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A remark on the L^s-regularity of the minima of functionals of the calculus of variations

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ABSTRACT. In this note we study the summability properties of the minima of some non differentiable functionals of Calculus of the Variations.

1. INTRODUCTION

Let us consider the two following functionals

$$I(u)=\int_G f(x, Du)\,dx-\int_G h(x)\,u(x)\,dx$$

$$J(u)=\int_G f(x, Du)\,dx-\sum_{i=1}^{n}\int_G g_i(x)\,u_{x_i}(x)\,dx$$

where:

i) G is a bounded open subset in R^n, $n\geq 2$

ii) $f=f(x,z)$ is a Caratheodory function

iii) $h(x)\in L^p(G)$ and $g=(g_1,\dots,g_n)$ denotes a vector field in G with $g_i\in L^p(G)$ for $i=1,\dots,n$, $p\geq 2$.

If $f(x,z)$ is differentiable with respect to the variable z then a minimum $\bar u\in H_0^1(G)$ of $I(u)$ has to satisfy the Euler equation

$$-\frac{\partial}{\partial x_i} f_{z_i}\ (x, Du)=h(x) \tag{1}$$

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and a minimum $\bar{u} \in H_0^1(G)$ of $J(u)$

$$\frac{\partial}{\partial x_i} f_{z_i}(x, Du) = \sum_{i=1}^{n} (g_i)_{x_i} \tag{2}$$

in the sense of distributions.

When (1) and (2) are linear elliptic equations, for example if $f(x, Du) = a_{ij}(x) u_{x_i} u_{x_j}$ with coefficients a_{ij} measurable bounded functions such that $a_{ij}(x)\xi_i \xi_j \geq \alpha |\xi|^2$, $\alpha > 0$ and $\xi \in R^n$, the summability properties of the weak solutions have been studied by Stampacchia (see [7], [8], [9], [10]) in the case $n \geq 2$ and NIRENBERG [5], if $n = 2$.

Moreover analogous summability properties for weak solutions of non linear elliptic equations have been studied by Boccardo and Giachetti in [2]. In this note we extend such results to the minima $\bar{u} \in H_0^1(G)$ of general functionals of the Calculus of Variations $I(u)$ and $J(u)$ under the only assumption on $f(x, z)$

$$\liminf_{\epsilon \to 0^+} \frac{f(x, z) - f(x, z - \epsilon z)}{\epsilon} \geq |z|^2 \tag{3}$$

Since some existence results of minima of integral functionals of the Calculus of Variations have been proved without the convexity assumption on the integrand function (See [5]), it makes sense to study also regularity of such minima.

Moreover we point out that in this paper no upper control on $f(x, z)$ is assumed.

More precisely we prove the following theorems.

Theorem 1. *Let* $\bar{u} \in H_0^1(G)$ *be a non negative minimum of* $I(u)$. *If* f *satisfies* (3) *and* $h(x) \in L^p(G)$, $2 \leq p < \frac{n}{2}$, *then* $\bar{u} \in L^q(G)$, $q = \frac{np}{n-2p}$.

Theorem 2. *Let* $\bar{u} \in H_0^1(G)$ *be a minimum of* $J(u)$. *If* f *satisfies* (3)*and* $g_i \in L^p(G)$, $2 < p < n$, *then* $\bar{u} \in L^q(G)$, $q = \frac{np}{n-p}$.

Remark 1. If, instead of assumption (3), we have

$$\liminf_{\epsilon \to 0} \frac{f(x, z) - f(x, z - \epsilon z)}{\epsilon} \geq |z|^r, \quad r > 1$$

with $r' < p < \frac{n}{r-1}$, then, by proceeding as in the proof of theorem 2, we obtain that $u \in L^s(G)$, $s = [p(r-1)]^*$, if $g_i \in L^{p(r-1)}$.

2. PROOF OF THE THEOREMS

Before proving the theorems we introduce some notations. For a given function u on G we denote by $u^{\#}(x)$ its spherically decreasing rearrangement which is defined on $G^{\#}$, the ball centered at the origin with the same measure than G.

For this definition and for the real functions we refer to [3]. In the proof of theorem 1 we use the following comparison result which is proved in a more general context in [6].

Theorem 3. *Let* $\bar{u} \in H_0^1(G)$ *be a non negative minimum of* $I(u)$. *If* f *satisfies* (3) *and* $h(x) \in L^p(G)$ $p \geq 2$, *then the rearrangement* $\bar{u}^{\#}$ of $\bar{u}$ *verifies the following estimate*

$$\bar{u}^{\#}(x) \leq v(x) \qquad \text{a.e. in } G^{\#} \tag{4}$$

where $v(x)$ is the minimum of the functional

$$\int_G \left[\frac{|Du|^2}{2} - g^{\#}(x)u(x)\right] dx. \tag{5}$$

Proof of Theorem 1. The result we prove now is a consequence of previous Theorem 3. Indeed let $\bar{u} \in H_0^1(G)$ be a minimum of $I(u)$, by theorem 3, $\bar{u}^{\#}$ satisfies the inequality $\bar{u}^{\#}(x) \leq v(x)$ a.e. in $G^{\#}$ and $v(x)$ has to satisfy the Euler equation of (5) which is $-\Delta v = g^{\#}$ in $G^{\#}$.

Since $\|g^{\#}\|_{L^p(G^{\#})} = \|g\|_{L^p(G)}$ (see [3]), we deduce by Agmon-Douglis-Nirenberg's theorem and by Sobolev inequality that $v \in L^q(G^{\#}) q = \frac{np}{n-2p}$. Consequently, by (4), $\bar{u}^{\#} \in L^q(G)$ and, by the above property of the rearrangements, $\bar{u} \in L^q(G)$.

Remark 2. The above arguments don't work when $g \in H^{-1,p'}(G)$ so that we need a different one to prove theorem 2.

Proof of Theorem 2

Consider a real continuous function of one variable $\phi(t)$ satisfying $\phi(0) = 0$ and $\phi'(t) \geq 0 \; \forall t$.

Let $\bar{u}\in H_0^1(G)$ be a minimum of $J(u)$ and consider for each $k>0$.

$$T_k(\bar{u})=\begin{cases} \bar{u} & \text{if } |\bar{u}|\leq k \\ k & \text{if } \bar{u}>k \\ -k & \text{if } \bar{u}<-k \end{cases}$$

the function $\phi(T_k\bar{u})\in H_0^1(G)$ for each $k>0$.

Obviously, for $\epsilon>0$, since $\bar{u}$ is a minimum of $J(u)$, we have

$$J(\bar{u})\leq J(\bar{u}-\epsilon\phi(T_k\bar{u}))$$

By using the definition of J and eliminating equal terms we get

$$\int_G f(x, D\bar{u})\,dx\leq\int_G f[x, D\bar{u}-\epsilon D(\phi(T_k\bar{u}))]\,dx$$
$$+\epsilon\sum_{i=1}^{n}\int_G g_i(x)\,(\phi(T_k\bar{u}))_{x_i}\,dx$$

Dividing for $\epsilon>o$, the previous inequality may be written in the following equivalent way:

$$\int_G \frac{f(x, D\bar{u})-f(x, D\bar{u}-\epsilon D\phi(T_k\bar{u}))}{\epsilon\phi'(T_k\bar{u})}\phi'(T_k\bar{u})\,dx\leq$$

$$\leq\sum_{i=1}^{n}\int_G g_i(x)\phi'(T_k\bar{u})\,(T_k\bar{u})_{x_i}\,dx$$

Now we use assumption (3), Fatou's lemma and Schwartz inequality, to get

$$\int_{|u|\leq k}|Du|^2\phi'(u(x))\,dx\leq\sum_{i=1}^{n}\int_{|u|\leq k} g_i(x)u_{x_i}(x)\phi'(u)\,dx\leq$$

$$\leq\left(\int_{|u|\leq k}|g|^2\phi'(u)\,dx\right)^{\frac{1}{2}}\left(\int_{|u|\leq k}|Du|^2\phi'(u)\,dx\right)^{\frac{1}{2}}$$

So we obtain

$$\int_{|u|\leq k}|Du|^2\phi'(u)\,dx\leq\int_{|u|\leq k}|g|^2\phi'(u)\,dx. \tag{6}$$

Now we choose $\phi(s)=\frac{1}{t+1}|s|^t s$ so that $\phi'(s)=|s|^t$ with t some positive real number which we shall precise in the following.

Such test functions have been introduced by Miranda in [4] and have been also used in [2] to study the regularity of solutions of non linear elliptic equations.

From (6) we have

$$\int_{|u|\leq k} |Du|^2 |u|^t dx \leq \int_{|u|\leq k} |g|^2 |u|^t dx$$

or equivalently

$$\left(\frac{2}{t+2}\right)^2 \int_{|u|\leq k} |D(|u|^{\frac{t}{2}+1})|^2 dx \leq \int_{|u|\leq k} |g|^2 |u|^t dx.$$

Let us denote by q^* the Sobolev exponent of any number $q \in]1, n[$, i.e. $q^* = \dfrac{nq}{n-q}$,

By using Sobolev and Holder inequalities, we get

$$\left(\int_{|u|\leq k} |u|^{(\frac{t}{2}+1)2^*} dx\right)^{\frac{2}{2^*}} \leq \frac{(t+2)^2}{4}\left(\int_{|u|\leq k} |g|^p\right)^{\frac{2}{p}}\left(\int_{|u|\leq k} |u|^{t\frac{p}{p-2}}\right)^{1-\frac{2}{p}} \tag{7}$$

Now choose t in such a way that $\alpha \equiv (\frac{t}{2}+1)2^* = \dfrac{tp}{p-2}$, i.e. $t = n\dfrac{(p-2)}{n-p}$, then $\alpha = p^* = \dfrac{np}{n-p}$. By easy calculations from (7) we have

$$\left(\int_{|u|\leq k} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \leq c\left(\int_{|u|\leq k} |g|^p dx\right)^{\frac{1}{p}} \leq c\left(\int_G |g|^p\right)^{\frac{1}{p}}$$

with $c = \dfrac{n}{2}\ \dfrac{p-2}{n-p} + 1$.

Consequently for $k \to +\infty$ we get the estimate

$$\left(\int_G |u|^{p^*} dx\right)^{\frac{1}{p^*}} \leq c\left(\int_G |g|^p\right)^{\frac{1}{p}}$$

Remark 3. The previous proof also works if we consider the functional

$$\int_G f(x, Du)\,dx - \int_G h(x)u(x)\,dx - \sum_{i=1}^n \int_G g_i(x)u_{x_i}(x)\,dx$$

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