

Orthogonality in normed linear spaces: a classification of the different concepts and some open problems

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ABSTRACT. Orthogonality in inner product spaces is a binary relation that can be expressed in many ways without explicit mention to the inner product of the space.

Great part of such definitions have also sense in normed linear spaces. This simple observation is at the base of many concepts of orthogonality in these more general structures.

Various authors introduced such concepts over the last fifty years, although the origins of some of the most interesting results that can be obtained for these generalized concepts are, as usual in Mathematics, in previous or parallel works about convex sets, ellipsis or ellipsoids, duality, etc. (see, e.g., Gruber's paper [13] about the prior contributions of Caratheodory, Blaschke or Radon).

DIFFERENT CONCEPTS OF ORTHOGONALITY

According to its greater frequency in the literature this exposition is limited to the case in which E is a real normed linear space.

When the norm of E is induced by an inner product, the orthogonality of two points x and y of E is equivalent to each one of next (main or secondary) propositions.

In the more general context of normed linear spaces any one of such propositions is a definition of orthogonality between x and y .

ROBERTS (1934): $\|x - \lambda y\| = \|x + \lambda y\|$, for every $\lambda \in \mathbb{R}$.

BIRKHOFF (1935): $\|x\| \leq \|x + \lambda y\|$, for every $\lambda \in \mathbb{R}$.

CARLSSON (1961): $\sum_{k=1}^m a_k \|b_k x + c_k y\|^2 = 0$, where $m \geq 2$ and a_k, b_k, c_k are real

numbers such that

$$\sum_{k=1}^m a_k b_k c_k \neq 0, \quad \sum_{k=1}^m a_k b_k^2 = \sum_{k=1}^m a_k c_k^2 = 0.$$

Obviously C-orthogonality is not a single concept of orthogonality but a family of them. Before and after Carlsson's paper [8] the following members of such family have been considered separately:

Isosceles (1945): $\|x-y\| = \|x+y\|$

Pythagorean (1945): $\|x-y\|^2 = \|x\|^2 + \|y\|^2$

both introduced by James [14].

a-Isosceles (1988): $\|x-ay\| = \|x+ay\|$

a-Pythagorean (1988): $\|x-ay\|^2 = \|x\|^2 + a^2\|y\|^2$

both for some fixed $a \neq 0$. They appear in [3] starting from a hint of [6].

ab (1978): $\|ax+by\|^2 + \|x+y\|^2 = \|ax+y\|^2 + \|x+by\|^2$

where $a, b \in (0,1)$, considered by Kapoor and Prasad [18].

a (1983): $(1+a^2)\|x+y\|^2 = \|ax+y\|^2 + \|x+ay\|^2$

where $a \neq 1$, considered by Diminnie, Freese and Andalaft [12].

UNITARY-CARLSSON: either $\|x\|\|y\| = 0$, or $\|x\|^{-1}x$ is Carlsson-orthogonal to $\|y\|^{-1}y$.

Apparently there is no general study of this family of concepts. However, as for Carlsson's one, some particular members of this family have been considered:

U-Isosceles (1957): either $\|x\|\|y\| = 0$, or $\|x\|^{-1}x$ is Isosceles-orthogonal to $\|y\|^{-1}y$; introduced by Singer [22].

U-Pythagorean (1986): either $\|x\|\|y\| = 0$, or $\|x\|^{-1}x$ is Pythagorean-orthogonal to $\|y\|^{-1}y$, considered by Diminnie, Andalaft and Freese [11].

DIMINNIE (1983): $\sup\{f(x)g(y)-f(y)g(x); f, g \in S'\} = \|x\|\|y\|$, where S' denotes the unit sphere of the topological dual of E .

AREA (1984): either $\|x\|\|y\| = 0$ or they are linearly independent and such that $x, y, -x, -y$ divide the unit ball of their own plane (identified to \mathbb{R}^2) in four equal areas. Introduced by Alonso [1].

As we have pointed out above all these propositions mean the same (orthogonality between x and y) when E is an inner product space, but, in general, they do not mean the same in normed linear spaces. Moreover, equivalence between almost any two of them is a characteristic property of inner product spaces. [1,2,9,17,20].

On the other hand, orthogonality in inner product spaces is symmetric ($x \perp y$ implies $y \perp x$), homogeneous ($x \perp y$ and $\lambda, \mu \in \mathbb{R}$ imply $\lambda x \perp \mu y$), additive ($x \perp y$ and $x \perp z$ imply $x \perp y+z$), existing (for every $x, y \in E$ there exists $\alpha \in \mathbb{R}$ such that $x \perp \alpha x + y$), unique (the above α is unique when $x \neq 0$), etc.

The study of such properties for generalized orthogonalities in normed linear spaces is one of the main topics of many papers. A description of current knowledge with an extensive bibliography can be seen in [2].

SOME OPEN PROBLEMS

Within the two families of the above-mentioned questions (relations between two different orthogonalities and properties of each orthogonality) some substantial problems remain open. Among them some of the most general and, perhaps, most interesting are the following ones:

Problem 1. It is known, e.g., that if Isosceles implies Pythagorean, or conversely, then E is an inner product space [9]. However nothing similar seems to be known for the general case of any two C-orthogonalities.

We conjecture that if a C-orthogonality implies another "essentially different" C-orthogonality then E is an inner product space.

Problem 2. It is known, e.g., that U-Isosceles is equivalent to U-Pythagorean when E is the space \mathbb{R}^2 endowed with a norm whose unit sphere is a regular octagon [1]. It is also known for other pairs of U-orthogonalities (even in two dimensional spaces) that if one implies the other then E is an inner product space [3].

Our conjecture is that if $\dim E = 3$ and if an U-orthogonality implies another "essentially different" U-orthogonality then E is an inner product space.

Problem 3. There are C-orthogonalities that are trivially symmetric (Isosceles, Pythagorean, ...) and other that are non symmetric (a-Isosceles, a-Pythagorean, with $a \neq 1$, ...). Moreover, it is known that if a-Isosceles or a-Pythagorean ($a \neq 1$) are symmetric then E is an inner product space [18,19].

We conjecture that either a C-orthogonality is "trivially symmetric" or such a property is characteristic of inner oproduct spaces.

Problem 4. It is known that additivity of Carlsson (on the left or on the right in non-symmetric cases) is equivalent to homogeneity and a characteristic property of inner product spaces [8].

It is known also that additivity of Diminnie [10], U-Pythagorean [7] and Birkhoff (on the left) [15] are characteristic properties of inner product spaces of dimension ≥ 3 . But additivity on the right of Birkhoff is characteristic of smooth spaces [16] and additivity of Roberts may be trivial since this orthogonality may be strongly non-existing (i.e. such that $x \perp y$ if and only if $\|x\| \|y\| = 0$) [14].

On the other hand, in two-dimensional spaces and for homogeneous orthogonalities (all of them excepting Carlsson), additivity is equivalent to uniqueness and hence it is either a general property of some orthogonalities as Area, U-Isosceles, ..., (in 2-dimensional spaces) or it is closely related to some geometric properties of the space (rotundity, smoothness, ...) which are weaker than its norm induced by an inner product.

Our conjecture is that additivity of Area and U-Carlsson (on the left and on the right in non symmetric cases) are characteristic properties of inner product spaces of dimension ≥ 3 .

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