R}^-$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\mathfrak{a}$ be its complexification. Now let G be the adjoint group of $\mathfrak{g}$ and let K be analytic subgroup of G with Lie algebra $\mathrm{ad}_\mathfrak{g}(\mathfrak{f})$. Also let M be the centralizer of $\mathfrak{a}$ in K. This paper is concerned with the action of K in $\mathfrak{g}$ given by the restriction of the Adjoint representation. If $S'(\mathfrak{g})$ denotes the ring of all polynomial functions on $\mathfrak{g}$ then clearly $S'(\mathfrak{g})$ is a G-module and a fortiori a K-module.

If L is a reductive complex linear algebraic group, V is a finite dimensional complex vector space and $\alpha: L \to GL(V)$ is a representation then, concerning the classification of the L-orbits in V, there are some criteria of «non-complicatedness». (See [K] or [M 1], p. 160). To state them, let us

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recall that V/L is the notation for the affine variety associated to $S'(V)^L$ and $V \xrightarrow{\pi} V/L$ is the projection corresponding to the inclusion of rings. Let $\mathfrak{N} = \mathfrak{N}(V, L)$ be the fiber $\pi^{-1}(\pi(0))$. The criteria are:

    A. $\mathfrak{N}$ is a finite union of orbits. Currently (V, L) is visible.

    B.   All the fibres of π are of the same dimension.

    C. $S'(V)$ is a $S'(V)^L$-free module. Currently (V, L) is cofree.

    D. $S'(V)^L$ is a polynomial ring. Currently (V, L) is coregular.

    E.   The isotropy subgroup L^x is non trivial for every $x \in V$.

In this paper we work out the classification of the pairs $(\mathfrak{g}, K)$ as above for which each criteria is satisfied; see propositions A, B, C, D, E below.

If L_1 and L_2 are groups acting on finite dimensional vector spaces V_1 and V_2 respectively, and if we look $L_1 \times L_2$ acting on $V_1 \times V_2$ in the obvious way then it is trivial that

$$S'(V_1 \times V_2)^{L_1 \times L_2} \simeq S'(V_1)^{L_1} \otimes S'(V_2)^{L_2}$$

so $(V_1 \times V_2, L_1 \times L_2)$ is coregular (resp., cofree) iff (V_1, L_1) and (V_2, L_2) are.

Furthermore, the isotropy subgroup $(L_1 \times L_2)^{(x, y)} \cong L_1^x \times L_2^y$, the orbit $(L_1 \times L_2)(x, y) \cong L_1 x \times L_2 y$ $(V_1 \times V_2)/(L_1 \times L_2) \cong V_1/L_1 \times V_2/L_2$ and if $\xi_i \in V_i/L_i$, then $\pi^{-1}(\xi_1, \xi_2) = \pi^{-1}(\xi_1) \times \pi^{-1}(\xi_2)$. So $(V_1 \times V_2, L_1 \times L_2)$ satisfies A (resp., B, E) iff (V_1, L_1) and (V_2, L_2) do. Thus we can restrict our attention to the irreducible pairs $(\mathfrak{g}, K)$. As a synthesis, we get for irreducible $\mathfrak{g}_{\mathbb{R}}$:

Theorem: *$(\mathfrak{g}, K)$ never satisfies criteria B nor E; it satisfies criteria A, C, D if and only if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(p, 1)$ or $\mathfrak{su}(p, 1)$.*

We will use the application of the Luna's Slice Etale Theorem to the Invariant theory developed in [KPV] and also used in [Sch 1] to classify all the (V, L) coregular with L simple. Note that we can replace K by any connected algebraic group K' with Lie algebra $\mathfrak{f}$ acting on $\mathfrak{g}$ with the same infinitesimal action as K. Being a case by case analysis, we will follow E. Cartan's list as it appears in [He], chapter IX. Furthermore, it is clear that it suffices to look at the types I and II, see [He] p. 327.

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2. PRELIMINARIES

Let V, L be as in the introduction, meaning of course by a representation a morphism of algebraic groups. For $x \in V$, the conjugacy class of the

isotropy subgroup L^x is called an isotropy class. If the orbit Lx is closed, L^x is reductive and the representation of L^x in $T_x(V)/T_x(Lx)$ is called the slice representation at x, where T_x notes the tangent space at x. We say that (L^x) is a closed isotropy class.

Lemma 1 ([KPV], [Sch 1]): *Let* $V = V_1 \oplus V_2$ *be a direct sum of finite dimensional L-modules. Then:*

i)   *If* (V, L) *is coregulat then* (V_1, L) *and* (V_2, L) *are.*
ii)   *If* (V, L) *is coregular then every its slice representation is.*
iii) *If* (H) *is a closed isotropy class of* V_1 *then* (V, L) *coregular implies* (V_2, H) *is.*
iv)   *In particular, if the image of H in GL* (V_2) *is a non-trivial finite subgroup of SL* (V_2) *then* (V, L) *is not coregular.*

Proof: *i)* is easy and *ii)* follows from Luna's Theorem (see [KPV]). *iii)* is an application of *i)* and *ii)*; *iv)* is a consequence of the well known Chevalley-Sheppard-Todd Theorem, as it was pointed out in [Sch 1]. ∎

The unique minimal closed isotropy class is called the principal isotropy class. For the Adjoint representation, it is a maximal torus. If V has a L-invariant non-degenerate bilinear symmetric form (V is L-orthogonalizable, for short) then the set of those $x \in V$ such that (L^x) is principal contains an open dense subset of V (see [L] and [R]). The hypothesis is certainly fulfilled for the pairs $(\mathfrak{g}, K)$, $(\mathfrak{p}, K)$, $(\mathfrak{g}, G)$ taking the Killing form. It is obvious that $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{p}$ is a K-module decomposition.

Lemma 2: *The principal isotropy class of* $(\mathfrak{p}, K)$ *is* (M).

Proof: By Lemma 20, in p. 803, of [K R] and in the notation therein, $M_\theta = (K_\theta)^x$ for all x «regular» in $\mathbf{a}$. But $M = M_\theta \cap K$, and $K^x = (K_\theta)^x \cap K$, $\forall x$ in the open dense subset of «regular» elements in $\mathbf{a}$. ∎

We denote by $\Pi(V)$ or $\Pi_L(V, \mathbf{b})$ the set of weights associated to the representation of L in V and a fixed Cartan subalgebra $\mathbf{b}$ of $\mathfrak{l}$, the Lie algebra of L.

The following result is a well-known consequence of the graded version of the Nakayama Lemma and in the present form is useful to establish that some graded ring is not regular.

Lemma 3: *Let* $A = A_0 \oplus A_1 \oplus \dots$ *be a graded ring with* $A_0 = F$ *a field;* $A_+ = A_1 \oplus \dots$ *is the maximal homogeneous ideal.*

i) A *is regular iff dim Krull* $A = \dim_F A_+/A_+^2$. *In such case, if* $t_1, \dots, t_n$ *are homogeneous elements of* A *such that their images in* A_+/A_+^2 *form an F-basis, then they are algebraically independent over F.*

ii)   If $A_1 = 0$ and $t_1, ..., t_s$ are F-linearly independent in A_2, then A regular implies $t_1, ..., t_s$ are F-a.i.

iii) If $A_1 = A_3 = 0$, $t_1, ... t_s$ is an F-basis of A_2 and $t_{s+1}, ..., t_r$ are F-1.i. in A_4 such that $A_2^2 \cap < t_{s+1}, ..., t_r > = 0$ then A regular implies $t_1, ..., t_s$, $t_{s+1}, ..., t_r$ are F-a.i. ∎

The non-coregularity of $(\mathfrak{g}, K)$ will follow in some cases from the following fact:

Lemma 4: *Assume that rank* $\mathfrak{g} =$ *rank* $\mathfrak{f}$; *that* $\mathfrak{f} = \mathfrak{f}_1 \oplus \mathfrak{f}_2$ *is a direct sum of Lie algebras where* $\mathfrak{f}_2 \cong \mathfrak{sl}(2, \mathbb{C})$; *and that as* $\mathfrak{f}$-*module,* $\mathfrak{p}$ *is* $\rho_1 \otimes \rho_2$ *where* ρ_2 *is the natural representation of* $\mathfrak{f}_2$ *in* $\mathbb{C}^2$ *and dim* $\rho_1 \geq 4$. *Then* $(\mathfrak{g}, K)$ *is not coregular.*

Proof: It is clear from Lemma 1 that it suffices to show that $(\mathfrak{p}, H)$ is not coregular, where H is a maximal torus of K, whose Lie algebra is isomorphic to $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$, a Cartan subalgebra of $\mathfrak{f}$, $\mathfrak{b}_j$ a Cartan subalgebra of $\mathfrak{f}_j$. Our first task is to descript $\Pi(\mathfrak{p}, \mathfrak{b})$. If σ is the weight of $\mathfrak{f}_2$ such that $\rho_2 \cong V(\sigma)$, then $\Pi(\rho_2) = \{\pm \sigma\}$. Then $\Pi(\mathfrak{p}, \mathfrak{b}) = \{\alpha \pm \sigma: \alpha \in \Pi(\rho_1)\}$ by abuse of notation. But $\mathfrak{b}$ is also a Cartan subalgebra of $\mathfrak{g}$ and then if $\lambda \in \Pi(\mathfrak{p}, \mathfrak{b})$, λ is a non-compact root in $\Phi(\mathfrak{g}, \mathfrak{b})$; so $-\lambda \in (\mathfrak{p}, \mathfrak{b})$. Thus if $\alpha \in \Pi(\rho_1)$, $-\alpha$ too.

Next, let $\{t_{\alpha, \sigma}, t_{\alpha, -\sigma}: \alpha \in \Pi(\rho_1)\}$ be a basis of $\mathfrak{p}$ such that $t_{\alpha, \pm\sigma}$ is a vector of weight $\alpha \pm \sigma$ and let $\{T_{\alpha, \sigma}, T_{\alpha, -\sigma}\}$ be the corresponding dual basis. Thus:

$$S'(\mathfrak{p})^H = \bigoplus_{j \geq o} S'(\mathfrak{p})_j^H = \bigoplus_{j \geq o} < \text{monomials in } T_{\alpha, \pm\sigma} \text{ of weight } 0 > = \bigoplus_{j \geq o} A_j$$

Clearly, if j is odd then $A_j = 0$. Also if $U_\alpha = T_{\alpha, \sigma} \cdot T_{-\alpha, -\sigma}$, then $\{U_\alpha: \alpha \in \Pi(\rho_1)\}$ is a basis of A_2. As dim $\rho_1 \geq 4$, there exist $\alpha, \beta \in \Pi(\rho_1)$ such that $\alpha \neq \pm \beta$. Put $S_{\alpha, \beta} = T_{\alpha, \sigma} T_{-\alpha, -\sigma} T_{\beta, -\sigma} T_{-\beta, -\sigma}$. Obviously, $A_2^2 \cap < S_{\alpha, \beta}, S_{\beta, \alpha} > = 0$. But U_α, $S_{\alpha, \beta}, S_{\beta, \alpha}$ are not a.i. because $S_{\alpha, \beta} S_{\beta, \alpha} = U_\alpha U_{-\alpha} U_\beta U_{-\beta}$ and Lemma 3 applies. ∎

3. THE CASE BY CASE ANALYSIS OF COREGULARITY

Types II, IV: Here $\mathfrak{l}$ is a simple Lie algebra over $\mathbb{C}$, $\mathfrak{g} = \mathfrak{l} \times \mathfrak{l}$ and $\theta(x, y) = (y, x)$. Then it is easy to see that $\mathfrak{f} \cong \mathfrak{l}$ and as $\mathfrak{f}$-module, $\mathfrak{g}$ is Ad $\oplus$ Ad. Looking at Schwarz tables in [Sch 1], we see that $(\mathfrak{g}, K)$ is coregular iff $\mathfrak{l} = \mathfrak{si}(2, \mathbb{C})$ (table 1.a.18).

Types I, III: The Classical Structures

Type AI: Here $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{f} = \mathfrak{so}(n, \mathbb{C})$ with $n \geq 3$. (For $n = 2$, it is isomoprhic to BDI, $p = 2$, $q = 1$). If $(\mathfrak{g}, K)$ were coregular, then by [Sch 1], table

3a, $\mathfrak{p}$ must be φ_1, the natural action in $\mathbb{C}^n$. We get a contradiction computing dim $\mathfrak{p} = (n^2+n)/2 - 1$.

Type AII: Here $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{C})$, $\mathfrak{k} \approx \mathfrak{sp}(n, \mathbb{C})$ with $n \geq 3$. (For $n=2$, it is isomorphic to BDI, $p=6$, $q=1$). The Schwarz notation for Ad is φ_1^2 so that if $(\mathfrak{g}, K)$ were coregular, by Table 4a, $\mathfrak{p}$ must be φ_1, the natural action on $\mathbb{C}^{2n}$. As dim $\mathfrak{p} = 2n^2 - n - 1$, we get a contradiction.

Type AIII: Here $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{C})$, $\mathfrak{k} = \{ \begin{pmatrix} A & O \\ O & B \end{pmatrix} \in \mathfrak{g} : A \in \mathbb{C}^{p \times p} \}$, $\mathfrak{p} = \{ \begin{pmatrix} O & C \\ D & O \end{pmatrix} \in \mathfrak{g} \}$. When $q=1$, corregularity of $(\mathfrak{g}, K)$ was proved by Cooper in [C]. So, let $q \geq 2$. We can choose $\mathfrak{a}$ as in [He], p. 368. As it was pointed out in the Introduction, we may assume that $K = \{ \begin{pmatrix} A & O \\ O & B \end{pmatrix} \in SL(p+q, \mathbb{C}), A \in \mathbb{C}^{p \times p} \}$ and then it is easy to see that $M = \{ \begin{pmatrix} A & O \\ O & B \end{pmatrix} \in K : B$ is diagonal, $A = \{ \begin{pmatrix} B & O \\ O & C \end{pmatrix} \} \}$. If we can show that $(\mathfrak{k}, M)$ is non-coregular, we are done.

Now, $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3$, where $\mathfrak{k}_1 = \{ \begin{pmatrix} A & O \\ O & O \end{pmatrix} \in \mathfrak{g} \} \approx \mathfrak{sl}(p, \mathbb{C})$, $\mathfrak{k}_2 \approx \mathfrak{sl}(q, \mathbb{C})$, and $\mathfrak{k}_3 \approx \mathbb{C}$ is the center of $\mathfrak{k}$. As M-module, $\mathfrak{k}_1$ admits a submodule isomorphic to $\mathfrak{k}_2 : \{ \begin{pmatrix} A & O \\ O & O \end{pmatrix} \in \mathfrak{g} : A = \begin{pmatrix} C & O \\ O & O \end{pmatrix}$, with $c \in \mathfrak{sl}(q, \mathbb{C}) \}$ and the action of M in $\mathfrak{k}_2$ is given by $B.(a_{ij}) = (b_i b_j^{-1} a_{ij})$ if B is the diagonal $(b_1, ..., b_q)$. Let V be the M-submodule of $\mathfrak{sl}(q, \mathbb{C})$, $V = \{ (a_{ij}) : a_{ii} = 0 \, \forall \, i \}$. Clearly, it suffices to show that $(V \oplus V, M)$ is not coregular. Note that $q \neq 1$ implies $V \neq 0$. Putting $S'(V)^M = A_o \oplus A_1 \oplus ...$, a_{ij}, b_{ij} the canonical coordinates of the first and the second copy of V, respectively, then $A_1 = 0$ and $A_2 = \langle a_{ij} a_{ji}, b_{ij} b_{ji}, a_{ij} b_{ji} \rangle$. Thus Lemma 3 applies.

Type BDI: Here $\mathfrak{g} = \mathfrak{so}(p+q, \mathbb{C})$, $\mathfrak{k} = \mathfrak{so}(p, \mathbb{C}) \oplus \mathfrak{so}(q, \mathbb{C}) = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{p} = \{ \begin{pmatrix} O & B \\ C & O \end{pmatrix} : B \in \mathbb{C}^{p \times q}, B + {}^t C = 0 \}$. We can choose $\mathfrak{a} = \{ \begin{pmatrix} O & B \\ C & O \end{pmatrix} : B$ is «diagonal», i.e. $b_{ij} = 0$ if $i \neq j \}$.

We may assume that $K = SO(p, \mathbb{C}) \times SO(q, \mathbb{C})$ and then it is easy to see that $M = \{ (A, B) \in K : B$ is the diagonal $(\epsilon_1, ..., \epsilon_q)$ with $\epsilon_i^2 = 1$, $\Pi \epsilon_i = 1$, and $A = \begin{pmatrix} B & O \\ O & C \end{pmatrix}$ with $C \in SO(p-q, \mathbb{C}) \}$. $q=1$: Then $(\mathfrak{g}, K)$ is coregular by Cooper [C], Benabdallah [B], or [Sch 1], Table 3 a.2.

$q \geq 3$: It follows from Lemmas 1 and 2 that $(\mathfrak{g}, K)$ coregular implies $(\mathfrak{k}_2, M)$ coregular. Note that the morphism $M \longrightarrow GL(\mathfrak{k}_2)$, say ρ, depends clearly only on $B = (\epsilon_1, ..., \epsilon_q)$ and $\rho(B)(X_{ij}) = (\epsilon_i \epsilon_j X_{ij})$. Then det $\rho(B) = \Pi_{i<j} \epsilon_i \epsilon_j = (\Pi_i \epsilon_i)^{q-1} = 1$. For $B = (1, -1, -1, 1, ..., 1)$, $\rho(B) \neq$ Id; therefore $\rho(M)$ is a finite, non trivial subgroup of $SL(\mathfrak{k}_2)$ and Lemma 1 applies.

$q=2$: Here $M \cong SO(p-2, \mathbb{C}) \times \{\pm I_2\}$, where I_2 is the identity of $GL(2, \mathbb{C})$.

Now, as M-module, $\mathfrak{f}_1 \cong \mathfrak{so}(p-2, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C}) \oplus \mathbb{C}^{p-2} \oplus \mathbb{C}^{p-2}$; where $\mathfrak{so}(p-2, \mathbb{C}) \cong \{\begin{pmatrix} O & O \\ O & A \end{pmatrix} \in \mathfrak{f}_1: \quad A \in \mathfrak{so}(p-2, \mathbb{C})\}$, $\mathfrak{so}(2, \mathbb{C})$ similarly $\mathbb{C}^{p-2} \oplus \mathbb{C}^{p-2} \cong \{\begin{pmatrix} O & X \\ Y & O \end{pmatrix} \in \mathfrak{f}_1: Y \in \mathbb{C}^{(p-2)\times 2}, x = -{}^t Y\} = V$. If $x \in \mathfrak{so}(p-2, \mathbb{C})$ is semisimple regular, then $M^x = T \times \{\pm I_2\}$ where T is a maximal torus of $\mathfrak{so}(p-2, \mathbb{C})$. Thus it suffices to show that $(V, T \times \{\pm I_2\})$ is not coregular. If $(A, \epsilon I) \in M^x$ and $(Y_1, Y_2) \in V$, the action is given by $(\epsilon A Y_1, \epsilon A Y_2)$. Then

$$S'(V)^{T X \{\pm I\}} = \oplus_{j,\, even} S'(V)_j^T = \oplus_{j,\, even} S'(V)_j^t$$

where using an appropiate characterization of $\mathfrak{so}(p-2, \mathbb{C})$, the Cartan subalgebra $\mathfrak{t}$ can be chosen $\{\begin{pmatrix} D & O \\ O & -B \end{pmatrix}: D$ is a diagonal $(d_1, ..., d_k)\}$, if $p-2 = k$ is even. (The argument when p is odd is similar).

If $v_1, ..., v_{2k}, w_1, ..., w_{2k}$ is the dual basis associated with $\{(e_j, 0), (0, e_j)\}$ then $S'(V)_2^T = \langle v_i v_{k+i}, w_i w_{k+i}, v_i w_{k+i}, v_{k+i} w_i \rangle$ and Lemma 3 applies.

Note that k must be ≥ 1, i.e. $p \geq 4$. The remaining cases are $(3,2)$ and $(2,2)$; respectively, $\mathfrak{sp}(2, \mathbb{R})$ (type CI) and $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ (type AI $\times$ type AI).

Type DIII: Here $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, $\mathfrak{f} \simeq \mathfrak{gl}(n, \mathbb{R})$ and as $\mathfrak{f}$-module, $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ where $\mathfrak{p}_i \simeq \mathfrak{so}(n, \mathbb{C})$ with actions $\sigma_1(Z)(U) = ZU + U^t Z$, $\sigma_2(Z)(U) = -{}^t Z U - U Z$.

We can choose $\mathfrak{a} = \{(V, V): V = \sum \lambda_j (e_{2j-1, 2j} - e_{2j, 2j-1}), \lambda_j \in \mathbb{C}\}$. We can assume that $K = GL(n, \mathbb{C})$ and then it easy to show that $M \simeq SL(2, \mathbb{C}) \times ... \times SL(2, \mathbb{C})$, h times, if $n = 2h$ is even and $M \simeq SL(2, \mathbb{C})^h \times \mathbb{C}^*$ if $n = 2h+1$ is odd. The isomorphism is realized by «blocks in the diagonal». By Lemmas 1 and 2 it suffices to study the pair $(\mathfrak{f}, M)$.

Consider the M-submodule of $\mathfrak{f}$

$$V = \{Z \in \mathfrak{f}: Z_{ij} = 0 \text{ if } i \geq 4 \text{ or } j \geq 4\}$$

Obviously $(V, M) \simeq (Ad \oplus V_1 \oplus V_2, \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}))$. Thus we look at $(V_1 \oplus V_2, T)$, where T is a maximal torus of $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$ and the action is given by $(t, r)(A, B) = (t A r^{-1}, r B t^{-1})$. Let a_i, b_i be the canonical basis of V_j, $j = 1, 2$.

If $S'(V_1 \oplus V_2)^T = A_0 \oplus A_1 \oplus ...$, then $A_1 = 0$, $A_2 = \langle a_1 a_4, a_2 a_3, b_1 b_4, b_2 b_3, a_1 b_1, a_2 b_3, a_3 b_2, a_4 b_4 \rangle$. Thus Lemma 3 applies.

This method works for $n \geq 4$. But for $n = 2, 3$ $\mathfrak{g}_\mathbb{R}$ is isomorphic to AIII and AI $\times$ AI, respectively.

Type CI: Here $\mathfrak{g} = \mathfrak{sp}\,(n,\,\mathbb{C})$, $\mathfrak{f} \simeq \mathfrak{gl}\,(n,\,\mathbb{C})$ and as $\mathfrak{f}$-module, $\mathfrak{p} \simeq \mathfrak{p}_1 \oplus \mathfrak{p}_2$ where $\mathfrak{p}_i = \{A \in \mathfrak{gl}\,(n,\,\mathbb{C}): A = {}^t A\}$ with actions $\sigma_1(Z)(A) = ZA + A^t Z$, σ_2 the dual of σ_1. We can choose $\mathfrak{a} = \{(D,\,D): D \text{ is diagonal}\}$ and if we assume that $K = GL(n,\,\mathbb{C})$, it is easy to see that $M = \{X \in K: X \text{ is a diagonal } (\epsilon_1, ..., \epsilon_n) \text{ with } \epsilon_i = \pm 1\}$. Looking at the pair $(\mathfrak{f},\,M)$ it is immediately that $\det \mathrm{Ad}\; m = \Pi_{i,j}(\epsilon_i \epsilon_j) = 1$, if $m = (\epsilon_1, ..., \epsilon_n) \in M$. But $m = (-1, 1, ..., 1)$ acts non trivially so that Lemma 1 iv) applies.

This method works for $n > 1$. For $n = 1, \mathfrak{sp}\,(1,\,\mathbb{C}) \simeq \mathfrak{sl}(2,\,\mathbb{C})$, trivially coregular.

Type CII: Here $\mathfrak{g} = \mathfrak{sp}\,(p+q,\,\mathbb{C})$; $\mathfrak{f} \simeq \mathfrak{sp}\,(p,\,\mathbb{C}) \oplus \mathfrak{sp}\,(q,\,\mathbb{C})$ and $\mathfrak{p} \simeq \mathbb{C}^{2p \times 2q}$ with the action $(Z_1, Z_2)\,X = Z_1\,X - X Z_2$.

We can choose $\alpha = \{\begin{pmatrix} A & O \\ O & A \end{pmatrix}: A \in \mathbb{C}^{p \times q},\ A = \sum \lambda_i e_{ii}\}$. We can assume that $K = SP\,(q,\,\mathbb{C}) \times SP(q,\,\mathbb{C})$ and then it is easy to see that $M = \{(X_1, X_2) \in K:$ $X_2 = \begin{pmatrix} A^1 & A^2 \\ A^3 & A^4 \end{pmatrix}$ with A^j diagonal in $GL(q,\,\mathbb{C})$, $A_{ii}^1 A_{ii}^4 - A_{ii}^3 A_{ii}^2 = 1$ and $X_1 = \begin{pmatrix} B^1 & B^2 \\ B^3 & B^4 \end{pmatrix}$ with $B^j = \begin{pmatrix} A^j & O \\ O & C^j \end{pmatrix}$, $\begin{pmatrix} C^1 & C^2 \\ C^3 & C^4 \end{pmatrix} \in SP(p-q,\,\mathbb{C})\}$.

That is, $M \simeq SL(2,\,\mathbb{C})^q \times SP\,(p-q,\,\mathbb{C})$. Now we can assume $q > 1$ because for $q = 1, p \geq 2$ we are in the situation of Lemma 4 and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}\,(1,1) \simeq \mathfrak{so}\,(4,1)$, implies $(\mathfrak{g},\,K)$ coregular.

It is clear that $\mathfrak{f}_1$ has a M-submodule isomorphic to $\mathfrak{f}_2$, so we are done proving the non coregularity of $(\mathfrak{f}_2 \oplus \mathfrak{f}_2,\,SL(2,\,\mathbb{C})^q)$.

$$\text{Put } V_{ij} = \langle e_{i,j} - e_{q+j,\,q+i},\ e_{j,i} - e_{q+i,\,q+j},\ e_{i,q+j} + e_{j,q+i},\ e_{q+i,j} + e_{q+j,i}\rangle$$

$$\text{if } i \neq j \text{ and } W_i = \langle e_{i,i} - e_{q+i,\,q+i},\ e_{q+i,i},\ e_{i,q+i}\rangle;\ \text{then}$$

$$\mathfrak{f}_2 = (\oplus_i W_i) \oplus (\oplus_{i<j} V_{ij}) \text{ and } \oplus_i W_i \simeq \mathrm{Ad}\,(SL(2,\,\overline{\mathbb{C}})^q).$$

So we can restrict our attention to the pair $(V_{12} \oplus V_{12},\,T)$ where $T = \{(X_1, ..., X_q): X_j \text{ is a diagonal in } \mathfrak{sl}(2,\,\mathbb{C})\}$. If $\alpha_s,\,\beta_r$ are the dual basis to the described above, and $S'(V_{12} \oplus V_{12})^T = A_0 \oplus A_1 \oplus ..$ then $A_1 = 0$, $A_2 = \langle \alpha_1 \alpha_2,\ \alpha_3 \alpha_4,\ \beta_1 \beta_2,\ \beta_3 \beta_4,\ \alpha_1 \beta_2,\ \alpha_2 \beta_1,\ \alpha_3 \beta_4,\ \alpha_4 \beta_3\rangle$ and Lemma 3 applies.

The Exceptional Structures

Most of the cases follows from Schwarz tables [Sch 1] or from Lemma 4. So we list them. The reference for the K-module structure of $\mathfrak{p}$ is [F de V].

Type	$\mathfrak{g}$	$\mathfrak{f}$	$\mathfrak{p}$	Method
EI	e_6	$\mathfrak{sp}(4,\mathbb{C})$	42	Table 4a.3, dim $\mathfrak{p} \neq 8$
EIV	e_6	$\mathfrak{f}_4$		” 5a.4
EV	e_7	$\mathfrak{sl}(8,\mathbb{C})$	70	” 1a.20, dim $\mathfrak{p} \neq 8$
EVIII	e_8	$\mathfrak{so}(16,\mathbb{C})$	128	” 3a.2, dim $\mathfrak{p} \neq 16$
FII	$\mathfrak{f}_4$	$\mathfrak{so}(9,\mathbb{C})$	16	” 3a.5, dim $\mathfrak{p} \neq 9$
EII	e_6	$\mathfrak{si}(6,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$	$\Lambda^3(\mathbb{C}^6)$	Lemma 4
EVI	e_7	$\mathfrak{so}(12,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$	λ_5 (spin)	”
EIX	e_8	$e_7 \times \mathfrak{sl}(2,\mathbb{C})$	λ_7	”
FI	$\mathfrak{f}_4$	$\mathfrak{sp}(3,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$	λ_3	”
G	g_2	$\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$	$V(3) = 3\lambda_1$	”

Note: under «$\mathfrak{p}$» we have listed dim $\mathfrak{p}$ for [Sch 1], ρ_1 for Lemma 4. Here λ_j means the *j*-fundamental weight, as in [Hu].

There are two remaining cases:

Type EIII: Here $\mathfrak{g} = e_6$, $\mathfrak{f} = \mathfrak{so}(10,\mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. As *K*-module, $\mathfrak{p}_-$ is dual to $\mathfrak{p}_+$; $\mathfrak{p}_+$ is λ_5 (spin) as $[\mathfrak{f}, \mathfrak{f}]$-module and $\mathbb{C} =$ center of $\mathfrak{f}$ acts by non-trivial scalars.

Type EVII: Here $\mathfrak{g} = e_7$, $\mathfrak{f} = e_6 \oplus \mathbb{C}$, $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, $\mathfrak{p}_+$ is λ_1, etc.

We develope an argument for both of them. Let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{d}$ be a Cartan subalgebra of $\mathfrak{f}$, where $\mathfrak{t}$ is a C. s. of $[\mathfrak{f}, \mathfrak{f}]$ and $\mathfrak{d}$ is the center; let H be the corresponding maximal torus. The goal is to prove the non-coregularity of $(\mathfrak{p}, H)$. Let $\sigma \in \mathfrak{d}^*$ associated to the action on $\mathfrak{p}_+$; $\sigma \neq 0$ because $\mathfrak{g}$ has trivial center. By abuse of notation we call also σ the extension to $\mathfrak{b}$ vanishing on $\mathfrak{t}$; the same convention for $\lambda \in \Pi(\mathfrak{p}_+, \mathfrak{t})$.

Then $\Pi(\mathfrak{p}_+, \mathfrak{b}) = \{\lambda + \sigma : \lambda \in \Pi(\mathfrak{p}_+, \mathfrak{t})\}$. As usual, let $\{x_\lambda\}$ be the basis of $\mathfrak{p}_+$ where x_λ is a vector of weight $\lambda + \sigma$, $\lambda \in \Pi(\mathfrak{p}_+, \mathfrak{t})$; let $\{y_\lambda\}$ be the basis of $\mathfrak{p}_-$ where y_λ is a vector of weight $-\lambda - \sigma$, and let $\{X_\lambda, Y_\lambda\}$ be the corresponding dual basis. If $S'(\mathfrak{p}, H) = \oplus_{i \geq 0} A_i$ then $A_m = \langle X_{\lambda_1} \dots X_{\lambda_r} X_{\lambda_{r+1}} \dots X_{\lambda_m} : \sum_{i \geq r}(\lambda_i + \sigma) + \sum_{i > r}(-\lambda_i - \sigma) = 0\rangle$. Thus $A_m = 0$ if m is odd and $A_2 = \langle X_\lambda Y_\lambda\rangle$. Now assume that there are some $\lambda_1, \dots, \lambda_4$ in $\Pi(\mathfrak{p}_+, \mathfrak{t})$ such that $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$ and $\lambda_1 \neq \lambda_3, \lambda_4$. Then $X_{\lambda_1} X_{\lambda_2} Y_{\lambda_3} Y_{\lambda_4}$, $X_{\lambda_3} X_{\lambda_4} Y_{\lambda_1} Y_{\lambda_2}$ do not belong to A_2^2 and Lemma 3 applies.

The preceding hypothesis is fulfilled in both cases, as we can see easily; note that, as rank $\mathfrak{g} =$ rank $\mathfrak{f}$, we may look at the non-compact roots in $\mathfrak{p}_+$.

From the preceding analysis, we have:

Proposition D: $(\mathfrak{g}, K)$ is coregular if and only if it corresponds to $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(p, 1)$ or $\mathfrak{su}(p, 1)$. ∎

4. THE OTHER CRITERIA

Here we assume that L is a semisimple complex algebraic group, and that V is L-orthogonalizable; see section 2.

Proposition E: *i) If every root it in the $\mathbb{Z}$-span of $\Pi(V)$, then the principal isotropy class of $(Ad \oplus V, L)$ is (the class of)*

$$Ker\ (L \longrightarrow GL(Ad \oplus V)).$$

ii) If L is simple and V is non trivial, then every root is in the $\mathbb{Z}$-span of $\Pi(V)$.

iii) If 𝔤 is simple, the principal isotropy class of (𝔤, K) is trivial, discarding the trivial case when $\mathfrak{p} = 0$.

iv) (𝔤, K) never satisfies criteria E.

Proof: *i)* Let H be the maximal torus of L whose Lie algebra is 𝔥 and pick any element $x \in 𝔥$ such that $L^x = H$. As $V = \oplus_{\lambda \in \Pi(V)} V_\lambda$, we can choose $y = \Sigma_\lambda y_\lambda$, $y_\lambda \in V_\lambda - 0$. It follows that

$$L^{x+y} = L^x \cap L^y = H \cap L^y = \{A \in H: Ay_\lambda = y_\lambda \text{ for all } \lambda \in \Pi(V)\}$$

Now such $A = \exp a$, for some $a \in 𝔥$, and

$$(Ad_L\ A)\ y_\lambda = (\exp a)\ y_\lambda = e^{\lambda(a)} y_\lambda = y_\lambda$$

Then $\lambda(a) \in 2\pi i \mathbb{Z}$ for all $\lambda \in \Pi(V)$, because $y_\lambda \neq 0$. By hypothesis, $\mu(a) \in 2\pi i \mathbb{Z}$ for every root μ and then $A \in \text{Ker Ad } L$.

As V is L-orthogonalizable, the same is true for $Ad \oplus V$. So, it only remains to show that the set $\{Z \in \mathfrak{l} \oplus V: L^Z = \text{Ker Ad } L\}$ is dense in $\mathfrak{l} \oplus V$.

Let U be a Zariski open non empty subset of $\mathfrak{l} \oplus V$; its image under the projection map $\mathfrak{l} \oplus V \longrightarrow \mathfrak{l}$ is open so it exists x regular semisimple such that for some $y \in V$, $x + y \in U$. Now $\mathfrak{l}^x$, the centralizer of x in $\mathfrak{l}$, is a Cartan subalgebra of $\mathfrak{l}$. From the conjugacy theorem, it follows that $\Phi(\mathfrak{l}, \mathfrak{l}^x)$ is contained in the $\mathbb{Z}$-span of $\Pi(V, \mathfrak{l}^x)$. $\{y \in V: x + y \in U\}$ and $\{y \in V: y_\lambda \neq 0 \ \forall\ \lambda \in \Pi(V, \mathfrak{l}^x)\}$ are both open non empty; taking y in the intersection, $x + y \in U$ and $L^{x+y} = \text{Ker Ad}$.

ii) Let W be the subgroup of $𝔥^*$ generated by $\Pi(V)$ and let $\Phi = \Phi(\mathfrak{l}, 𝔥)$. We claim that $\Phi = (\Phi \cap W) \cup (\Phi \cap W^\perp)$. It suffices to show that $\Phi - W^\perp \subset W$. If $\alpha \in \Phi - W^\perp$, there is some $\mu \in \Pi(V)$ such that $(\alpha, \mu) \neq 0$. The α-string through

μ is $\mu\text{-}r\alpha, ..., \mu+q\alpha$ with $r-q=(\alpha,\mu)\neq 0$; thus $\mu\pm\alpha\in\Pi$ *(V)* and $\alpha\in W$. Since $\mathfrak{l}$ is simple, Φ is irreducible; as $V\neq 0$, $W\neq 0$ and $\Phi=\Phi\cap W$.

iii) Let L be the connected subgroup of K with Lie algebra $\mathfrak{l}=[\mathfrak{f},\mathfrak{f}]$, let $V=\mathfrak{p}$ and let $\mathfrak{h}$, W, Φ be as in the proof of *ii)*. Then $\Phi=(\Phi\cap W)\cup(\Phi\cap W^\perp)$. Let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be the ideals of $\mathfrak{l}$ such that: if $\mathfrak{h}_j$ is a Cartan subalgebra of $\mathfrak{f}_j$ given by $\mathfrak{h}_j=\mathfrak{h}\cap\mathfrak{f}_j$, then the root systems $\Phi(\mathfrak{f}_1,\mathfrak{h}_1)$ and $\Phi(\mathfrak{f}_2,\mathfrak{h}_2)$ are identified with $\Phi\cap W$ and $\Phi\cap W^\perp$ respectively. If $\lambda\in\Pi(\mathfrak{p},\mathfrak{h})$, $\lambda(\mathfrak{h}_2)=0$. Thus the action of $\mathfrak{f}_2$ in $\mathfrak{p}$ is trivial. Now Jacobi implies that $[\mathfrak{p},\mathfrak{p}]$ is an ideal of $\mathfrak{f}$ and that $[\mathfrak{f}_2,[\mathfrak{p},\mathfrak{p}]]=0$. Then if $\delta=$ center of $\mathfrak{f}$, $[\mathfrak{f}_1+\delta+\mathfrak{p},\mathfrak{f}_2]=0$ and $\mathfrak{f}_1+\delta+\mathfrak{p}$, $\mathfrak{f}_2$ are ideals of $\mathfrak{g}$. By hypothesis $\mathfrak{f}_2=0$ and $\Phi=\Phi\cap W$. Assume here that dim $\delta=1$; as $\mathfrak{f}$-module, $\mathfrak{p}=\mathfrak{p}_+\oplus\mathfrak{p}_-$ and δ acts in $\mathfrak{p}_+$(in $\mathfrak{p}_-$) via $\sigma\neq 0$ (via $-\sigma$). Also $\Pi(\mathfrak{p}_-,\mathfrak{h})=-\Pi(\mathfrak{p}_+,\mathfrak{h})$. Recalling that $\Phi\cup(\Pi(\mathfrak{p}_+,\mathfrak{h})\times\{\sigma\})\cup(\Pi(\mathfrak{p}_-,\mathfrak{h})\times\{-\sigma\})=\Phi(\mathfrak{g},\mathfrak{h}+\delta)$ it is also true that $\{\alpha\in\Phi:(\alpha,\Pi(\mathfrak{p}_+,\mathfrak{h}))>0\}=\Phi_+$ for some choice of a base Δ.

Pick $x\in\mathfrak{f}, c\in\delta, y\in\mathfrak{p}$ such that $K^{x+c}=H\times Z$ is a maximal torus of K. We want to show that $K^{x+c+y}=K^{x+c}\cap K^y=\text{Ker }\text{Ad}_{\mathfrak{g}}(K)$. Let $H_1\in\mathfrak{h}$, $H_2\in\delta$ such that $\exp(H_1+H_2)\in K^y$. Then $\forall\lambda\in\Pi(\mathfrak{p}_+,\mathfrak{h})$ $\lambda(H_1)+\sigma(H_2)\in 2\pi i$. If $\alpha\in\Phi_+$, $\alpha=\lambda_1-\lambda_2$, for some $\lambda_i\in\Pi(\mathfrak{p}_+,\mathfrak{h})$ (look at the α-string). Then $\alpha(H_1)\in 2\pi i\mathbb{Z}$. If $\alpha\in\Phi_+$, $\alpha=\lambda_1-\lambda_2$, for some $\lambda_i\in\Pi(\mathfrak{p}_+,\mathfrak{h})$ (look at the α-string). Then $\alpha(H_1)\in 2\pi i\mathbb{Z}$ and we can follow the line of the proof of *i)*.

iv) For types II-IV it follows from *ii)*; in other case from *iii)*. ∎

Next we will study the dimension of $\mathfrak{g}/K$. We return to the assumption: «*L* reductive».

From Algebraic Geometry we know, for $\zeta\in V/L$:

$$\dim \pi^{-1}(\zeta)+V/L\geq\dim V. \qquad [1]$$

Furthermore, there exists an open dense subset U of V such that $\forall\zeta\in\pi(U)$, the equality in [1] holds.

Lemma 5: *dim* $\mathfrak{g}/K=$ *dim* $\mathfrak{p}$

Proof: If V/L has generically closed orbits (i.e., the union of the closed orbits contains a non empty open set) then it is follows from [1] that $\dim V/L=\dim V-\dim L+\dim H$, where (H) is a principal isotropy class. Being $\dim H=0$ from Proposition E, $\dim\mathfrak{g}/K=\dim\mathfrak{g}/K=\dim\mathfrak{g}-\dim K=\dim\mathfrak{p}$. ∎

Our following task is to compute the dimension of $\mathfrak{N}$, the cone of unstable points in Mumford's terminology, using the ideas exposed in [Sch 2], via the

Hilbert-Mumford criterion. For convenience, we will summaryze them. See also [M 1], Ch. II or [M 2], p. 41.

Let $\Lambda: \mathbb{C}^* \to L$ be a morphism of algebraic groups, briefly a 1-PS. Put $Z_\Lambda = \{v \in V: \Lambda(z)v \to 0 \text{ if } z \to 0\}$. From the well known characterization $\mathfrak{N}(V, L) = \{v \in V: f(v) = 0 \; \forall \; f \in S'(V)^L \text{ homogeneous of positive degree}\}$ it follows that $\mathfrak{N}$ contains the various Z_Λ. In fact, the Hilbert-Mumford criterion insures that $\mathfrak{N}(V, L) = \cup_{\Lambda, 1\text{-PS}} Z_\Lambda$. Now if T is a maximal torus of L and Λ is a 1-PS, IM Λ is conjugated to a subgroup of T and

$$\mathfrak{N}(V, L) = \cup_{\Lambda, 1\text{-PS } in \, T} L \cdot Z_\Lambda.$$

Let $\mathfrak{t}$ be the Cartan subalgebra of the Lie algebra of L, $\mathfrak{l}$, corresponding to T. If Λ is a 1-PS in T, note by λ its infinitesimal generator. If $V = \oplus_{\mu \in \Pi(V, \mathfrak{t})} V_\mu$, then $\mu(\lambda) \in \mathbb{Z}$ and $\forall v \in V_\mu$, $z \in \mathbb{C}^*$: $\Lambda(z)v = z^{\mu(\lambda)} v$. So $Z_\Lambda = \oplus_{\mu: \mu(\lambda) > 0} V_\mu$; thus $\mathfrak{N}(V, L)$ is union of a finite number of $L \cdot Z_\Lambda$. Call $c_\Lambda = \text{codim } L \cdot Z_\Lambda$; then

$$\text{codim } \mathfrak{N} = \inf\{c_\Lambda: \Lambda \text{ is a 1-PS in } T\}.$$

Now let $\mathfrak{p}_\Lambda$ be the (parabolic) subalgebra of $\mathfrak{l}$ that normalizes Z_Λ, $\mathfrak{u}_\Lambda$ the subalgebra of $\mathfrak{l}$ generated by the root vectors not in $\mathfrak{p}_\Lambda$, U_Λ the connected algebraic subgroup of L corresponding to $\mathfrak{u}_\Lambda$. Following [Sch 2] we have $\mathfrak{l} = \mathfrak{p}_\Lambda \oplus \mathfrak{u}_\Lambda$ and

$$c_\Lambda = \dim V - \dim Z_\Lambda - \dim U_\Lambda + e_\Lambda \geq \dim V - \dim Z_\Lambda - \dim U_\Lambda \quad [2]$$
$$\text{where } e_\Lambda = \dim U_\Lambda - \sup\{\dim(T_z(U_\Lambda z) + Z_\Lambda)/Z_\Lambda: z \in Z_\Lambda\}$$

Furthermore, $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{d}$, where $\mathfrak{b}$ is a Cartan subalgebra of $[\mathfrak{l}, \mathfrak{l}]$. Then $\lambda = \lambda_{\mathfrak{b}} + \lambda_{\mathfrak{d}}$ (obvious notation). Call φ_λ the unique element in $\mathfrak{b}^*$ such that $\varphi_\lambda(H) = \text{Killing }(\lambda_{\mathfrak{b}}, H) \; \forall h \in \mathfrak{b}$. Now, $\forall \mu \in \Phi([\mathfrak{l}, \mathfrak{l}], \mathfrak{b}): (\varphi_\lambda, \mu) = \mu(\lambda) \in \mathbb{Z}$ and then $\varphi_\lambda \in E = \mathbb{R}\text{-span of } \Phi([\mathfrak{l}, \mathfrak{l}], \mathfrak{b})$ in $\mathfrak{b}^*$. (See [Hu], p. 40 and p. 67).

Finally, $\gamma(\mathfrak{t}) = \{\lambda \in \mathfrak{t}: \lambda = d\Lambda(1) \text{ for some } 1-PS \; \Lambda \text{ in } \mathfrak{t}\}$ is isomorphic to $\Gamma(T) = \{\Lambda: \lambda \; 1-PS \text{ in } T\}$ via $\Lambda \to \lambda$; then it is isomorphic to $\mathbb{Z}^d$, $d = \dim \mathfrak{t}$. Moreover, $\gamma(\mathfrak{t})$ is a lattice in $\mathfrak{t}$ and then identifying $\gamma(\mathfrak{b})$ with $\{\varphi_\lambda: \lambda \in \gamma(\mathfrak{b})\}$, $\gamma(\mathfrak{b})$ meets every open cone in E. (See [Ch], 9-06). As usual, rk denotes the rank.

Lemma 6: *codim* $\mathfrak{N} = 1/2 \; (\dim \mathfrak{p} + \text{rk } \mathfrak{g} + \text{rk } \mathfrak{f})$

Proof: Let $\mathfrak{t}_\mathfrak{g}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t} = \mathfrak{t}_\mathfrak{g} \cap \mathfrak{f}$ is a Cartan subalgebra of $\mathfrak{f}$. As above, $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{d}$, with $\mathfrak{b}$ a C. s. of $\mathfrak{f}' = [\mathfrak{f}, \mathfrak{f}]$. Put $\phi = \phi(\mathfrak{g}, \mathfrak{t}_\mathfrak{g})$.

i) Let first $L = K$ acting on $V = \mathfrak{f}$ by Ad. Let Λ be a 1-PS in T. If λ is regular (i.e., φ_λ lies in the interior of some Weyl chamber) then $Z_\Lambda = \mathfrak{f}_+$ for

the ordering defined by φ_λ. If not, an easy argument shows that $Z_\Lambda \subseteq \mathfrak{f}_+$ for some $\mathfrak{f}_+$. Now the Chevalley Restriction Theorem guarantees that dim $(\mathfrak{f}/K) = \mathrm{rk}\ \mathfrak{f}$. For λ regular $\mathfrak{p}_\Lambda = \mathfrak{f}_+ \oplus \mathfrak{t}$, $\mathfrak{u}_\Lambda = \mathfrak{f}_-$. Then from [1] and [2]:

$$\mathrm{rk}\ \mathfrak{f} \geq \mathrm{codim}\ \mathfrak{N}(\mathfrak{f}, K) \geq \dim \mathfrak{f} - \dim \mathfrak{f}_+ - \dim \mathfrak{f}_- = \mathrm{rk}\ \mathfrak{f}.$$

All of this is well known; the profit for us is that $e_\Lambda = 0$; so there exists $Z \in \mathfrak{f}_+$ such that dim $(T_z(U_\Lambda Z) + \mathfrak{f}_+)/\mathfrak{f}_+ = \dim \mathfrak{f}_-$.

ii) Let now $(V, L) = (\mathfrak{g}, K)$ and let F be the $\mathbb{R}$-span of ϕ. If rk $\mathfrak{g} = \mathrm{rk}\ \mathfrak{f}$, then $\mathfrak{t}_\mathfrak{g} = \mathfrak{t}$ it is clear that there are $1 - \mathrm{PS}$ in T, regular in both $\mathfrak{t}$ and $\mathfrak{t}_\mathfrak{g}$. We claim that the preceding is true even if rk $\mathfrak{f} < \mathrm{rk}\ \mathfrak{g}$.

For $\mu \in \mathfrak{t}^*$, put $\alpha_\mu \in \mathfrak{t}_\mathfrak{g}^*$ as follows: μ in $\mathfrak{t}$, 0 in $\mathfrak{t}_\mathfrak{g} \cap \mathfrak{p}$. θ induces $\phi \longrightarrow \phi$, $\alpha \longrightarrow \alpha\theta$ and hence $F \longrightarrow F$, called also θ. Clearly $\{x \in F: \theta x = x\} = \{x \in F: x = \alpha_\mu$ for $\mu = x_{|\mathfrak{t}}\}$.

Next for $\alpha \in \phi$, put $\beta = \alpha_{|\mathfrak{t}}$. If $\alpha = \alpha\theta$, $\alpha = \alpha_\beta$ and $\mathfrak{g}_\alpha = \mathfrak{g}_\beta$. If not, put $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha\theta} = \mathfrak{g}_\beta$; it is θ-stable and $\alpha_\beta = 1/2(\alpha + \alpha\theta)$. Under the above map, $\phi(\mathfrak{f}, \mathfrak{t})$ is contained in F, hence E. We identify E with its image.

Now $\{x \in F: \theta x = x\}^\perp = \{x \in F: \theta x = -x\} = \{x \in F: x_{|\mathfrak{t}} = 0\} \supseteq <\{1/2(\alpha - \alpha\theta): \alpha \in \phi\}>$. As the Killing form on F is non degenerate, $E = \{x \in F: \theta x = x\}$, $E \oplus E^\perp = F$ and the restriction of the Killing form on F to E is still non degenerate.

We must prove that the Zariski open cone in E, $E \cap \{H \in F: H$ is regular$\}$ is non empty. If not, putting $P_\alpha = \{H \in F: (\alpha, H) = 0\}$, $\alpha \in \phi$, we have $E \subseteq \cup_\alpha P_\alpha$ and by irreducibility, $E \subseteq P_\alpha$ for some α. Now $\alpha = \alpha_1 + \alpha_2$, $\alpha_1 \in E$, $\alpha_2 \in E^\perp$. Thus $(\alpha_1, E) = 0$, hence $\alpha_1 = 0$ and $\alpha \in E^\perp$. That is, $\alpha_{|\mathfrak{t}} = 0$, $\alpha\theta = -\alpha$. Pick $X \in \mathfrak{s}_\alpha \cap \mathfrak{f}$; $X = X_+ = X_-$ with $X_+ \subset \mathfrak{g}_\alpha$, $X_- \in \mathfrak{g}_{-\alpha}$. $\forall H \in \mathfrak{t}[H, X] = [H, X_+] + [H, X_-] = 0$; then $\mathfrak{s}_\alpha \cap \mathfrak{f} \subseteq \mathfrak{s}_\alpha \cap \mathfrak{t} \subseteq \mathfrak{s}_\alpha \cap \mathfrak{t}_\mathfrak{g} = 0 \therefore \mathfrak{s}_\alpha \subseteq \mathfrak{p}$. Then $\forall y \in \mathfrak{t}_\mathfrak{g} \cap \mathfrak{p}$, $[y, \mathfrak{s}_\alpha] \subseteq [\mathfrak{p}, \mathfrak{p}]$ $\cap \mathfrak{s}_\alpha \subseteq \mathfrak{f} \cap \mathfrak{p} = 0$. Then $\alpha_2 = 0$, a contradiction.

iii) From the preceeding and as in *i)*, it can be shown that $\mathfrak{N}$ is the union of the various K. Z_Λ with φ_λ in the open cone $\{X \in E: X$ is regular in both $\mathfrak{t}$ and $\mathfrak{t}_\mathfrak{g}\}$. Clearly, $\phi \cup 0 \longrightarrow \Pi(\mathfrak{g}, \mathfrak{t})$, $\alpha \longrightarrow \alpha_{|\mathfrak{t}}$ is surjective. Thus $Z_\Lambda = \oplus_{\beta:\beta(\lambda)>0} \mathfrak{g}_\beta = \mathfrak{g}_+$ for the order defined by λ. Even more, $Z_\Lambda \supseteq \oplus_{\beta:\beta(\lambda)>0} \mathfrak{f}_\beta = \mathfrak{f}_+$, $\mathfrak{p}_\Lambda = \mathfrak{f}_+ \oplus \mathfrak{t}$ and $\mathfrak{u}_\Lambda = \mathfrak{f}_-$.

Pick $Z \in \mathfrak{f}_+$ such that dim $(T_Z(U_\Lambda Z) + \mathfrak{f}_+)/\mathfrak{f}_+ = \dim \mathfrak{f}_-$; then dim $(T_Z(U_\Lambda Z) + \mathfrak{g}_+)/\mathfrak{g}_+ = \dim \mathfrak{f}_-$; so $e_\Lambda = 0$ and codim $\mathfrak{N} = \dim \mathfrak{g} - \dim \mathfrak{g}_+ - \dim \mathfrak{f}_- = 1/2(\dim \mathfrak{p} + \mathrm{rk}\ \mathfrak{g} + \mathrm{rk}\ \mathfrak{f})$. ∎

Lemma 7 ([Sch 3], p. 129): *(V, L) is cofree $\Leftrightarrow$ (V, L) is coregular and* codim $\mathfrak{N}(V/L) = \dim\ V/L$. ∎

Proposition B: *(𝔤, K) satisfies criteria B ⇔ $\underline{\mathfrak{g}_R}$ = 𝔰𝔬 (n, 1) or 𝔰𝔲 (n, 1).*

Proposition C: *(𝔤, K) is cofree ⇔ $\mathfrak{g}_R$ = 𝔰𝔬 (n, 1) or 𝔰𝔲 (n, 1).*

Proofs: If (V, L) is visible, $\mathfrak{N}(V, L)$ is the closure of an orbit and then it codim $\mathfrak{N} = \dim(\mathfrak{g}/K)$ holds iff $\mathfrak{g}_R$ = 𝔰𝔬 (n, 1) or 𝔰𝔲 (n, 1). In view of Lemma 7 and Proposition D, this implies Proposition C. Also, we have ⟹ in Proposition B. But cofreeness implies flatness and then all the fibres have the same dimension. ■■

Remark: The cofreeness in case 𝔰𝔬 (n, 1) is also proved in [Sch 2].

Proposition A: *(𝔤, K) is never visible.*

Proof: If (V, L) is visible, $\mathfrak{N}(V, L)$ is the closure of an orbit and then it follows easily that codim $\mathfrak{N} = \dim(V/L)$. (See [K], Lemma 3.5). Furthermore, for L linear reductive (V/L) visible implies that the multiplicity of any non-zero weight is at most 1. ([K], 3.4).

These two facts show the non-visibility of $(\mathfrak{g}, K)$ in most the cases, in view of Lemma 6 and the following well known fact:

If rk $\mathfrak{g} >$ rk $\mathfrak{k}$, then there is some $\alpha \in \Pi_{\mathfrak{k}}(\mathfrak{g})$ with multiplicity greather than one. (In the notation of Lemma 6, we must pick $\alpha \in \phi(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})$ such that $\alpha_{|\mathfrak{t}_{\mathfrak{g}} \cap \mathfrak{p}} \neq 0$).

There are two remaining cases:

$\mathfrak{g}_R$ = 𝔰𝔲 (n, 1): Here $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, $\mathfrak{k} = \{ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} \in \mathfrak{g}: A \in \mathfrak{gl}(n, \mathbb{C}) \}$ and hence we may assume that $K = \{ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \in SL(n+1, \mathbb{C}): X \in GL(n, \mathbb{C}) \}$. Choosing as usual $\mathfrak{b} = \{ \sum_{i=1}^{n+1} H_i e_{i,i}: \sum H_i = 0 \}$ as Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{k}$, it is well known that $\phi(\mathfrak{g}, \mathfrak{b}) = \{ \alpha_{i,j}: \alpha_{i,j}(H) = H_i - H_j$ if $H = \sum H_i e_{i,i}, i \neq j \}$. Take $\phi_+ = \{ \alpha_{i,j}: (i < j$ and $j \leq n$ or $i < n)$ or $(i = n+1, j = n) \}$. It corresponds to the 1–PS Λ given by $\Lambda(z) =$ the diagonal $(z, z^2, z^3, ..., z^{n+1}, z^n)$. Thus, $\mathfrak{g}_+ \subseteq \mathfrak{N}$. Put for $c \in \mathbb{C}$: $y_c = \begin{pmatrix} T & u \\ v & 0 \end{pmatrix}$ where $T = \sum_i e_{i,i+1}$, $u = e_{n-1}$ and $v = c e_n$. We claim that: $y_c \in K y_d \Rightarrow c = d$.

Let $\begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \in K$ such that $\begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} y_c = y_d$. Then $XT = TX$, $Xe_{n-1} = xe_{n-1}$, $cxe_n = de_n X$. Now it is easy to show that $X = xI_{n+1} + be_{1,n+1}$ and thus $c = d$.

$\mathfrak{g}_R$ = 𝔰𝔬 (2n, 1): Here $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$; we will follow the notation of [Hu], p. 3. Then $\mathfrak{k} = \{ x \in \mathfrak{g}: b_1 = b_2 = 0 \}$, $\mathfrak{p} = \{ x \in \mathfrak{g}: m = n = p = 0 \}$ and we assume that $K = \{ \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in SL(2n+1, \mathbb{C}): {}^t XsX = s \}$

Choosing $\mathfrak{b} = \{ H \in \mathfrak{g}: H$ is diagonal $\}$ as Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{f}$, it is known that there is some ordering for which $\mathfrak{g}_+ = \{ x \in \mathfrak{g}: b_1 = 0, p = 0$ and m is upper triangular $\} \subseteq \mathfrak{N}$.

Put for $c \in \mathbb{C}$: $y_c = (b_2 = e_n, m = T$ as above, $n = c (e_{n-1,n} - e_{n,n-1}))$. Then it is not so difficult to prove that $y_c = y_d$ iff $y_c \in K y_d$.

5. SOME REMARKS ON THE UNSTABLE CONE

As a corollary of the proof of Lemma 6, we can state: $\mathfrak{N}(\mathfrak{g}, K)$ *is the union of the various* $K.\mathfrak{g}_+$. *Furthermore, codim* $(\mathfrak{N}(\mathfrak{g}, K)) = codim\ K.\mathfrak{g}_+$ *for every such* $\mathfrak{g}_+$. This suggests us that the irreducible components of $\mathfrak{N}$ are those $K.\mathfrak{g}_+$. Actually, this follows from a general fact (as in [G], Corollary 2, p. 142). Let (V, L) be as above, P a parabolic subgroup of L, W a linear subspace of V such that $P.W \subseteq W$. Then $L.W$ is closed (because of the completeness of L/P).

The following step is to compute $c_{\mathfrak{N}}$, the number of irreducible components of $\mathfrak{N}$. Assume first that rank $\mathfrak{f} =$ rank $\mathfrak{g}$; then $\mathfrak{N}(\mathfrak{g}, K) = U_{every\, \mathfrak{g}_+} K.\mathfrak{g}_+$. From ([G], Corollary 2) we also know that $K.\mathfrak{g}_+ = K.\mathfrak{g}_+'$ if and only if there is some $\sigma \in W(\mathfrak{f}, \mathfrak{t})$ such that $\sigma(\mathfrak{g}_+) = \dot{\mathfrak{g}}_+'$. (Use Bruhat decomposition). Thus.

$$c_{\mathfrak{N}} = |W(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})| / |W(\mathfrak{f}, \mathfrak{t})|$$

Assume now rank $\mathfrak{f} <$ rank $\mathfrak{g}$. We prove now some easy facts in order to compute $c_{\mathfrak{N}}$. As usual $N_L(S)$ (resp. $C_L(S)$) is the normalizer (resp., the centralizer) of S in L.

i) $N_G(\mathfrak{t}) \subseteq N_G(\mathfrak{t}_{\mathfrak{g}})$

Proof: Let $Z \in N_G(\mathfrak{t})$, $\beta \in \Pi(\mathfrak{g}, \mathfrak{t})$. Then $Z.\mathfrak{g}_\beta \subseteq \mathfrak{g}_{\beta Z^{-1}}$. In particular, $Z.\mathfrak{g}_o = Z.\mathfrak{t}_{\mathfrak{g}} \subseteq \mathfrak{t}_{\mathfrak{g}}$.

ii) $C_K(\mathfrak{t}) = N_K(\mathfrak{t}) \cap C_G(\mathfrak{t}_{\mathfrak{g}})$

Proof: We only need to show $C_K(\mathfrak{t}) \subseteq C_G(\mathfrak{t}_{\mathfrak{g}})$. By i), $C_K(\mathfrak{t}) \subseteq N_G(\mathfrak{t}_{\mathfrak{g}})$. Let $Z \in C_K(\mathfrak{t})$ and call ζ its class in $N_G(\mathfrak{t}_{\mathfrak{g}})/C_G(\mathfrak{t}_{\mathfrak{g}}) = W(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})$. As ζ fixes every λ in E, regular in $\mathfrak{g}$, then $\zeta =$ id; i.e. $Z \in C_G(\mathfrak{t}_{\mathfrak{g}})$.

From the preceding, we get the following injections of finite groups:

$$W(\mathfrak{f}, \mathfrak{t}) = N_K(\mathfrak{t})/C_K(\mathfrak{t}) \rightarrow N_G(\mathfrak{t})/(N_G(\mathfrak{t}) \cap C_G(\mathfrak{t}_{\mathfrak{g}})) \rightarrow N_G(\mathfrak{t}_{\mathfrak{g}})/C_G(\mathfrak{t}_{\mathfrak{g}}) = W(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})$$

Call W_1 the group in the middle. (Note that all of this can be done if rank $\mathfrak{f} =$ rank $\mathfrak{g}$; then $W_1 = W(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})$).

Pick λ, μ in the open cone of regular elements both in $\mathfrak{g}$ and in $\mathfrak{f}$, included in E; call $\mathfrak{g}_+{}^\lambda$, $\mathfrak{g}_+{}^\mu$ the respective maximal nilpotent subalgebras of $\mathfrak{g}$. If $\sigma \in W_1$, $\sigma\mathfrak{g}_+{}^\lambda = \mathfrak{g}_+{}^{\sigma\lambda}$.

iii) If $\sigma \in W(\mathfrak{g}, \mathfrak{t}_\mathfrak{g})$ sends $\mathfrak{g}_+{}^\lambda$ to $\mathfrak{g}_+{}^\mu$ then $\sigma \in W_1$.

Proof: Pick $w \in W(\mathfrak{f}, \mathfrak{t})$ such that $w(\mathfrak{f}_+{}^\lambda) = \mathfrak{f}_+{}^\mu$; then σw^{-1} sends $\mathfrak{g}_+{}^{w\lambda}$ to $\mathfrak{g}_+{}^\mu$ so we can replace λ by $w\lambda$ and assume that $\mathfrak{f}_+{}^\lambda = \mathfrak{f}_+{}^\mu$ I.e., $\sigma(\Phi_+{}^\lambda(\mathfrak{f}, \mathfrak{t})) = = \Phi_+{}^\mu(\mathfrak{f}, \mathfrak{t})$. But then $\sigma(\Phi^\lambda(\mathfrak{f}, \mathfrak{t})) = \Phi^\mu(\mathfrak{f}, \mathfrak{t})$ and σ normalizes $\mathfrak{t} = \sum_{\alpha \in \Phi^\lambda}[\mathfrak{f}_\alpha, \mathfrak{f}_{-\alpha}]$; i.e. $\sigma \in W_1$.

We summarize the preceding in:

Lemma 8: *The irreducible components of $\mathfrak{N}(\mathfrak{g}, K)$ are the $K.\mathfrak{g}_+$ where $\mathfrak{g}_+$ corresponds to some $\lambda \in E$ regular both in $\mathfrak{g}$ and in $\mathfrak{f}$. The number of components is $c_{\mathfrak{N}} = |W_1| / |W(\mathfrak{f}, \mathfrak{t})|$.* ∎

Finally, we list some information about W_1 and $c_{\mathfrak{N}}$ for those $(\mathfrak{g}, K)$ satisfying rank $\mathfrak{f} <$ rank $\mathfrak{g}$. We left to the reader the task to verify it.

Type	$\mathfrak{g}$	$\mathfrak{f}$	W_1	$c_{\mathfrak{N}}$
AI, $n=2k$	$\mathfrak{sl}(n+1, \mathbb{C})$	$\mathfrak{so}(n+1, \mathbb{C})$	$\mathbb{Z}_2^k \times_{sd} \mathfrak{S}_k$	1
AI, $n=2k+1$	$\mathfrak{sl}(n+1, \mathbb{C})$	$\mathfrak{so}(n+1, \mathbb{C})$	$\mathbb{Z}_2^k \times_{sd} \mathfrak{S}_k$	2
AII	$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{sp}(2n, \mathbb{C})$	$\mathbb{Z}_2^n \times_{sd} \mathfrak{S}_n$	1
BDI, $p=2r+1$, $q=2s+1$	$\mathfrak{so}(p+q, \mathbb{C})$	$\mathfrak{so}(p, \mathbb{C}) X$ $\mathfrak{so}(q, \mathbb{C})$	$\mathbb{Z}_2^{r+s} \times_{sd} \mathfrak{S}_{r+s}$	$\binom{r+s}{s}$
EI	e_6	$\mathfrak{sp}(8, \bar{\mathbb{C}})$		3
EIV	e_6	f_4		1
II	$\mathfrak{l} \times \mathfrak{l}$, $\mathfrak{l}$ simple	diag $(\mathfrak{l})$	$W(\mathfrak{l})$	1

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