REVISTA MATEMATICA de la
Universidad Complutense de Madrid.
Volumen 2, número suplementario, 1989.

On minimality and l^p-complemented subspaces of Orlicz function spaces

FRANCISCO L. HERNÁNDEZ and BALTASAR RODRIGUEZ-SALINAS

ABSTRACT. Several properties of the class of minimal Orlicz function spaces L^F are described. In particular, an explicitly defined class of non-trivial minimal functions is showed, which provides concrete examples of Orlicz spaces without complemented copies of l^p-spaces.

A classical topic in Banach spaces is the study of the existence of l^p-complemented subspaces. It is well-known that from the existence of l^p-subspaces in a Banach space E does not follow that E contains a *complemented* copy of some l^p-space ($1<p<\infty$). This happens even when we restrict ourselves to reflexive Banach lattices E. The natural counter-examples for this are inside the class of minimal Orlicz sequence spaces studied by Lindenstrauss and Tzafriri ([L-T$_1$], [L-T$_2$], [L-T$_3$] pp. 164):

Theorem 1. *Given $1<\alpha\leqslant\beta<\infty$ arbitrary. There exists a minimal Orlicz sequence space l^F with indices α and β which does not have any complemented subspace isomorphic to l^p for $p\geqslant 1$, in spite of the fact that l^F contains isomorphic copies of l^p for any $\alpha\leqslant p\leqslant\beta$.*

Recall that an Orlicz function F is *minimal* at 0 ([L-T$_1$]) if for every function $G\in E_{F,1}$ it happens that $E_{G,1}=E_{F,1}$ where $E_{F,1}$ is the compact set $E_{F,1}=\overline{\{F(\lambda t)/F(\lambda):\ 0<\lambda\leqslant 1\}}$ in C[0,1]. The existence of minimal functions at 0 (different of the multiplicative ones t^p $1<p<\infty$) is proved by means of Zorn Lemma.

The examples given in ([L-T$_1$], [L-T$_2$]) of minimal functions are not explicitly defined in terms of elementary functions. In fact, all minimal functions are obtained, up to equivalence, via the method of contructing Orlicz functions F_ρ associated to 0-1 valued sequences $\rho=(\rho(n))_{n=1}^{\infty}$. This method due also

Supported in part by CAICYT grant 0338-84.
1980 Mathematics Subject Classification (1985 revision): 46A45, 46E30.
Editorial de la Universidad Complutense. Madrid, 1989.

to Lindenstrauss and Tzafriri ([L-T$_2$], [L-T$_3$] pp. 161), is a useful technique but rather sophisticated and uneasy to handle.

One of the goals of this lecture, which collects several results in [H-R.S$_1$] and [H-R.S$_2$], is to present a suitable class of minimal Orlicz spaces for which the minimal functions are *explicitly* defined. As far as we know these functions are the first examples of non-trivial minimal functions defined in a elementary form and without appealing to the above mentioned 0-1 valued sequence method.

We refer to ([L-T$_3$], [L-T$_4$]) for the definitions and terminology used on Orlicz and Banach spaces.

The class of *minimal Orlicz function spaces* $L^F(\mu)$ was introduced by V. Peirats and the first named author in [H-P$_1$], showing the existence of reflexive function spaces $L^F(\mu)$ without any complemented copy of l^p for any $p \neq 2$. (The Rademacher functions span a complemented subspace isomorphic to l^2).

Recall that a function F is *minimal at* ∞ ([H-P$_1$]) if $E^\infty_{F,1} = E^\infty_{G,1}$ for every function $G \in E^\infty_{F,1}$, where $E^\infty_{F,1}$ is the compact subset of the continuous function space C[0,∞) defined by

$$E^\infty_{F,1} = \overline{\left\{ \frac{F(\lambda t)}{F(\lambda)} : \lambda \geqslant 1 \right\}}$$

This notion of minimality at ∞ is slightly stronger than the minimality at 0. Fixed a minimal function M at 0 it is always possible to find a minimal function F at ∞ in such a way that its restriction to the [0,1] interval coincides with the function M.

Minimal function spaces $L^F(\mu)$ have several interesting properties (see [H-P$_2$], [H-P$_3$], [P]). For instance, a minimal space $L^F(0,1)$ contains always a complemented copy of the sequence space l^F, and moreover the projection from $L^F(0,1)$ on l^F is contractive. Also it holds that the associated indices to F at 0 and at ∞ are the same, i.e. $\alpha^\infty_F = \alpha_F$ and $\beta^\infty_F = \beta_F$.

The following result was proved in [H-P$_1$] for the cases of indices placed on the same side of 2. Afterwards in [H-R.S$_1$] this restriction was removed:

Theorem 2. *Given $1 < \alpha \leq \beta < \infty$ arbitrary. There exists a minimal Orlicz function space $L^F(0,1)$ with indices $\alpha^\infty_F = \alpha$ and $\beta^\infty_F = \beta$ which does not have any complemented subspace isomorphic to l^p for any $p \neq 2$.*

The proof of this result makes bassicaly use of the fact that a minimal Orlicz function space $L^F(0,1)$ contains a complemented copy of l^p for $p \neq 2$ if and only if the minimal Orlicz sequence space l^F does the same.

We shall show here that inside the suitable class of explicit minimal functions there are concrete examples of Orlicz (function and sequence) spaces without complemented copies of l^p-spaces.

Before going further, we would like to offer the motivation for the appearance of this class of functions and some related questions:

W. Johnson, B. Maurey, G. Schechtman and L. Tzafriri in ([J-M-S-T] pp. 235) consider the function $F(t)=t^p \exp(f(\log t))$ for $p>1$ where f is defined by

$$f(x)=\sum_{k=1}^{\infty}\left(1-\cos\frac{\pi x}{2^k}\right),$$

obtaining that the associated Orlicz function spaces $L^F(0,1)$ and $L^F(0,\infty)$ are isomorphic spaces. This gave a counterexample to a Mityagin's conjecture ([M]) saying that any Orlicz space (and more generally any symmetric space) with the above property has to be necessarily an L^p-space, $(1\leq p\leq\infty)$. Before that, Nielsen in [N] had proved that the Mityagin conjecture is true for the restricted class of Orlicz functions with slowly variation at ∞.

In ([N] pp. 256) it appears also the question whether the fact that two Orlicz function spaces $L^G(0,\infty)$ and $L^F(0,\infty)$are isomorphic implies that the corresponding Orlicz sequence spaces l^F and l^G have to be also isomorphic (or even more, the same space). A counterexample to this is obtained by considering the above Johnson et al. function F and as G the function defined by

$$G(t)=\begin{cases} t^2 & \text{if } 0\leq t\leq 1 \\ 2F(t)-1 & \text{if } t>1 \end{cases}$$

Then, using ([J-M-S-T], pp. 216), we have that

$$L^F(0,\infty)\approx L^F(0,1)\approx L^G(0,\infty),$$

but l^F and l^G are clearly not isomorphic.

When we restrict to minimal functions the above question has a positive answer:

Proposition 3. *If $L^F(0,\infty)$ and $L^G(0,\infty)$ are isomorphic for F and G minimal functions then l^F and l^G are also isomorphic.*

We present now the class of explicit minimal spaces. (In particular we get that the Johnson et al. function is minimal):

Theorem 4. *Given $p>1$ and q arbitrary. If $F_{p,q}$ is the function $F_{p,q}(0)=0$ and*

$$F_{p,q}(t)=t^p exp(qf(logt)) \quad if\ t>0,$$

then $L^{F_{p,q}}(\mu)$ is a minimal Orlicz space.

Scketch of the Proof: First notice that for $q=0$ we get the L^p-spaces, so the result is obvious.

Let us consider $F_{p,q}\equiv F$ for $q\neq 0$. If $G\in E^{\infty}_{F,1}$ and G is not equivalent to F, there exists a sequence $(s_n)\nearrow\infty$, such that

$$G(t)=\lim_{n\to\infty}\frac{F(e^{s_n}t)}{F(e^{s_n})}=t^p e^{q\,g(\log t)}$$

uniformly on the compact subsets of $[0,\infty)$ and where the function g is defined by

$$g(x)=\lim_{n\to\infty}[f(s_n+x)-f(s_n)]$$

$$=\lim_{n\to\infty}\sum_{k=1}^{\infty}(\cos\frac{\pi s_n}{2^k}-\cos\frac{\pi(x+s_n)}{2^k}).$$

Now for each $m\in\mathbb{N}$ we can take an scalar $0\leq s_n^{(m)}\leq 2^{m+1}$ with $s_n\equiv s_n^{(m)}$ (mod. 2^{m+1}). So, there exists a subsequence converging to a $\sigma_m\in[0,2^{m+1}]$. Thus, using the Cantor Diagonal method, we obtain a subsequence, denoted also by (s_n), such that $s_n^{(m)}\to\sigma_m$ and $0\leq\sigma_m\leqslant 2^{m+1}$ for each $m\in\mathbb{N}$.

Using the uniform convergence it can be deduced the following expression for the function g:

$$g(x)=\sum_{k=1}^{\infty}(\cos\frac{\pi\sigma_k}{2^k}-\cos\frac{\pi(x+\sigma_k)}{2^k}).$$

Now it rests to show that the function $F\in E^{\infty}_{G,1}$. By considering the sequence $(r_n)=(2^{n+1}-\sigma_n)$ and the uniform convergence, it is found out that

$$\lim_{n\to\infty}g[(r_n+x)-g(r_n)]=\sum_{k=1}^{\infty}(1-\cos\frac{\pi x}{2^k})=f(x)$$

So

$$\lim_{n\to\infty}\frac{G(e^{r_n}t)}{G(e^{r_n})}=t^p e^{q f(\log t)}=F(t)$$

and $F \in E^{\infty}_{G,1}$. This implies that $E^{\infty}_{F,1} \subset E^{\infty}_{G,1} \subset E^{\infty}_{F,1}$, and F is minmal at ∞.
q.e.d.

A direct consequence is that the sequence spaces $l^{F_{p,q}}$ are also minimal spaces (As far as we know the first examples defined explicitly).

More properties of this class of minimal spaces are the following:

Proposition 5. *Fixed $p>1$. For any q it holds that:*
(a) The associated indices at 0 and at ∞ to the function $F_{p,q}$ are equal to p.
(b) The spaces $L^{F_{p,q}}(0,1)$ and $L^{F_{p,q}}(0,\infty)$ are Riesz-isomorphic.
(c) Two spaces $L^{F_{p,r}}$ and $L^{F_{p,q}}$ are isomorphic if and only if $q=r$.

The proof of (b) is analogous to ([J-M-S-T], pp. 236): The function $F_{p,q} \equiv F$ is such that there exists a constant $K>0$ and an increasing sequence (r_n) with

$$\sum \frac{1}{F(r_n)} = 1 \text{ and } K^{-1} F(t) \leq \frac{F(r_n t)}{F(r_n)} \leq KF(t)$$

for every $n \in \mathbb{N}$ and $0 \leq t < \infty$. Now, let us consider a disjoint interval sequence (A_n) in $(0,1)$ with measure $\mu(A_n) = \dfrac{1}{F(r_n)}$ and φ_n the increasing affine mapping from A_n onto $[n, n+1)$. Then the operator $T: L^F(0,\infty) \rightarrow L^F(0,1)$ defined by

$$T(f) = \sum_{n=1}^{\infty} r_n \chi_{A_n} f(\varphi_n)$$

is a Riesz-isomorphism.

The statement (c) is obtained using the uniqueness of the symmetric structure for reflexive Orlicz function spaces ([J-M-S-T]) and the fact that the function $f(x)$ is not bounded at $\pm\infty$.

We pass now to study the embedding of l^p as a complemented subspace into the spaces $L^{F_{p,q}}$. It is still unknown a characterization of when an Orlicz (sequence or function) space contains a complemented copy of l^p. However there exist some necessary or sufficient conditions (see [L-T_3], [K], [L], [H-P_2]).

The following definition is an extension to the function space case of the Lindenstrauss and Tzafriri's one given for the Orlicz sequence space setting:

Fixed $\sigma>0$, the function t^p is called σ-*strongly non-equivalent* to $E^{\infty}_{F,1}$ if there exist two sequences of numbers (K_n) and integers (m_n), so that for $n \rightarrow \infty$ $K_n \rightarrow \infty$ and $m_n = o(K_n^{\sigma})$; and m_n-points $t_i \in (0,1)$ such that for every $\lambda \in [\max_i t_i^{-1}, \infty)$ there is at least one index i, $1 \leq i \leq m_n$ for which

$$\frac{F(\lambda t_i)}{F(\lambda)t_i^p} \notin [\frac{1}{K_n}, K_n]$$

For reflexive function spaces the above condition gives an useful criterion:

Theorem 6. *Given a reflexive space $L^F(0,1)$ and $p \neq 2$. If t^p is σ-strongly non-equivalent to $E^{\infty}_{F,1}$ for some $\sigma < \frac{1}{\beta^{\infty}_F}$, then $L^F(0,1)$ does not contain a complemented copy of l^p.*

The proof of this result has two different parts. The first step is to show using the thecniques developped in ([L-T$_2$], pp. 360) that under the hypothesis of the Theorem, *no weighted Orlicz sequence space* $l^F(w)$, with $\sum w_n < \infty$ (cf. [H-P$_2$]), contains a complemented subspace isomorphic to l^p.

The other fact needed is the following Lemma proved in [H-R.S$_1$] by using the disjointification Kadec-Pelczynski method (cf. [L-T$_4$] Proposition 1.c.8).

Proposition 7. *Let $L^F(0,1)$ be a reflexive space. Then $L^F(0,1)$ contains a complemented copy of l^p for $p \neq 2$ if and only if l^p is isomorphic to a complemented subspace of a weighted Orlicz sequence space $l^F(w)$ with $\sum w_n < \infty$.*

Let us apply these results to the above class of minimal spaces. In order to do it we need to consider an oscilation constant γ_f associated to the function $f(x) = \sum_{k=1}^{\infty} (1 - \cos\frac{\pi x}{2^k})$, defined as follows

$$\gamma_f = \lim_{n\to\infty} \frac{\gamma_n}{n} ,$$

where

$$\gamma_n = \inf_{s>0} \omega_n^f(s)$$

and

$$\omega_n^f(s) = \max_{0 \le x,y \le 2^n} [f(x+s) - f(y+s)] .$$

It can be proved that γ_f satisfies $0 < \gamma_f \le 2$. The following result holds ([H-R.S$_2$]):

Theorem 8. *Let $1 < p \neq 2$ and q verifying that*

$$\frac{p}{|q|} < \frac{\gamma_f}{2 \log_e 2}$$

Then the space $L^{F_{p,q}}$ does not contain any complemented copy of l^p.

As a consequence we easily obtain a result of Lindenstrauss and Tzafriri ([L-T_3], pp. 163) proved by using the method of 0-1 valued sequences:

Corollary 9. *For any $p>1$ there exists a minimal reflexive Orlicz sequence space l^F with indices $\alpha_F=\beta_F=p$ which does not have any complemented copy of l^p.*

Proof. Fixed $p>1$, we take q as

$$q=\frac{4\,p\log 2}{\dot{\gamma}_f}$$

Then considering the function $F_{p,q}\equiv F$ we deduce, from Theorem 8, that L^F does not contain a complemented copy of l^p. Since L^F is a minimal space, we conclude that l^F does not contain a complemented copy of l^p, either.

A natural open question is to determine values $p\neq 2$ and q verifying that the Orlicz space $L^{F_{p,q}}$ contains a complemented subspace isomorphic to l^p.

Any positive result in this direction would imply automatically that Problem 4.b.8 in ([L-T_3] has a negative solution, i.e. the existence of minimal Orlicz sequence spaces which are not prime.

Finally another open question is whether for any minimal function F the associated Orlicz spaces $L^F(0,1)$ and $L^F(0,\infty)$ are isomorphic.

References

[H-P_1] F.L. Hernández V. Peirats: *Orlicz function spaces without complemented copies of l^p.* Israel J. Math 56 (1986), 355-360.

[H-P_2] F.L. Hernández, V. Peirats: *Weighted sequence subspace of Orlicz function spaces isomorphic to l^p.* Arch. Math 50 (1988), 270-280.

[H-P_3] F.L. Hernández, V. Peirats: *Minimal Orlicz function spaces.* Proc. Int. conf. on Function Spaces (Poznán 1986). Teubner-Texte Zur Math. *103* (1988), 26-33.

[H-R.S_1] F.L. Hernández, B. Rodriguez-Salinas: *On l^p-complemented copies in Orlicz spaces.* Israel J. Math *62* (1988), 37-55.

[H-R.S_2] F.L. Hernández, B. Rodriguez-Salinas: *On l^p-complemented copies in Orlicz spaces II.* Israel J. Math. 68 (1989), 27-55.

[J-M-S-T] W.B. JOHNSON, B. MAUREY, G. SCHECHTMANN and L. TZAFRIRI: *Symmetric sructures in Banach spaces.* Mem. Amer. math. Soc. N. 217 (1979).

[K] N. J. KALTON: *Orlicz sequence spaces without local convexity.* Math. Proc. Camb. Phil. Soc. *81* (1977), 253-277.

[L] K. LINDBERG: *On subspaces of Orlicz sequence spaces.* Studia Math. *45* (1973), 119-146.

[L-T_1] J. LINDENSTRAUSS, L. TZAFRIRI: *On Orlicz sequence spaces II.* Israel J. Math *11* (1972), 355-379.

[L-T_2] J. LINDENSTRAUSS, L. TZAFRIRI: *On Orlicz sequence spaces III.* Israel J. Math. *14* (1973), 368-389.

[L-T_3] J. LINDENSTRAUSS, L. TZAFRIRI: *Classical Banach spaces I. Sequence spaces.* Springer-Verlag (1979).

[L-T_4] J. LINDENSTRAUSS, L. TZAFRIRI: *Classical Banach spaces II. Function spaces.* Springer-Verlag (1979).

[M] B . MITYAGIN: *The homotopy structure of the linear group of a Banach space.* Russian Math. Surv. *25* (1970), 59-103.

[N] N.J. NIELSEN: *On the Orlicz function spaces $L_M(0,\infty)$.* Israel J. Math. *20* (1975), 237-259.

[P] V. PEIRATS: *Espacios modulares de sucesiones vectoriales. Subespacios.* Doctoral Thesis. Univ. Complutense Madrid. 1986.

Dpto. de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense
28040-MADRID