Generalized Polarized Manifolds

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ABSTRACT

We introduce and develop the notion of generalized Poisson manifolds and analyze their main properties. Several generalized Hamiltonian maps for polarized Poisson manifolds and vectorial Hamiltonian maps for systems in dimension smaller or equal than 4 are given.

Key words: Hamiltonian systems, Poisson manifolds, symplectic structures, generalized Hamiltonian dynamics of Nambu.

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Introduction

k-symplectic structures were introduced by the author in 1984 ([2, 21]), and later generalized by Puta Mircea in 1988 [21]. Analogous properties in this context were analyzed by different authors, see, e.g., [15, 20].

A polarized structure on an even dimensional smooth manifold M is a pair (θ, E) constituted by a closed differential 2-form θ of maximum rank and by an n-codimensional integrable subbundle E of TM which is Lagrangian with respect to the 2-form θ. Locally, there exists a coordinate system $(x^i, y^i)_{1\leq i\leq n}$ (the Darboux coordinate system) such that

$$\theta = \sum_{i=1}^{n} dx^i \wedge dy^i,$$

and the subbundle E is defined by $dy^1 = \cdots = dy^n = 0$.

The notion of a polarized manifold plays an important role in the theory of Kostant-Souriau's geometric quantization, see for e.g.[18, 22]. Some important properties are given by A. Weinstein, P. Dazord, J. M. Morvan, P. Molino, P. Libermann.

Let us recall that one of the main motivations which led to introduce the notion of k-symplectic structure as an extension of the geometry of polarization ([10]) was to propose a geometric support for the equations of Nambu ([19]), in analogy with the well known symplectic geometry and classical Hamiltonian formalism. Some properties of the Poisson structure subordinate to a k-symplectic manifold have led us to introduce the notion of vectorial polarized Poisson structure. For a fixed finite dimensional real vector space V, this structure is defined on a foliated manifold $(M, \mathfrak{F})$ by a pair $(\mathfrak{H}(M, \mathfrak{F}), P)$, where $\mathfrak{H}(M, \mathfrak{F})$ is a submodule of the space $\mathcal{C}^\infty(M, V)$ of V-valued smooth functions on M, over the ring of basic functions for the foliation $\mathfrak{F}$, and P is a $\mathcal{C}^\infty(M)$-antisymmetric bilinear mapping

$$P : \bigwedge\nolimits_1(M, V) \times \bigwedge\nolimits_1(M, V) \longrightarrow \mathcal{C}^\infty(M, V)$$

that coincides with the classical case for $V = \mathbb{R}$.

A remarkable feature of the Hamiltonian description of classical dynamics is Liouville's theorem, which states that the volume of phase space occupied by an ensemble of systems is remains invariant. The theorem plays, amongst other things, a fundamental role in statistical mechanics. On the other hand, Hamiltonian dynamics is not the only formalism that makes statistical mechanics possible. Any set of equations which leads to Liouville theorem in a suitably defined phase space will do (provided, of course, that ergodicity may be assumed). Nambu proposes a possible generalization of the Hamiltonian dynamics for a 3-dimensional space.

In this context, k-symplectic geometry represents a geometrical tool which comprises differential 2-forms $\theta^1, \ldots, \theta^k$, such that the Hamiltonian map is $\mathbb{R}^k$-valued H, the components of which are related to Hamiltonian systems X_H by means of the identity

$$i(X_H)\theta^p = -dH^p,$$

in order to find Nambu-Hamilton equations preserving the specific features of the classical symplectic geometry.

From this perspective, a k-symplectic structure is a $(k+1)$-tuple $(\theta^1, \ldots, \theta^k; E)$ such that $\theta^1, \ldots, \theta^k$ constitute a non degenerate system vanishing on the tangent vector fields to leaves.

The generalized Darboux theorem shows that there exists about each point x_0 of M a local coordinate system $(x_i^p, y^i)_{1\le p\le k, 1\le i\le n}$ defined on a neighborhood such that

$$\theta^p_{|U} = dx_i^p \wedge dy^i$$

and E is given by $dy^1 = \cdots = dy^n = 0$.

The study of an exterior system in high dimension, underlines the existence of an infinite number of systems not algebraically equivalent. k-symplectic systems are defined directly by conditions of regularity; they can be interpreted as models of exterior systems of maximum rank vanishing on the tangent vector fields of the generalized Lagrangian foliation.

In this work, we introduce the notion of a vectorial polarized structure on a manifold M. This structure is given by (θ, E), where θ is a closed vectorial valued 2-form on M vanishing on the section of a subbundle E. The polarized manifolds and the k-symplectic manifolds are particular vectorial polarized manifolds.

1. Generalized polarized manifolds

1.1. Definition

Let $(M, \mathfrak{F})$ be a foliated manifold, M an m-dimensional manifold endowed with a p-dimensional foliation, and let V be a real k-dimensional vector space.

Let us fix a basis $(e_r)_{1\leq r\leq k}$ of V, and let $\bigwedge_2(M,V) = \bigwedge_2(M)\otimes V$ be the space of V-valued differential two forms; that is, the space of

$$\theta = \theta^\alpha \otimes e_\alpha = \theta^1 \otimes e_1 + \cdots + \theta^k \otimes e_k$$

where $\theta^1, \dots, \theta^k \in \bigwedge_2(M)$.

We say that (θ, E) is a vectorial polarized structure, or generalized polarized manifold, on M, if the following conditions hold:

(i) The V-valued 2-form θ is non degenerate, that is,

$$\forall x \in M,\ \ \forall X \in T_xM, \quad i(X)\theta = 0 \Longrightarrow X = 0.$$

(ii) Each leaf of $\mathfrak{F}$ is maximal totally isotropic with respect to θ.

1.2. Hamiltonian systems

Suppose that M is endowed with a vectorial polarized structure $(\theta; E)$ and let

$$j : \mathfrak{X}(M) \longrightarrow \Lambda_1(M)\otimes V$$

be defined by

$$j(X) = i(X)\theta,\ \forall X \in \mathfrak{X}(M).$$

A vector field X on M is called a vectorial polarized Hamiltonian system if it is an infinitesimal automorphism for the vectorial polarized structure $(\theta; E)$, that is, if the following conditions are satisfied:

(i) X is foliate;

(ii) $i(X)\theta$ is closed.

We denote by $\mathfrak{L}(M,\mathfrak{F})$ the $\mathcal{C}^\infty(M)$-module of infinitesimal automorphisms of the pair (θ, E).

Let X be a vectorial polarized Hamiltonian system. By Poincaré lemma, for every $x \in M$, there exists an open neighborhood U of x and a (smooth) mapping $H : U \to V$ satisfying the relationship

$$i(X)\theta_{|U} = -dH_{|U}.$$

Conversely, if a smooth mapping $H : M \to V$ satisfies

$$dH = dH^\alpha \otimes e_\alpha \in j(\mathfrak{L}(M,\mathfrak{F})),$$

there exists a unique vector field on M, denoted X_H and called the vectorial polarized Hamiltonian system associated to H, such that

$$i(X_H)\theta = -dH$$

on M. The vector field X_H is called *a (strongly) Hamiltonian system.*

A smooth mapping $H : M \to V$ satisfying $dH \in j(\mathfrak{L}(M,\mathfrak{F}))$ is called a vectorial polarized Hamiltonian mapping of the vectorial polarized structure $(\theta; E)$.

1.3. Poisson bracket of a vectorial polarized structure

Let H and K be two vectorial polarized Hamiltonian mappings and X_H, X_K the associated polarized Hamiltonian systems. The Lie bracket $[X_H, Y_K]$ is a polarized Hamiltonian system. More precisely, the mapping of M into V defined by

$$\{H,K\} = \theta(X_H, X_K) = \theta^\alpha(X_H, X_K)e_\alpha$$

satisfies

$$[X_H, X_K] = X_{\{H,K\}}.$$

The mapping $\{H,K\}$ is called the (vectorial) *Poisson bracket* of the (vectorial) Hamiltonian mappings H and K. We denote the space of all vectorial polarized Hamiltonian mappings by $\mathfrak{H}(M,\mathfrak{F},V)$.

2. Polarized manifolds

A real polarization on M is a vectorial polarization (θ, E) such that $m = 2p$ and $V = \mathbb{R}$.

Theorem 2.1 (Darboux theorem)**.** *Every point of M has an open neighborhood U with local coordinates system $(x^1,\dots,x^n,y^1,\dots,y^n)$ such that*

$$\theta = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$$

and $\mathfrak{F}$ is defined by equations $dy^1 = \cdots = dy^n = 0$.

And with respect to an adapted coordinates system $(x^1,\dots,x^n,y^1,\dots,y^n)$, the polarized Hamiltonian mapping H takes the form

$$H=\sum_{i=1}^{n} a_i(y^1,\dots,y^n)x^i+b(y^1,\dots,y^n)$$

where $a_1,\dots,a_n,b$ are basic functions for $\mathfrak{F}$. Recall that, by the symplectic duality $\zeta : X\mapsto i(X)\theta$ between the tangent bundle TM and the cotangent bundle T^*M, we associate to θ a non degenerate bivector P (the Poisson tensor) defined by

$$P(\alpha,\beta)=\theta(\zeta^{-1}(\alpha),\zeta^{-1}(\beta))\quad \text{for all } \alpha,\beta\in \bigwedge\nolimits_1(M),$$

and we have an antisymmetric linear mapping $\underline{P} : \bigwedge_1(M)\to \mathfrak{X}(M)$, given by

$$\langle \beta, \underline{P}(\alpha)\rangle = P(\alpha,\beta).$$

3. k-symplectic manifolds

A k-symplectic structure on M is a vectorial polarization (θ,E) such that $m=n(k+1)$ and $p=nk$.

3.1. Canonical k-symplectic structure on $\mathbb{R}^{n(k+1)}$

Consider $\mathbb{R}^{n(k+1)}$ endowed with its Cartesian coordinates $(x_i^\alpha,y^i)_{1\le\alpha\le k,1\le i\le n}$. Let E be the subbundle of $T\mathbb{R}^{n(k+1)}$ defined by the equations

$$dy^1=0,\ \dots,\ dy^n=0$$

and

$$\theta=\theta^\alpha\otimes e_\alpha=(dx_i^\alpha\wedge dy^i)\otimes e_\alpha$$

The pair (θ,E) defines a k-symplectic structure on $\mathbb{R}^{n(k+1)}$ called *the canonical k-symplectic structure*. This structure induces a natural k-symplectic structure on the torus $\mathbb{T}^{n(k+1)}$.

3.2. The generalized Darboux theorem

Let M be an $n(k+1)$-dimensional manifold. If $(\theta=\theta^\alpha\otimes e_\alpha,E)$ is a k-symplectic structure on M then for every point p of M there exists an open neighborhood U of M containing p endowed with local coordinates $(x_i^\alpha,y^i)_{1\le\alpha\le k,1\le i\le n}$ called an adapted coordinate system, such that the V-valued differential form θ is represented on U by

$$\theta=\theta^\alpha\otimes e_\alpha=(dx_i^\alpha\wedge dy^i)\otimes e_\alpha,$$

and E is defined by the equations $dy^1=0,\dots,dy^n=0$.

Proposition 3.1. *Let $H = (H^\alpha)_{1\leq\alpha\leq k}$ be a vectorial polarized Hamiltonian mapping and let X_H be the associated polarized Hamiltonian system. With respect to an adapted coordinates system $(x_i^\alpha, y^i)_{1\leq\alpha\leq k, 1\leq i\leq n}$, the components H^p of H and X_H can be written as*

$$H^\alpha = x_j^\alpha f^j(y^1,\dots,y^n) + g^\alpha(y^1,\dots,y^n)$$

and

$$\begin{aligned} X_H &= -\left(x_j^\alpha \frac{\partial f^j}{\partial y^s}(y^1,\dots,y^n) + \frac{\partial g^\alpha}{\partial y^s}(y^1,\dots,y^n)\right)\frac{\partial}{\partial x_s^\alpha} \\ &\quad + f^s(y^1,\dots,y^n)\frac{\partial}{\partial y^s}, \end{aligned}$$

where f^j and g^α are smooth basic functions on U.

Remark 3.2. Assume that $k \geq 2$. It follows from the proof of the previous proposition that, if the Pfaffian forms $i(X)\theta^1,\dots,i(X)\theta^k$ are closed (or equivalently $L_X\theta^1 = \cdots = L_X\theta^k = 0$), then the vector field X is necessarily an infinitesimal automorphism of $\mathfrak{F}$.

With respect to an adapted coordinate system $(x_i^\alpha, y^i)_{1\leq\alpha\leq k,1\leq i\leq n}$, the components $\{H,K\}^\alpha$ of $\{H,K\}$ are given by

$$\{H,K\}^\alpha = \sum_{s=1}^n \left(\frac{\partial H^\alpha}{\partial x_s^\alpha}\frac{\partial K^\alpha}{\partial y^s} - \frac{\partial H^\alpha}{\partial y^s}\frac{\partial K^\alpha}{\partial x_s^\alpha}\right),$$

Let $\mathfrak{H}(M)$ be the set of Hamiltonian mappings of the k-symplectic structure $(\theta^1,\dots,\theta^k;E)$. The correspondence $(H,K)\to\{H,K\}$, of $\mathfrak{H}(M)\times\mathfrak{H}(M)$ into $\mathfrak{H}(M)$, is a skew-symmetric $\mathbb{R}$-bilinear mapping satisfying the Jacobi identity.

Proposition 3.3. $(\mathfrak{H}(M), \{\ ,\ \})$ *is an infinite-dimensional Lie algebra.*

4. Nambu's statistical mechanics

Let (x,y,z) be a triplet of dynamical variables (a canonical triplet) which spans a 3-dimensional phase space M. This is a formal generalization of conventional phase space spanned by a canonical pair (p,q). Next, we will introduce two functions H and G depending on (x,y,z) which serve as a pair of "Hamiltonians" to determine the motion of points in phase space. More precisely Nambu has postulated the following Hamilton equations:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{D(H,G)}{D(y,z)}, \\ \dfrac{dy}{dt} = \dfrac{D(H,G)}{D(z,x)}, \\ \dfrac{dz}{dt} = \dfrac{D(H,G)}{D(x,y)}, \end{cases}$$

where $D(H,G)/D(y,z)$ denote the Jacobian

$$\frac{D(H,G)}{D(y,z)} = \frac{\partial H}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial H}{\partial z}\frac{\partial G}{\partial y}.$$

The above equations are called *Nambu's equations of motion*, and the vector field whose integral curves are given by Nambu's equations of motion will be denoted by $X^n_{(H,G)}$ and called the *dynamical system of Nambu.*

Consider the space $M = \mathbb{R}^3$ endowed with its canonical 2-symplectic structure $(\theta^1, \theta^2; E)$ defined by

$$\begin{cases} \theta^1 = dx \wedge dz, \\ \theta^2 = dy \wedge dz, \\ E = \ker dz. \end{cases}$$

The Hamiltonian mapping of the 2-symplectic structure is the mapping

$$H : M \longrightarrow \mathbb{R}^2$$

whose components are given by

$$\begin{cases} H^1 = f(z)x + g^1(z), \\ H^2 = f(z)y + g^2(z), \end{cases}$$

where f, g^1, and g^2 are smooth real functions depending only on the variable z. The integral curves of the Hamiltonian system X_H of the 2-symplectic structure are given by the following equations:

$$\frac{dx}{dt} = -\frac{\partial H^1}{\partial z}, \qquad \frac{dy}{dt} = -\frac{\partial H^2}{\partial z},$$

and

$$\frac{dz}{dt} = \frac{\partial H^1}{\partial x} = \frac{\partial H^2}{\partial y}.$$

Theorem 4.1. *Let $H = (H^1, H^2)$ be a Hamiltonian mapping of the 2-symplectic structure, where $H^1 = f(z)x + g^1(z)$ and $H^2 = f(z)y + g^2(z)$). Then the Hamiltonian system X_H and the dynamical system of Nambu X^n_H are related by*

$$X^n_H = f(z)X_H.$$

Corollary 4.2. *The mapping*

$$(f(z))^{-1}H = (x + h^1(z), y + h^2(z))$$

is a solution of Nambu's equations of motions on a domain where $f(z)$ is a non-vanishing function and

$$h^1(z) = (f(z))^{-1}g^1(z), \qquad h^2(z) = (f(z))^{-1}g^2(z).$$

5. Vectorial polarized Poisson manifolds

5.1. Definition

Let $(M, \mathfrak{F})$ be a foliated manifold, let M be an n-dimensional manifold endowed with a p-dimensional foliation, and let V be a k-dimensional real vector space.

Let us fix a basis $(e_r)_{1\leq r\leq k}$ of V with dual basis $(\omega^r)_{1\leq r\leq k}$, and let $\bigwedge_1(M,V) = \bigwedge_1(M)\otimes V$ be the space of V-valued differential forms of degree 1, that is, the space of

$$\alpha = \alpha^1 \otimes e_1 + \cdots + \alpha^k \otimes e_k$$

where $\alpha^1, \dots, \alpha^k \in \bigwedge_1(M)$. Locally, on an open neighborhood U endowed with local coordinates system $(x^1, \dots, x^n)$, each element $\alpha \in \bigwedge_1(M,V)$ has the form

$$\alpha_{|U} = \sum_{r=1}^{k}\sum_{i=1}^{n} \alpha_i^r dx^i \otimes e_r$$

where $\alpha_i^r : U \to \mathbb{R}$ are smooth mappings.

We denote by E_V^o the annihilator of the subbundle E in $\bigwedge_1(M,V)$, that is, the space of V-valued 1-forms on M vanishing on the cross sections of E.

Definition 5.1. Let $(M, \mathfrak{F})$ be a foliated manifold, let $\mathfrak{H}(M, \mathfrak{F})$ be a submodule of $\mathcal{C}^\infty(M,V)$ over the ring $\mathfrak{B}(M,\mathfrak{F})$ of basic functions for the foliation $\mathfrak{F}$ and let

$$P : \bigwedge\nolimits_1(M,V) \times \bigwedge\nolimits_1(M,V) \longrightarrow \mathcal{C}^\infty(M,V)$$

be an antisymmetric $\mathcal{C}^\infty(M)$-bilinear mapping. We say that $(\mathfrak{H}(M,\mathfrak{F}), P)$ is a vectorial polarized Poisson structure on M, if the following properties hold:

(i) $P(\alpha,\beta) = 0$ for all $\alpha,\beta \in E_V^o$,

(ii) for all $H, K \in \mathfrak{H}(M,\mathfrak{F})$, $P(dH, dK) \in \mathfrak{H}(M,\mathfrak{F})$,

(iii) the correspondence $(H,K) \to \{H,K\} = P(dH,dK)$, from $\mathfrak{H}(M,\mathfrak{F}) \times \mathfrak{H}(M,\mathfrak{F})$ with values in $\mathfrak{H}(M,\mathfrak{F})$, gives to $\mathfrak{H}(M,\mathfrak{F})$ a Lie algebra law,

(iv) each $H \in \mathfrak{H}(M,\mathfrak{F})$ corresponds to a vector field X_H such that

$$\langle dK, X_H\rangle = \{H,K\},$$

for all $K \in \mathfrak{H}(M,\mathfrak{F})$.

P will be called a vectorial polarized Poisson tensor.

Let us consider an open neighborhood U of M endowed with an adapted local coordinates system $(x^1, \dots, x^p, y^1, \dots, y^q)$. Since P is zero on the annihilator E_V^o of the subbundle E in $\bigwedge_1(M,V)$, then the tensor P has the form

$$P = A_{pq}^{ijr}\Big(\Big(\frac{\partial}{\partial x^i}\otimes\omega^p\Big)\wedge\Big(\frac{\partial}{\partial x^j}\otimes\omega^q\Big)\Big)\otimes e_r + B_{pq}^{ijr}\Big(\Big(\frac{\partial}{\partial x^i}\otimes\omega^p\Big)\wedge\Big(\frac{\partial}{\partial y^j}\otimes\omega^q\Big)\Big)\otimes e_r \quad (1)$$

where $A_{pq}^{ijr}, B_{pq}^{ijr} : U \to \mathbb{R}$ are differential mappings. The Jacobi identity implies that

$$
\begin{aligned}
&\frac{\partial A_{uv}^{abr}}{\partial x^l} A_{rw}^{lcv} - \frac{\partial A_{uv}^{abr}}{\partial y^m} B_{wr}^{cmv} + \frac{\partial A_{vw}^{bcr}}{\partial x^l} A_{ru}^{lav} \\
&\qquad\qquad - \frac{\partial A_{vw}^{bcr}}{\partial y^m} B_{ur}^{amv} + \frac{\partial A_{wu}^{car}}{\partial x^l} A_{rv}^{lbv} - \frac{\partial A_{wu}^{car}}{\partial y^m} B_{vr}^{bmv} = 0, \\
&\frac{\partial A_{uv}^{abr}}{\partial x^l} B_{rw}^{lcv} + \frac{\partial B_{vw}^{bcr}}{\partial x^l} A_{ru}^{lav} - \frac{\partial B_{vw}^{bcr}}{\partial y^m} B_{ur}^{amv} - \frac{\partial B_{uw}^{acr}}{\partial x^l} A_{rv}^{lbv} + \frac{\partial B_{uw}^{acr}}{\partial y^m} B_{vr}^{bmv} = 0, \\
&\qquad\qquad\qquad\qquad \frac{\partial B_{uv}^{abv}}{\partial x^l} A_{rv}^{lbv} - \frac{\partial B_{uw}^{acr}}{\partial x^l} B_{rv}^{lbv} = 0.
\end{aligned}
\tag{2}
$$

For each element $\alpha \in \bigwedge_1(M,V)$ we can associate a $\mathcal{C}^\infty(M)$-linear mapping

$$P(\alpha,\cdot) : \bigwedge\nolimits_1(M,V) \longrightarrow \mathcal{C}^\infty(M,V)$$

such that $P(\alpha,\cdot)(\beta) = P(\alpha,\beta)$ for each $\beta \in \bigwedge_1(M,V)$. The linear mapping $P(\alpha,\cdot)$ coincides with the vector field $\underline{P}(\alpha)$ for $k=1$.

6. Model vectorial polarized Poisson manifolds

Let us consider the model space $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ endowed with the p-dimensional model foliation $\mathfrak{F}$ defined by the equations $dy^1 = \cdots = dy^q = 0$, where (x^i, y^j), with $i = 1,\dots,p$ and $j = 1,\dots,q$, are the Cartesian coordinates system and let $V = \mathbb{R}^k$ be the real space where the canonical basis $(e_r)_{1\le r\le k}$ with dual basis $(\omega^r)_{1\le r\le k}$ is fixed.

Let $(\mathfrak{H}(M,\mathfrak{F}), P)$ be a vectorial polarized Poisson structure on $\mathbb{R}^n$. The vectorial polarized Poisson bivector P takes the form (1) and satisfies the Jacobi identity (2), and $\mathfrak{H}(M,\mathfrak{F})$ is a submodule of $\mathcal{C}^\infty(\mathbb{R}^n, V)$ over the ring $\mathfrak{B}(\mathbb{R}^n,\mathfrak{F})$ of basic functions. Now we give some examples of polarized Poisson structures widening the space of vectorial polarized Hamiltonian mappings subordinate to k-symplectic manifolds.

(i) Let $\mathfrak{H}(\mathbb{R}^n,\mathfrak{F})$ be the $\mathfrak{B}(M,\mathfrak{F})$-submodule of $\mathcal{C}^\infty(\mathbb{R}^n,V)$ spanned by $e_1, \dots, e_k$. Thus $\mathfrak{H}(\mathbb{R}^n,\mathfrak{F}) = \mathfrak{B}(\mathbb{R}^n,\mathfrak{F}) \times \cdots \times \mathfrak{B}(\mathbb{R}^n,\mathfrak{F})$ (k times) and for all vectorial polarized Poisson bivector P, the associated Lie algebra $(\mathfrak{B}(M,\mathfrak{F}), \{\ ,\ \})$ is Abelian.

(ii) For $V = \mathbb{R}^2$ we consider the $\mathfrak{B}(M,\mathfrak{F})$-submodule $\mathfrak{H}(\mathbb{R}^n,\mathfrak{F})$ of $\mathcal{C}^\infty(\mathbb{R}^n,V)$ spanned by the mappings

$$
\begin{aligned}
X^I &: (x,y) \longmapsto x^I e_1 && (I = 1,\dots,p),\\
X^{ij} &: (x,y) \longmapsto x^i x^j e_2 && (i,j = 1,\dots,p),
\end{aligned}
$$

and by the vectors e_1, e_2. The components H^1 and H^2 of each element H of $\mathfrak{B}(M, \mathfrak{F})$ take the form

$$H^1 = \sum_{i=1}^{p} f_i(y^1, \ldots, y^q)x^i + g^1(y^1, \ldots, y^q) \qquad (i = 1, \ldots, p),$$
$$H^2 = \sum_{i,j=1}^{p} f_{ij}(y^1, \ldots, y^q)x^i x^j + g^2(y^1, \ldots, y^q) \qquad (i, j = 1, \ldots, p),$$

where $f_i, f_{ij}, g^1, g^2 \in \mathfrak{B}(M, \mathfrak{F})$.

(iii) Suppose that $p = mk$. We denote by $(x, y) = (x^{ra}, y^1, \ldots, y^q)_{1\leq a\leq m,\, 1\leq r\leq k}$ the Cartesian coordinates system of $\mathbb{R}^n$. Let $\mathfrak{H}(\mathbb{R}^n, \mathfrak{F})$ be the $\mathfrak{B}(M, \mathfrak{F})$-submodule of $\mathcal{C}^\infty(\mathbb{R}^n, V)$ spanned by the mappings

$$X^a : (x, y) \longmapsto \sum_{r=1}^{k} x^{ra} e_r \qquad (a = 1, \ldots, m)$$

and by the vectors $e_1, \ldots, e_k$. The component H^r of each element H of $\mathfrak{H}(M, \mathfrak{F})$ takes the form

$$H^r = \sum_{a=1}^{m} f_a(y^1, \ldots, y^n)x^{ra} + g^r(y^1, \ldots, y^n) \qquad (r = 1, \ldots, k),$$

where $f_a, g^r \in \mathfrak{B}(M, \mathfrak{F})$.

(iv) In the previous notations, we consider the $\mathfrak{B}(M, \mathfrak{F})$-submodule $\mathfrak{H}(\mathbb{R}^n, \mathfrak{F})$ of $\mathcal{C}^\infty(\mathbb{R}^n, V)$ spanned by the mappings

$$X^{ab} : (x, y) \longmapsto \sum_{r=1}^{k} x^{ra} x^{rb} e_r \qquad (a, b = 1, \ldots, m),$$

and by the vectors $e_1, \ldots, e_k$. The component H^r of each element H of $\mathfrak{B}(M, \mathfrak{F})$ has the form

$$H^r = \sum_{a,b=1}^{m} f_{ab}(y^1, \ldots, y^n)x^{ra} x^{rb} + b^r(y^1, \ldots, y^n) \qquad (r = 1, \ldots, k),$$

where $f_{ab}, g^r \in \mathfrak{B}(M, \mathfrak{F})$.

(v) In the previous notations, we consider the $\mathfrak{B}(M, \mathfrak{F})$-submodule $\mathfrak{H}(\mathbb{R}^n, \mathfrak{F})$ of

$\mathcal{C}^\infty(\mathbb{R}^n, V)$ spanned by the mappings

$$X^a : (x,y) \longmapsto \sum_{r=1}^{k} x^{ra} e_r \qquad (a = 1, \dots, m),$$

$$X^{ab} : (x,y) \longmapsto \sum_{r=1}^{k} x^{ra} x^{rb} e_r \qquad (a, b = 1, \dots, m)$$

and by the vectors $e_1, \dots, e_k$. The components H^r of each element H of $\mathfrak{B}(M, \mathfrak{F})$ take the form

$$H^r = \sum_{a=1}^{m} f_a(y^1, \dots, y^n) x^{ra} + \sum_{a,b=1}^{m} f_{ab}(y^1, \dots, y^n) x^{ra} x^{rb} + b^r(y^1, \dots, y^n),$$

where $r = 1, \dots, k$ and $f_a, f_{ab}, g^r \in \mathfrak{B}(M, \mathfrak{F})$.

7. Local models of vectorial polarized systems

For polarized manifolds and k-symplectic manifolds, there is a unique model: the Darboux model and its generalization.

In the case of the vectorial polarized systems, there is not a unique model.

Here we give, some local models of the vectorial polarized systems and the corresponding vectorial polarized Hamiltonian mappings, in the some cases where the dimension of the space is less or equal than 4.

7.1. For $k = 2$ and $m = 3$

If every $\mathbb{R}^2$-valued form $\theta = \theta^1 \otimes e_1 + \theta^2 \otimes e_2$ in $\mathbb{R}^3$ admits a maximal solutions of dimension 2, then it is a 2-symplectic system and can be written under the following local form:

$$\begin{cases} \theta^1 = dx^1 \wedge dx^3, \\ \theta^{32} = dx^2 \wedge dx^3. \end{cases}$$

7.2. For $k = 3$ and $m = 3$

Consider the $\mathbb{R}^3$-valued form in $\mathbb{R}^3$ given by

$$\theta = \theta^1 \otimes e_1 + \theta^2 \otimes e_2 + \theta^3 \otimes e_3,$$

with rank 3, such that the system $\{\theta^1, \theta^2, \theta^3\}$ is not algebraically equivalent to the 2-system. Then, locally, this system is algebraically equivalent to the following model:

$$\begin{cases} \theta^1 = dx^2 \wedge dx^3, \\ \theta^2 = dx^3 \wedge dx^1, \\ \theta^3 = dx^1 \wedge dx^2. \end{cases}$$

This system has a 1-dimensional maximal solutions.

The vectorial polarized Hamiltonian mappings take the following expressions:

$$\begin{cases} H^1 = ax^2x^3 + b_3x^2 - b_2x^3 + \alpha_1, \\ H^2 = -ax^1x^3 + b_1x^3 - b_3x^1 + \alpha_2, \\ H^3 = -ax^1x^2 + b_2x^1 - b_1x^2 + \alpha_3. \end{cases}$$

where a, b_1, b_2, b_3, α_1, α_2, α_3 are real numbers.

7.3. For $k = 2$ and $m = 4$

Let $\theta = \theta^1 \otimes e_1 + \theta^2 \otimes e_2$ be a $\mathbb{R}^2$-valued form in $\mathbb{R}^4$ with maximum rank and with maximal solutions of dimension 2. Then locally we have

$$\begin{cases} \theta^1 = dx^1 \wedge dy^1, \\ \theta^2 = dx^2 \wedge dy^2. \end{cases}$$

This system has a 2-dimensional maximal solutions (the foliation defined by $dy^1 = dy^2 = 0$).

The vectorial polarized Hamiltonians mappings take the following expressions:

$$\begin{cases} H^1 = f^1(y^1, y^2)x^1 + g^1(y^1, y^2), \\ H^2 = f^2(y^1, y^2)x^2 + g^2(y^1, y^2), \end{cases}$$

where f^1, f^2, g^1, g^2 are basic functions for the foliation defined by $dy^1 = dy^2 = 0$.

7.4. For $k = 3$ and $m = 4$

We consider only

(i) The 3-symplectic system

$$(S^1) \quad \begin{cases} \theta^1 = dx^1 \wedge dy, \\ \theta^2 = dx^2 \wedge dy, \\ \theta^3 = dx^3 \wedge dy. \end{cases}$$

(ii) The system

$$(S^2) \quad \begin{cases} \theta^1 = dx^1 \wedge dy^2 + dx^2 \wedge dy^1, \\ \theta^2 = dx^1 \wedge dy^1, \\ \theta^3 = dx^2 \wedge dy^2. \end{cases}$$

This system has a 2-dimensional maximal solutions (the foliation defined by $dy^1 = dy^2 = 0$).

The vectorial polarized Hamiltonian mappings take the following expressions:

$$\begin{cases} H^1 = f^1(y^1, y^2)x^1 + g^1(y^1, y^2), \\ H^2 = f^2(y^1, y^2)\ x^2 + g^2(y^1, y^2), \\ H^3 = f^2(y^1, y^2t)x^1 - f^1(y^1, y^2)x^2 + g^3(y^1, y^2). \end{cases}$$

where f^1, f^2, f^3, g^1, g^2, g^3 are basic functions for the foliation defined by $dy^1 = dy^2 = 0$.

(iii) The system

$$(S^3) \qquad \begin{cases} \theta^1 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4, \\ \theta^2 = dx^1 \wedge dx^3 - dx^2 \wedge dx^4, \\ \theta^3 = dx^1 \wedge dx^4 + dx^2 \wedge dx^3. \end{cases}$$

Locally, this system has a 1-dimensional maximal solution.

The vectorial polarized Hamiltonian mappings take the following expressions:

$$\begin{cases} H^1(x_1, x_2, x_3, x_4) = ax_1 + bx_2 + cx_3 + dx_4 + \alpha_1, \\ H^2(x_1, x_2, x_3, x_4) = -dx_1 - cx_2 + bx_3 + ax_4 + \alpha_2, \\ H^3(x_1, x_2, x_3, x_4) = cx_1 - dx_2 - ax_3 + bx_4 + \alpha_3. \end{cases}$$

where a, b, c, d, α_1, α_2, α_3 are real numbers.

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