Tempered Radon Measures

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ABSTRACT

A tempered Radon measure is a σ-finite Radon measure in $\mathbb{R}^n$ which generates a tempered distribution. We prove the following assertions. A Radon measure μ is tempered if, and only if, there is a real number β such that $(1+|x|^2)^{\frac{\beta}{2}}\mu$ is finite. A Radon measure is finite if, and only if, it belongs to the positive cone $\overset{+}{B}{}^0_{1\infty}(\mathbb{R}^n)$ of $B^0_{1\infty}(\mathbb{R}^n)$. Then $\mu(\mathbb{R}^n) \sim \|\mu \mid B^0_{1\infty}(\mathbb{R}^n)\|$ (equivalent norms).

Key words: Radon measure, tempered distributions, Besov spaces

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Introduction

A substantial part of fractal geometry and fractal analysis deals with Radon measures in $\mathbb{R}^n$ (also called fractal measures) with compact support. One may consult [5] and the references given there. In the present paper we clarify the relation between arbitrary σ-finite Radon measure in $\mathbb{R}^n$, tempered distributions and weighted Besov spaces. It comes out that a σ-finite Radon measure μ in $\mathbb{R}^n$ can be identified with a tempered distribution $\mu \in S'(\mathbb{R}^n)$ if and only if there is a real number β such that

$$\mu_\beta(\mathbb{R}^n) < \infty, \quad \text{where} \quad \mu_\beta = (1+|x|^2)^{\frac{\beta}{2}}\mu.$$

Radon measures μ with $\mu(\mathbb{R}^n) < \infty$ are called finite. These finite Radon measures can be identified with the positive cone $\overset{+}{B}{}^0_{1\infty}(\mathbb{R}^n)$ of the distinguished Besov space $B^0_{1\infty}(\mathbb{R}^n)$ and

$$\|\mu \mid B^0_{1\infty}(\mathbb{R}^n)\| \sim \mu(\mathbb{R}^n)$$

(equivalent norms).

This paper is organised as follows. In section 1 we collect the definitions and preliminaries. We introduce the well-known weighted Besov spaces $B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)$ and prove that for fixed p, q with $0 < p, q \le \infty$

$$S(\mathbb{R}^n) = \bigcap_{\alpha,s\in\mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)$$

and

$$S'(\mathbb{R}^n) = \bigcup_{\alpha,s\in\mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha).$$

Although known to specialists we could not find an explicit reference. In section 2 we prove in the Theorems 2.1 and 2.2 the above indicated main results.

1. Definitions and preliminaries

Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$, is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\|f \mid L_p(\mathbb{R}^n)\| = \Bigg(\int_{\mathbb{R}^n} |f(x)|^p\, dx\Bigg)^{\frac{1}{p}}, \quad 0 < p < \infty$$

with the standard modification if $p = \infty$.

If $\varphi \in S(\mathbb{R}^n)$ then

$$\hat\varphi(\xi) = F\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi}\, dx, \quad \xi \in \mathbb{R}^n,$$

denotes the Fourier transform of φ. The inverse Fourier transform is given by

$$\check\varphi(x) = F^{-1}\varphi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(\xi) e^{ix\xi}\, d\xi, \quad x \in \mathbb{R}^n.$$

One extends F and F^{-1} in the usual way from S to S'. For $f \in S'(\mathbb{R}^n)$,

$$Ff(\varphi) = f(F\varphi), \quad \varphi \in S(\mathbb{R}^n).$$

Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1, \quad |x| \le 1 \qquad \text{and} \qquad \varphi_0(x) = 0, \quad |x| \ge \frac{3}{2}, \tag{1}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}. \tag{2}$$

Then, since

$$1 = \sum_{j=0}^{\infty} \varphi_j(x) \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{3}$$

the φ_j form a dyadic resolution of unity in $\mathbb{R}^n$. $(\varphi_k \hat{f})\check{}$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$. In particular, $(\varphi_k \hat{f})\check{}(x)$ makes sense pointwise.

Definition 1.1. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the dyadic resolution of unity according to (1)–(3), $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$, and

$$\|f \mid B^s_{pq}(\mathbb{R}^n)\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_k \hat{f})\check{} \mid L_p(\mathbb{R}^n)\|^q \right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$). Then the Besov space $B^s_{pq}(\mathbb{R}^n)$ consists of all $f \in S'(\mathbb{R}^n)$ such that $\|f \mid B^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty$.

We denote by $L_p(\mathbb{R}^n, \langle x \rangle^\alpha)$, where

$$\langle x \rangle^\alpha = (1 + |x|^2)^{\frac{\alpha}{2}},$$

the weighted L_p-space quasi-normed by

$$\|f \mid L_p(\mathbb{R}^n, \langle x \rangle^\alpha)\| = \|\langle \cdot \rangle^\alpha f \mid L_p(\mathbb{R}^n)\|.$$

Definition 1.2. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the dyadic resolution of unity according to (1)–(3), $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. Then the weighted Besov space $B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)$ is a collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \mid B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_k \hat{f})\check{} \mid L_p(\mathbb{R}^n, \langle x \rangle^\alpha)\|^q \right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$) is finite.

Remark 1.3. If $\alpha = 0$ then we have the space $B^s_{pq}(\mathbb{R}^n)$ as introduced in Definition 1.1. It is also known from [1, ch. 4.2.2] that the operator $f \mapsto \langle x \rangle^\alpha f$ is an isomorphic mapping from $B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)$ onto $B^s_{pq}(\mathbb{R}^n)$. In particular,

$$\|\langle \cdot \rangle^\alpha f \mid B^s_{pq}(\mathbb{R}^n)\| \sim \|f \mid B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)\|.$$

Next we review some special properties of weighted Besov spaces.

Proposition 1.4. *For fixed* $0 < p, q \le \infty$

$$S(\mathbb{R}^n) = \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha) \tag{4}$$

and

$$S'(\mathbb{R}^n) = \bigcup_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha).$$

Proof. Step 1. The inclusion

$$S(\mathbb{R}^n) \subset \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)$$

is clear.

To prove that any $f \in \bigcap_{\alpha,s\in\mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)$ belongs to $S(\mathbb{R}^n)$, it is sufficient to show that for any fixed $N \in \mathbb{N}$ there are $\alpha(N) \in \mathbb{R}$ and $s(N) \in \mathbb{R}$ such that

$$\sup_{|\beta|\le N} \sup_{x\in\mathbb{R}^n} \langle x\rangle^{2N} |D^\beta f(x)| \le c\|f \mid B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)\|.$$

For any multiindex β there are polynomials P^β_γ, $\deg P^\beta_\gamma \le 2N$ such that

$$\langle x\rangle^{2N} D^\beta f(x) = \sum_{\gamma\le\beta} D^\gamma[(P^\beta_\gamma f)(x)].$$

Hence

$$\begin{aligned}
\sup_{|\beta|\le N} \sup_{x\in\mathbb{R}^n} \langle x\rangle^{2N} |D^\beta f(x)| &= \sup_{|\beta|\le N} \sup_{x\in\mathbb{R}^n} \Big|\sum_{\gamma\le\beta} D^\gamma[(P^\beta_\gamma f)(x)]\Big| \\
&\le \sup_{|\beta|\le N} \sum_{|\gamma|\le N} \sup_{x\in\mathbb{R}^n} |D^\gamma[(P^\beta_\gamma f)(x)]| \\
&\le \sup_{|\beta|\le N} \sum_{|\gamma|\le N} \|P^\beta_\gamma f \mid C^N(\mathbb{R}^n)\|.
\end{aligned} \tag{5}$$

Due to the embedding theorems [3, ch. 2.7.1],

$$\begin{aligned}
\|P^\beta_\gamma f \mid C^N(\mathbb{R}^n)\| &\le c\Big\|P^\beta_\gamma f \;\Big|\; B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)\Big\| \\
&= c\Big\|\frac{P^\beta_\gamma}{\langle x\rangle^{2N}} \langle x\rangle^{2N} f \;\Big|\; B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)\Big\|
\end{aligned} \tag{6}$$

for any $\varepsilon > 0$. $\frac{P^\beta_\gamma}{\langle x\rangle^{2N}}$ is a pointwise multiplier for $B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)$ [3, ch. 2.8.2]. Therefore

$$\begin{aligned}
&\Big\|\frac{P^\beta_\gamma}{\langle x\rangle^{2N}} \langle x\rangle^{2N} f \;\Big|\; B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)\Big\| \\
&\qquad \le c\Big\|\frac{P^\beta_\gamma}{\langle x\rangle^{2N}} \;\Big|\; \mathcal{C}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n)\Big\| \cdot \Big\|\langle x\rangle^{2N} f \;\Big|\; B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)\Big\|.
\end{aligned} \tag{7}$$

According to Remark 1.3

$$\left\| \langle x \rangle^{2N} f \;\Big|\; B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n) \right\| \sim \left\| f \;\Big|\; B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^{2N}) \right\|. \tag{8}$$

Combining (5)–(8), one gets

$$\begin{aligned} \sup_{|\beta|\le N} \sup_{x\in\mathbb{R}^n} \langle x \rangle^{2N} |D^\beta f(x)| &\le c \sum_{|\gamma|\le N} \left\| \langle x \rangle^{2N} f \;\Big|\; B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n) \right\| \\ &\le c \left\| f \;\big|\; B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^{2N}) \right\| \end{aligned} \tag{9}$$

and it follows (4).

Step 2. Let $1 < p \le \infty$, $1 < q \le \infty$ and let p' and q' be defined in the standard way by

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

The inclusion

$$\bigcup_{\alpha,s\in\mathbb{R}} B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha) \subset S'(\mathbb{R}^n)$$

is evident.

As far as the opposite inclusion is concerned, we recall that $f \in S'(\mathbb{R}^n)$ if and only if there are $l \in \mathbb{N}$ and $m \in \mathbb{N}$ such that

$$|f(\varphi)| \le c \sup_{|\alpha|\le m} \sup_{x\in\mathbb{R}^n} \langle x \rangle^l |D^\alpha \varphi(x)|,$$

for all $\varphi \in S(\mathbb{R}^n)$. By (9),

$$\sup_{|\alpha|\le m} \sup_{x\in\mathbb{R}^n} \langle x \rangle^l |D^\alpha \varphi(x)| \le c \left\| \varphi \;\Big|\; B_{p'q'}^{m+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^l) \right\|.$$

According to our choice of p and q, it follows that $1 \le p' < \infty$ and $1 \le q' < \infty$. Thus, by [3, ch. 2.11.2],

$$f \in \left(B_{p'q'}^{m+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^l) \right)' = B_{pq}^{-(m+\frac{n}{p}+\varepsilon)}(\mathbb{R}^n, \langle x \rangle^{-l}).$$

This means

$$S'(\mathbb{R}^n) \subset \bigcup_{\alpha,s\in\mathbb{R}} B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha).$$

Step 3. Let $0 < p \le 1$, $1 < q \le \infty$. By the arguments above, for $f \in S'(\mathbb{R}^n)$ there are $\alpha \in \mathbb{R}$ and $s \in \mathbb{R}$ such that

$$f \in B_{\infty q}^s(\mathbb{R}^n, \langle x \rangle^\alpha).$$

We want to show that
$$f \in B^s_{pq}(\mathbb{R}^n, \langle x\rangle^{\alpha-\gamma}), \quad \gamma > \frac{n}{p}.$$
Indeed,
$$\begin{aligned}
\|f \mid B^s_{pq}(\mathbb{R}^n, \langle x\rangle^{\alpha-\gamma})\| &= \Big(\sum_{j=0}^{\infty} 2^{jsq} \|\langle x\rangle^{\alpha-\gamma} (\varphi_j \hat{f})\check{} \mid L_p(\mathbb{R}^n)\|^q\Big)^{\frac{1}{q}} \\
&\le \Big(\sum_{j=0}^{\infty} 2^{jsq} \sup_{x\in\mathbb{R}^n} [\langle x\rangle^{\alpha} |(\varphi_j \hat{f})\check{}(x)|]^q \Big(\int_{\mathbb{R}^n} \langle x\rangle^{-\gamma p}\, dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \\
&\le c\|f \mid B^s_{\infty q}(\mathbb{R}^n, \langle x\rangle^{\alpha})\|.
\end{aligned}$$

Step 4. When $0 < q \le 1$, first we may find $\alpha \in \mathbb{R}$ and $s \in \mathbb{R}$ such that
$$f \in B^s_{pq^*}(\mathbb{R}^n, \langle x\rangle^{\alpha}),$$
$q^* > 1$, and then use the fact that
$$B^s_{pq*}(\mathbb{R}^n, \langle x\rangle^{\alpha}) \subset B^{s-\varepsilon}_{pq}(\mathbb{R}^n, \langle x\rangle^{\alpha}), \quad \varepsilon > 0. \qquad \square$$

Next we recall some notation. A measure μ is called σ-finite in $\mathbb{R}^n$ if for any $R > 0$,
$$\mu(\{x : |x| < R\}) < \infty.$$
A measure μ is a Radon measure if all Borel sets are μ measurable and

(i) $\mu(K) < \infty$ for compact sets $K \subset \mathbb{R}^n$,

(ii) $\mu(V) = \sup\{\, \mu(K) :\ K \subset V$ is compact$\}$ for open sets $V \subset \mathbb{R}^n$,

(iii) $\mu(A) = \inf\{\, \mu(V) :\ A \subset V,\ V$is open$\}$ for $A \subset \mathbb{R}^n$.

Let μ be a positive Radon measure in $\mathbb{R}^n$. Let T_μ,
$$T_\mu : \varphi \longmapsto \int_{\mathbb{R}^n} \varphi(x)\, \mu(dx), \quad \varphi \in S(\mathbb{R}^n),$$
be the linear functional generated by μ.

Definition 1.5. A positive Radon measure μ is said to be tempered if $T_\mu \in S'(\mathbb{R}^n)$.

Proposition 1.6. *Let μ^1 and μ^2 be two tempered Radon measures. Then*
$$T_{\mu^1} = T_{\mu^2} \text{ in } S'(\mathbb{R}^n) \quad \text{if, and only if,} \quad \mu^1 = \mu^2.$$

Proof. The Proposition is valid by the arguments in [5, p. 80]. $\square$

This justifies the identification of μ and correspondent tempered distribution T_μ and we may write $\mu \in S'(\mathbb{R}^n)$.

Definition 1.7. $f \in S'(\mathbb{R}^n)$ is called a positive distribution if

$$f(\varphi) \geq 0 \qquad \text{for any } \varphi \in S(\mathbb{R}^n) \text{ with } \varphi \geq 0.$$

If $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ then $f \geq 0$ means $f(x) \geq 0$ almost everywhere.

Remark 1.8. If f is a positive distribution, then $f \in C_0(\mathbb{R}^n)'$ and it follows from the Radon-Riesz theorem that there is a tempered Radon measure μ such that

$$f(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\, \mu(dx)$$

[2, pp. 61, 62, 71, 75].

2. Main assertions

Our next result refers to tempered measures.

Theorem 2.1.

(i) *A Radon measure μ in $\mathbb{R}^n$ is tempered if, and only if, there is a real number β such that $\langle x\rangle^\beta \mu$ is finite.*

(ii) *Let μ be a tempered Radon measure in $\mathbb{R}^n$. Let $j \in \mathbb{N}$,*

$$A_j = \{\, x : 2^{j-1} \leq |x| \leq 2^{j+1} \,\}, \quad A_0 = \{\, x : |x| \leq 2 \,\}.$$

Then for some $c > 0$, $\alpha \geq 0$,

$$\mu(A_k) \leq c 2^{k\alpha} \quad \textit{for all } k \in \mathbb{N}_0.$$

Proof. Step 1. First we prove part (ii). Suppose that the assertion does not hold. Then for $c = 1$ and $l \in \mathbb{N}$ there is $k_l \in \mathbb{N}_0$ such that

$$\mu(A_{k_l}) > 2^{k_l l}. \tag{10}$$

As soon as it is found one k_l with (10), it follows that there are infinitely many k_l^m, $m \in \mathbb{N}$, that satisfy (10).

With $j \in \mathbb{N}$,

$$A_j^* = \{\, x : 2^{j-2} \leq |x| \leq 2^{j+2} \,\}, \quad A_0^* = \{\, x : |x| \leq 4 \,\}.$$

For $l = 1$ take any of k_1^m, let it be k_1. For $l = 2$ choose $k_2 \gg k_1$ in such a way that $A_{k_1}^*$ and $A_{k_2}^*$ have an empty intersection. For arbitrary $l \in \mathbb{N}$ take

$$k_l \gg k_{l-1} \quad \text{and} \quad A_{k_{l-1}}^* \cap A_{k_l}^* = \emptyset.$$

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Let φ_0 be a C^∞ function on $\mathbb{R}^n$ with

$$\varphi_0(x) = 1, \quad |x| \le 2 \qquad \text{and} \qquad \varphi_0(x) = 0, \quad |x| \ge 4.$$

Let $k \in \mathbb{N}$ and

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+3}x), \quad x \in \mathbb{R}^n.$$

Then we have

$$\operatorname{supp} \varphi_k \subset A_k^*$$

and

$$\varphi_k(x) = 1, \quad x \in A_k.$$

Let

$$\varphi(x) = \sum_{l=1}^{\infty} 2^{-lk_l} \varphi_{k_l}(x).$$

For any fixed $N \in \mathbb{N}_0$

$$\begin{aligned}
\sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} &(1+|x|^2)^N |D^\alpha \varphi(x)| \\
&= \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N \left| D^\alpha \left(\sum_{l=1}^{\infty} 2^{-lk_l} \varphi_{k_l}(x) \right) \right| \\
&\le \sup_{l\in\mathbb{N}} \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} 2^{-lk_l} 2^{-|\alpha|k_l} 2^{|\alpha|} (1+|x|^2)^N |(D^\alpha \varphi_1)(2^{-k_l+1}x)|.
\end{aligned}$$

The last inequality holds, since the functions φ_{k_l} have disjoint supports. With the change of variables

$$x' = 2^{-k_l+1}x$$

one gets

$$\begin{aligned}
\sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} &(1+|x|^2)^N |D^\alpha \varphi(x)| \\
&\le \sup_{l\in\mathbb{N}} \sup_{|\alpha|\le N} 2^{-lk_l} 2^{-|\alpha|k_l} 2^{|\alpha|} 2^{2(k_l-1)N} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N |D^\alpha \varphi_1(x)| \\
&\le c \sup_{l\in\mathbb{N}} \sup_{|\alpha|\le N} 2^{-k_l(l+|\alpha|-2N)} \le c \sup_{l\in\mathbb{N}} 2^{-k_l(l-2N)}.
\end{aligned}$$

Since N is fixed and l is tending to infinity, $2^{-k_l(l-2N)}$ is bounded. Thus $\varphi \in S(\mathbb{R}^n)$.

According to the definition of tempered Radon measures

$$\int_{\mathbb{R}^n} \psi(x)\, \mu(dx) < +\infty$$

for any $\psi \in S(\mathbb{R}^n)$, but

$$\int_{\mathbb{R}^n} \varphi(x)\,\mu(dx) \geq \sum_{l=1}^{\infty} \int_{A_{k_l}} \varphi(x)\,\mu(dx) \geq \sum_{l=1}^{\infty} 2^{-lk_l} 2^{lk_l} = +\infty.$$

This means that our assertion (10) is false.

Step 2. We prove part (i). Since $\langle x\rangle^{\beta}\mu$ is finite, it is tempered. Then μ is also tempered. To prove the other direction we take $\beta = -(\alpha+1)$. Then we get

$$\begin{aligned}\langle\cdot\rangle^{\beta}\mu(\mathbb{R}^n) &= \int_{\mathbb{R}^n} \langle x\rangle^{-(\alpha+1)}\,\mu(dx) \leq \sum_{k=0}^{\infty} \int_{A_k} \langle x\rangle^{-(\alpha+1)}\,\mu(dx) \\ &\leq c\sum_{k=0}^{\infty} 2^{-k(\alpha+1)} \int_{A_k} \mu(dx) \leq c\sum_{k=0}^{\infty} 2^{-k(\alpha+1)} 2^{k\alpha} < \infty. \qquad \square\end{aligned}$$

In order to characterize finite Radon measures we define the positive cone $\overset{+}{B}{}^{s}_{pq}(\mathbb{R}^n)$ as the collection of all positive $f \in B^s_{pq}(\mathbb{R}^n)$.

Theorem 2.2. *Let $M(\mathbb{R}^n)$ be the collection of all finite Radon measures. Then*

$$M(\mathbb{R}^n) = \overset{+}{B}{}^{0}_{1\infty}(\mathbb{R}^n)$$

and

$$\mu(\mathbb{R}^n) \sim \|\mu \mid B^0_{1\infty}(\mathbb{R}^n)\|, \quad \mu \in M(\mathbb{R}^n). \tag{11}$$

Proof. By the proof in [5, pp. 82, 83, Proposition 1.127],

$$\|\mu \mid B^0_{1\infty}(\mathbb{R}^n)\| \leq \mu(\mathbb{R}^n) \quad \text{if} \quad \mu \in M(\mathbb{R}^n).$$

In order to prove the converse inequality, one use the characterisation of Besov spaces via local means. Let k_0 be a C^∞ non-negative function with

$$\operatorname{supp} k_0 \subset \{\, x : |x| \leq 1 \,\} \quad \text{and} \quad \widehat{k_0(0)} \neq 0.$$

If $f \in \overset{+}{B}{}^{0}_{1\infty}(\mathbb{R}^n)$, then $f = \mu$ is a tempered measure. By [5, p. 10, Theorem 1.10],

$$\|\mu \mid B^0_{1\infty}(\mathbb{R}^n)\| \geq c\|k_0(1,\mu)|L_1(\mathbb{R}^n)\| = c\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_0(x-y)\,d\mu(y)\,dx.$$

Applying Fubini's theorem, one gets

$$\|\mu \mid B^0_{1\infty}(\mathbb{R}^n)\| \geq c\mu(\mathbb{R}^n). \qquad \square$$

Corollary 2.3. *Let $f \in L_1(\mathbb{R}^n)$ and $f(x) \geq 0$ almost everywhere. Then*

$$\|f \mid L_1(\mathbb{R}^n)\| \sim \|f \mid B^0_{1\infty}(\mathbb{R}^n)\|.$$

Proof. Let $\mu = f\mu_L$, where μ_L is the Lebesgue measure. Then

$$\mu(\mathbb{R}^n) = \int_{\mathbb{R}^n} f(x)\,\mu_L(dx) = \|f \mid L_1(\mathbb{R}^n)\|$$

and

$$\|\mu | B^0_{1\infty}(\mathbb{R}^n)\| = \|f \mid B^0_{1\infty}(\mathbb{R}^n)\|.$$

From (11) follows the statement in the Corollary. □

The question arises whether Corollary 2.3 can be extended to all $f \in L_1(\mathbb{R}^n)$. We have

$$L_1(\mathbb{R}^n) \hookrightarrow B^0_{1\infty}(\mathbb{R}^n), \quad \text{hence} \quad \|f \mid B^0_{1\infty}(\mathbb{R}^n)\| \leq c\|f \mid L_1(\mathbb{R}^n)\|$$

for all $f \in L_1(\mathbb{R}^n)$. But the converse is not true even for functions $f \in L_1(\mathbb{R}^n)$ with compact support in the unit ball.

Proposition 2.4. *There are functions $f_j \in L_1(\mathbb{R}^n)$ with*

$$\operatorname{supp} f_j \subset \{\, y : |y| \leq 1 \,\}, \quad j \in \mathbb{N},$$

such that $\{f_j\}$ is a bounded set in $B^0_{1\infty}(\mathbb{R}^n)$, but

$$\|f_j \mid L_1(\mathbb{R}^n)\| \to \infty \quad \text{if} \quad j \to \infty.$$

Proof. We may assume $n = 1$.

Let $a \in C^1(\mathbb{R})$ be an odd function with

$$\operatorname{supp} a \subset \{\, x : |x| \leq 2 \,\}, \quad a(x) \geq 0, \quad x \geq 0$$

and

$$\max_{-2 \leq x \leq 2} |a(x)| = |a(-1)| = a(1) = 1.$$

If $c = \max_{-2\leq x\leq 2} |a'(x)|$, then $c \geq 1$. Define $a_0 \in C^1(\mathbb{R})$ by

$$a_0(x) = c^{-1} a(x).$$

Then one has for any $x \in \mathbb{R}$,

$$|a_0(x)| \leq c^{-1} \leq 1, \quad |a_0'(x)| \leq 1, \quad \text{and} \quad \int_{\mathbb{R}} a_0(x)\,dx = 0.$$

Define a function a_ν, $\nu \in \mathbb{N}$, by

$$a_\nu(x) = 2^\nu a_0(2^\nu x).$$

Then

$$\operatorname{supp} a_\nu \subset [-2^{-\nu+1}, 2^{-\nu+1}]$$

and

$$|a_\nu(x)| \le c^{-1} 2^\nu, \quad |a_\nu'(x)| \le 2^{2\nu}, \quad \int\limits_{\mathbb{R}} a_\nu(x)\, dx = 0.$$

According to [5, p. 12, Definition 1.15], a_0 is an 1_1-atom and a_ν are $(0,1)_{1,1}$-atoms. It follows from [4, Theorem 13.8] that $\sum_{\nu=1}^{\infty} a_\nu(x)$ converges in $S'(\mathbb{R}^n)$ and represents an element of $B^0_{1\infty}(\mathbb{R}^n)$. Let $f \overset{S'}{=} \sum_{\nu=1}^{\infty} a_\nu$.

Let

$$f_j(x) = \sum_{\nu=1}^{j} a_\nu(x).$$

Then $\operatorname{supp} f_j \subset [-1,1]$,

$$\begin{aligned} \|f_j \mid L_1(\mathbb{R}^n)\| &\ge \int\limits_0^{+\infty} f_j(x)\, dx = \int\limits_0^{+\infty} \sum_{\nu=1}^{j} a_\nu(x)\, dx \\ &= j \int\limits_0^{+\infty} a_0(x)\, dx \to \infty, \qquad j \to \infty. \end{aligned}$$

On the other hand one has by the above atomic argument

$$\|f_j \mid B^0_{1\infty}(\mathbb{R})\| \le 1 \quad \text{for} \quad j \in \mathbb{N}. \qquad \square$$

Corollary 2.5. *Not any characteristic function of a measurable subset of $\mathbb{R}^n$ is a pointwise multiplier in $B^0_{1\infty}(\mathbb{R}^n)$.*

Proof. Let $f \in L_1(\mathbb{R}^n)$ real. Let M_+ be a set of points x such that $f(x) \ge 0$ and $M_- = \{\, x : f(x) < 0 \,\}$. Then

$$\|f \mid L_1(\mathbb{R}^n)\| = \|\chi_{M_+} f \mid L_1(\mathbb{R}^n)\| + \|\chi_{M_-} f \mid L_1(\mathbb{R}^n)\|,$$

where χ_{M_+}, χ_{M_-} are characteristic functions of sets M_+ and M_- respectively. One may apply Corollary 2.3 to the functions $\chi_{M_+} f$ and $\chi_{M_-} f$ and get

$$\|f \mid L_1(\mathbb{R}^n)\| \le c\|\chi_{M_+} f \mid B^0_{1\infty}(\mathbb{R}^n)\| + c\|\chi_{M_-} f \mid B^0_{1\infty}(\mathbb{R}^n)\|.$$

If any characteristic function of a set in $\mathbb{R}^n$ would be a pointwise multiplier in $B^0_{1\infty}(\mathbb{R}^n)$, then

$$\|\chi_{M_+} f \mid B^0_{1\infty}(\mathbb{R}^n)\| \leq c\|f \mid B^0_{1\infty}(\mathbb{R}^n)\|, \quad \|\chi_{M_-} f \mid B^0_{1\infty}(\mathbb{R}^n)\| \leq c\|f \mid B^0_{1\infty}(\mathbb{R}^n)\|,$$

hence

$$\|f \mid L_1(\mathbb{R}^n)\| \leq c\|f \mid B^0_{1\infty}(\mathbb{R}^n)\|.$$

Since for any function $f \in L_1(\mathbb{R}^n)$ holds

$$\|f \mid B^0_{1\infty}(\mathbb{R}^n)\| \leq c\|f \mid L_1(\mathbb{R}^n)\|,$$

one gets

$$\|f \mid L_1(\mathbb{R}^n)\| \sim \|f \mid B^0_{1\infty}(\mathbb{R}^n)\|, \quad \text{for real } f \in L_1(\mathbb{R}^n).$$

This can be also extended to complex functions $f \in L_1(\mathbb{R}^n)$. But acoording to the Proposition 2.4 this is not true. □

References

[1] D. E. Edmunds and H. Triebel, *Function spaces, entropy numbers, differential operators*, Cambridge Tracts in Mathematics, vol. 120, Cambridge University Press, Cambridge, 1996.

[2] P. Malliavin, *Integration and probability*, Graduate Texts in Mathematics, vol. 157, Springer-Verlag, New York, 1995.

[3] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.

[4] ______, *Fractals and spectra*, Monographs in Mathematics, vol. 91, Birkhäuser Verlag, Basel, 1997.

[5] ______, *Theory of function spaces*, III, Monographs in Mathematics, vol. 100, Birkhäuser Verlag, Basel, 2006.