Weighted Composition Operators between Weighted Bergman Spaces and Weighted Banach Spaces of Holomorphic Functions

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ABSTRACT

We characterize boundedness and compactness of weighted composition operators acting between weighted Bergman spaces $A_{w,p}$ and weighted Banach spaces H_v^∞ of holomorphic functions.

Key words: weighted Bergman space, weighted composition operator, weighted Banach space of holomorphic functions.

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Introduction

Let v and w be strictly positive bounded continuous functions (*weights*) on the open unit disk D in the complex plane and $H(D)$ be the space of holomorphic functions. We are interested in operators acting between the weighted Bergman space

$$A_{w,p} = \left\{ f \in H(D);\ \|f\|_{w,p} := \left(\int_D |f(z)|^p w(z)\, dA(z) \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 \le p < \infty,$$

where $dA(z)$ is the area measure on D normalized so that area of D is 1 and the weighted Banach space of holomorphic functions (weighted Bergman space of infinite order)

$$H_v^\infty := \{ f \in H(D);\ \|f\|_v := \sup_{z \in D} v(z)|f(z)| < \infty \}.$$

Let $\phi : D \to D$ and $\psi : D \to \mathbb{C}$ be analytic mappings. Such maps induce a linear weighted composition operator $\psi C_\phi(f) = \psi(f \circ \phi)$ between spaces of holomorphic functions of the type defined above.

Composition operators and weighted composition operators have been studied on various spaces of holomorphic functions, see, e.g., [2–5, 10]. For more general information on composition operators we refer to the monographs [6, 9]. In this article we want to characterize boundedness and compactness of weighted composition operators acting between spaces of the type defined above in terms of the weights. Our result is similar to the one Sharma and Sharma obtained in [10] for standard weights.

1. Preliminaries

We denote by B_v^∞ the closed unit ball of H_v^∞. The so-called *associated weights* are an important tool to handle weighted spaces of holomorphic functions. For a weight v the associated weight $\tilde{v}$ is defined as follows

$$\tilde{v}(z) := \frac{1}{\sup\{\, |f(z)|;\ f \in H_v^\infty, \|f\|_v \leq 1\,\}} = \frac{1}{\|\delta_z\|_{H_v^{\infty\prime}}}, \quad z \in D,$$

where δ_z is the point evaluation of z. The associated weights are also continuous and $\tilde{v} \geq v > 0$ (see [1]). Furthermore, for each $z \in D$ there is $f_z \in H_v^\infty$, $\|f\|_v \leq 1$, such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$. A weight is called *essential* if there is a constant $C > 0$ with

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \quad \text{for every } z \in D.$$

For examples of essential weights and conditions when weights are essential see [1–3].

We fix $a \in D$ and consider the automorphism $\varphi_a(z) := \frac{z-a}{1-\bar{a}z}$, $z \in D$, which interchanges 0 and a. Furthermore we use the fact that

$$\varphi_a'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}, \quad z \in D.$$

2. Results

We first need the following auxiliary result. The following lemma is well-known for standard weights (see [7] or [8]) but to the best of our knowledge not known for other weights.

Lemma 2.1. *Let w be a weight of the form $w = |u|$ where u is a holomorphic function without any zeros on D. Then*

$$|f(z)| \leq \frac{1}{(1-|z|^2)^{\frac{2}{p}} w(z)^{\frac{1}{p}}} \|f\|_{w,p}$$

for all $z \in D$, $f \in A_{w,p}$.

Proof. Let $\alpha \in D$ be an arbitrary point. Consider the map

$$T_\alpha : A_{w,p} \longrightarrow A_{w,p}, \qquad T_\alpha(f(z)) = f(\varphi_\alpha(z))\varphi'_\alpha(z)^{\frac{2}{p}} \frac{u(\varphi_\alpha(z))^{\frac{1}{p}}}{u(z)^{\frac{1}{p}}}.$$

This map is an isometry since a change of variables yields

$$\begin{aligned} \|T_\alpha f\|_{w,p}^p &= \int_D w(z)|f(\varphi_\alpha(z))|^p|\varphi'_\alpha(z)|^2 \frac{w(\varphi_\alpha(z))}{w(z)}\, dA(z) \\ &= \int_D |f(\varphi_\alpha(z))|^p|\varphi'_\alpha(z)|^2 w(\varphi_\alpha(z))\, dA(z) \\ &= \int_D w(t)|f(t)|^p\, dA(t) = \|f\|_{w,p}^p. \end{aligned}$$

Now put $g(z) = T_\alpha(f(z))$. By the mean-value property we obtain

$$w(0)|g(0)|^p \leq \int_D w(z)|g(z)|^p\, dA(z) = \|g\|_{w,p}^p = \|f\|_{w,p}^p.$$

Hence

$$w(0)|g(0)|^p = |f(\alpha)|^p(1-|\alpha|^2)^2 w(\alpha) \leq \|f\|_{w,p}^p.$$

Thus $|f(\alpha)| \leq \frac{\|f\|_{w,p}}{(1-|\alpha|^2)^{\frac{2}{p}} w(\alpha)^{\frac{1}{p}}}$. Since α was arbitrary, the claim follows. □

The proof of the following result was inspired by [10].

Theorem 2.2. *Let w be a weight of the form $w = |u|$ where u is a holomorphic function without any zeros on D. Then the weighted composition operator $\psi C_\phi : A_{w,p} \to H_v^\infty$ is bounded if and only if*

$$\sup_{z\in D} \frac{v(z)|\psi(z)|}{(1-|\phi(z)|^2)^{\frac{2}{p}}\tilde{w}(\phi(z))^{\frac{1}{p}}} < \infty.$$

Proof. By [1, Example 1.4] we know that under the given assumptions $w = \tilde{w}$. First suppose that $M = \sup_{z\in D} \frac{v(z)|\psi(z)|}{(1-|\phi(z)|^2)^{\frac{2}{p}} w(\phi(z))^{\frac{1}{p}}} < \infty$. By the lemma we know that

$$|f(z)| \leq \frac{\|f\|_{w,p}}{(1-|z|^2)^{\frac{2}{p}} w(z)^{\frac{1}{p}}}$$

for all $z \in D$, independent of $f \in A_{w,p}$. Thus, for $z \in D$, we get

$$\begin{aligned} \|\psi C_\phi f\|_v &= \sup_{z\in D} v(z)|\psi(z)||f(\phi(z))| \\ &\leq \sup_{z\in D} \frac{v(z)|\psi(z)|}{w(\phi(z))^{\frac{1}{p}}(1-|\phi(z)|^2)^{\frac{2}{p}}} \|f\|_{w,p}. \end{aligned}$$

For the converse let $a \in D$ be arbitrary. There exists $f_a^p \in B_w^\infty$ such that $|f_a(a)|^p = \frac{1}{\tilde{w}(a)}$. Now put $g_a(z) := f_a(z)\varphi_a'(z)^{\frac{2}{p}}$. Then a change of variables yields

$$\begin{aligned}\|g_a\|^p &= \int_D |g_a(z)|^p w(z)\, dA(z) = \int_D |f_a(z)|^p |\varphi_a'(z)|^2 w(z)\, dA(z)\\ &\leq \sup_{z\in D} w(z)|f_a(z)|^p \int_D |\varphi_a'(z)|^2\, dA(z) \leq \int_D |\varphi_a'(z)|^2\, dA(z) = \int_D dA(t) = 1.\end{aligned}$$

Next, we assume that there is a sequence $(z_n)_{n\in\mathbb{N}} \subset D$ such that $|\phi(z_n)| \to 1$ and

$$\frac{|\psi(z_n)|v(z_n)}{\tilde{w}(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{\frac{2}{p}}} \geq n$$

for every $n \in \mathbb{N}$. Thus consider now $g_n(z) := g_{\phi(z_n)}(z)$ for every $n \in \mathbb{N}$ as defined above. Then we obtain that $(g_n)_n$ lies in the closed unit ball of $A_{w,p}$ and

$$c \geq v(z_n)|\psi(z_n)||g_n(\phi(z_n))| = \frac{v(z_n)|\psi(z_n)|}{\tilde{w}(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{\frac{2}{p}}} \geq n$$

for every $n \in \mathbb{N}$, which is a contradiction. □

Examples 2.3.

(i) Consider $p = 1$, $w(z) = |1-z|$, $\psi(z) = \frac{1-z}{2} = 1 - \frac{z+1}{2}$, $v(z) = (1-|z|^2)^2$, and $\phi(z) = \frac{z+1}{2}$. By [1, Example 1.4] we know that $w = \tilde{w}$. Then we obtain

$$\sup_{z\in D} \frac{v(z)|\psi(z)|}{(1-|\phi(z)|^2)^2 w(\phi(z))} = \sup_{z\in D} \frac{(1-|z|^2)^2}{(1-|\frac{z+1}{2}|^2)^2} < \infty.$$

Hence the corresponding weighted composition operator is bounded.

(ii) Consider $p = 1$, $w(z) = |1-z|$, $\psi(z) = \frac{1-z}{2} = 1 - \frac{z+1}{2}$, $v(z) = 1 - |z|^2$, and $\phi(z) = \frac{z+1}{2}$. By [1, Example 1.4] we know that $w = \tilde{w}$. Then for $z = r \in \mathbb{R}$ by using the rule of L'Hospital we get

$$\frac{v(r)|\psi(r)|}{(1-|\phi(r)|^2)^2 w(\phi(r))} = 16\frac{1-r^2}{9-12r-2r^2+4r^3+r^4} \to \infty \qquad \text{if } r \to 1.$$

Hence the operator is not bounded.

The proof of the following result was inspired by [10].

Theorem 2.4. *Let w be a weight of the form $w = |u|$ where u is a holomorphic function without any zeros on D. Moreover let $\phi : D \to D$ be analytic with $\|\phi\| = 1$ and $\psi \in H_v^\infty$. Then the weighted composition operator $\psi C_\phi : A_{w,p} \to H_v^\infty$ is compact if and only if*

$$\lim_{r\to 1} \sup_{\{z;|\phi(z)|>r\}} \frac{v(z)|\psi(z)|}{(1-|\phi(z)|^2)^{\frac{2}{p}}\tilde{w}(\phi(z))^{\frac{1}{p}}} = 0. \qquad (*)$$

Proof. First, we assume that (*) holds. Let $(f_n)_n$ be a bounded sequence in $A_{w,p}$ that converges to zero uniformly on compact subsets of D. Let $M = \sup_n \|f_n\|_{w,p} < \infty$. Given $\varepsilon > 0$ there is $r > 0$ such that if $|\phi(z)| > r$, and then

$$\frac{v(z)|\psi(z)|}{(1-|\phi(z)|^2)^{\frac{2}{p}}\tilde{w}(\phi(z))^{\frac{1}{p}}} < \varepsilon.$$

By Lemma 2.1 we have

$$|f_n(z)| \leq \frac{\|f_n\|_{w,p}}{(1-|z|^2)^{\frac{2}{p}} w(z)^{\frac{1}{p}}}.$$

Thus, for $z \in D$, we obtain

$$v(z)|\psi C_\phi f_n(z)| = v(z)|\psi(z)||f_n(\phi(z))| \leq \frac{v(z)|\psi(z)|}{(1-|\phi(z)|^2)^{\frac{2}{p}} w(\phi(z))^{\frac{1}{p}}}\|f_n\|_{w,p} \leq \varepsilon M$$

for all n.

On the other hand, since $f_n \to 0$ uniformly on $\{u;\ |u| \leq r\}$, there is an $n_0 \in \mathbb{N}$ such that, if $|\phi(z)| \leq r$ and $n \geq n_0$, then $|f_n(\phi(z))| < \varepsilon$. By assumption we know $\psi \in H_v^\infty$. Thus we have $N = \sup_{z\in D} v(z)|\psi(z)| < \infty$ and hence

$$v(z)|\psi C_\phi f_n(z)| = v(z)|\psi(z)||f_n(\phi(z))| \leq N\varepsilon.$$

Conversely, suppose that $\psi C_\phi : A_{w,p} \to H_v^\infty$ is compact and that $(*)$ does not hold. Then there are $\delta > 0$ and $(z_n)_n \subset D$ with $|\phi(z_n)| \to 1$ such that

$$\frac{v(z_n)|\psi(z_n)|}{w(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{\frac{2}{p}}} \geq \delta.$$

for all n. Since $|\phi(z_n)| \to 1$ there exist natural numbers $\alpha(n)$ with $\lim_{n\to\infty} \alpha(n) = \infty$ and such that $|\phi(z_n)|^{\alpha(n)} \geq \frac{1}{2}$ for all n. For each n consider the function g_n

$$g_n(z) := f_{\phi_n}(z)\varphi'_{\phi_n}(z)^{\frac{2}{p}} z^{\alpha(n)}.$$

Then $(g_n)_n$ is norm bounded and $g_n \to 0$ pointwise because of the factor $z^{\alpha(n)}$. Thus, it follows that a subsequence of $(\psi C_\phi g_n)_n$ tends to 0 in H_v^∞. On the other hand

$$\begin{aligned}\|\psi C_\phi g_n\|_v &\geq v(z_n)|\psi C_\phi g_n(z_n)| = v(z_n)|\psi(z_n)||g_n(\phi(z_n))| \\ &= \frac{v(z_n)|\psi(z_n)||\phi(z_n)|^{\alpha(n)}}{(1-|\phi(z_n)|^2)^{\frac{2}{p}} w(\phi(z_n))^{\frac{1}{p}}} \geq \frac{1}{2}\delta,\end{aligned}$$

which is a contradiction. □

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