Coincidence of Some Classes of Universal Functions

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ABSTRACT

Let Ω be a domain in the complex plane such that Ω satisfies appropriate geo-metrical and topological properties. We prove that if f is a holomorphic function in Ω, then its Taylor series, with center at any $\xi \in \Omega$, is universal with respect to overconvergence if and only if its Cesàro (C, k)-means are universal for any real $k > -1$. This is an extension of the same result, proved recently by F. Bayart, for any integer $k \geq 0$. As a consequence, several classes of universal functions introduced in the related literature are shown to coincide.

Key words: Universal series, overconvergence, Taylor series, Cesàro means, Ostrowski gaps.

2000 Mathematics Subject Classification: 30B30, 30B10, 30E10.

1. Introduction

The first example of a universal Taylor series in the complex plane $\mathbb{C}$ was defined by Seleznev in [32]. He proved that there exists a sequence of complex numbers (a_n) such that, for any compact set $K \subset \mathbb{C}$ with $0 \notin K$ and connected complement and for any function h, continuous on K and holomorphic in the interior of K, there exists a sequence of natural numbers (λ_n) such that $\sum_{k=1}^{\lambda_n} a_k z^k$ converges to h uniformly on K. Of course, the radius of convergence of such a power series is necessarily equal to

zero. Later on, universal Taylor series with strictly positive radius of convergence were defined, first independently by Luh [20] and Chui and Parnes [8] and finally, in their strongest sense, by Nestoridis [28]. Moreover, Melas and Nestoridis have studied in [24] the existence of universal Taylor series with respect to summability methods (see also [5, 7]). Next, in [2], F. Bayart proved, among others, that if f is a holomorphic function in a "suitable" simply connected domain Ω, then its Taylor series is universal if and only if its Cesàro (C, k)-means are universal for any integer $k \geq 0$. Our aim here is to give a new proof of this result, extending this to every *real $k > -1$*.

Let us first recall some notations and definitions. Let Ω be a simply connected domain in the complex plane, $\Omega \neq \mathbb{C}$. We denote by $H(\Omega)$ the set of holomorphic functions on Ω, endowed with the topology of uniform convergence on compact sets. For $f \in H(\Omega)$ we denote by $S_n(f, \xi)$ the nth-partial sum of the Taylor series of f with center at $\xi \in \Omega$, that is

$$S_n(f, \xi)(z) = \sum_{\nu=0}^{n} a_\nu (z - \xi)^\nu \quad (z \in \mathbb{C}).$$

Then, for real $k > -1$, the (C, k)-means of order n of the Taylor expansion of f with center ξ, are given by

$$\sigma_n^k(f, \xi)(z) = \sum_{\nu=0}^{n} \frac{A_{n-\nu}^k}{A_n^k} a_\nu (z - \xi)^\nu,$$

where

$$A_m^j = \frac{\Gamma(m + j + 1)}{\Gamma(m + 1)\Gamma(j + 1)} = \frac{(j + 1)(j + 2) \cdots (j + m)}{m!},$$

i.e., A_m^j equals to $\binom{m+j}{m}$, or, if j is an integer, to $\binom{m+j}{j}$. The Γ in the above formula is Euler's Gamma function. We refer to [33] for standard properties of Cesàro means of series. Further, for K a subset of $\mathbb{C}$, we denote by K° its interior and by K^c its complement. Finally, $D(w, r)$ and $\overline{D}(w, r)$ stand for the open and closed disk, respectively, with center $w \in \mathbb{C}$ and radius $r > 0$.

In the following we shall prove our main results for universal Taylor series in the sense of Nestoridis. Then, the corresponding results for universal Taylor series in the sense of Luh and Chui - Parnes follow easily.

Definition 1.1. Let $f \in H(\Omega)$, $\xi \in \Omega$, $j \in (-1, +\infty)$ and let K be a compact set in $\mathbb{C}$ with $K \cap \Omega = \emptyset$, K^c connected and $h : K \to \mathbb{C}$ be a continuous function on K and holomorphic in K°.

- If, for every K and h as above, there exists a sequence (λ_n) of natural numbers such that
$$\sup_{z \in K} |S_{\lambda_n}(f, \xi)(z) - h(z)| \to 0, \ as \ n \to \infty,$$
  then we say that f belongs to the class $U(\xi, \Omega^c)$.

- If, for every K and h as above, there exists a sequence (λ_n) of natural numbers such that, for every compact set $N \subset \Omega$,

$$\sup_{\xi \in N} \sup_{z \in K} |S_{\lambda_n}(f, \xi)(z) - h(z)| \to 0, \ as \ n \to \infty,$$

then we say that f belongs to the class $U(\Omega, \Omega^c)$.

- If, for every K and h as above, there exists a sequence (λ_n) of natural numbers such that

$$\sup_{z \in K} |\sigma^j_{\lambda_n}(f, \xi)(z) - h(z)| \to 0, \ as \ n \to \infty,$$

then we say that f belongs to the class $U_{Ces(j)}(\xi, \Omega^c)$.

- If, for every K and h as above, there exists a sequence (λ_n) of natural numbers such that, for every compact set $N \subset \Omega$,

$$\sup_{\xi \in N} \sup_{z \in K} |\sigma^j_{\lambda_n}(f, \xi)(z) - h(z)| \to 0, \ as \ n \to \infty,$$

then we say that f belongs to the class $U_{Ces(j)}(\Omega, \Omega^c)$.

- If, for every K and h as above, there exists a sequence (λ_n) of natural numbers such that, for every $j > -1$,

$$\sup_{z \in K} |\sigma^j_{\lambda_n}(f, \xi)(z) - h(z)| \to 0, \ as \ n \to \infty,$$

then we say that f belongs to the class $U_{Ces}(\xi, \Omega^c)$.

- If, for every K and h as above, there exists a sequence (λ_n) of natural numbers such that, for every compact set $N \subset \Omega$ and for every $j > -1$,

$$\sup_{\xi \in N} \sup_{z \in K} |\sigma^j_{\lambda_n}(f, \xi)(z) - h(z)| \to 0, \ as \ n \to \infty,$$

then we say that f belongs to the class $U_{Ces}(\Omega, \Omega^c)$.

Definition 1.2. If in Definition 1.1 the compact set K does not meet the boundary of Ω, i.e. $K \cap \overline{\Omega} = \emptyset$, then we have the definitions of the corresponding classes

$$U(\xi, \overline{\Omega}^c), \ U(\Omega, \overline{\Omega}^c), \ U_{Ces(j)}(\xi, \overline{\Omega}^c), \ U_{Ces(j)}(\Omega, \overline{\Omega}^c), \ U_{Ces}(\xi, \overline{\Omega}^c) \ and \ U_{Ces}(\Omega, \overline{\Omega}^c),$$

in the sense of Luh and Chui - Parnes.

Remark 1.3. In all the above definitions it is equivalent to require $\lambda_n < \lambda_{n+1}$. Furthermore, suitable applications of Baire's Theorem and Mergelyan's Theorem show that all the above classes are dense and G_δ subsets of $H(\Omega)$ ([3, 13, 24, 26, 28, 29]). For the importance of generic results and the role of Baire's Theorem in Complex, Harmonic

and Functional Analysis we refer to [14, 15]. Properties of the universal functions belonging to these classes can be found in [1, 2, 4, 9–12, 16, 19, 23–28, 30, 31]. For example, if $\Omega = \mathbb{D} = D(0,1)$ we have $U(0, \mathbb{D}^c) \neq U(0, \overline{\mathbb{D}}^c)$ ([26]). In general the classes $U(\xi, \Omega^c)$ and $U(\Omega, \Omega^c)$ contain functions, which have wild properties; in particular they cannot be smooth on the boundary ([24], [25], [26]) and they are holomorphic exactly in Ω ([24], [31]). On the contrary the classes $U(\xi, \overline{\Omega}^c)$ and $U(\Omega, \overline{\Omega}^c)$ contain also smooth functions up to the boundary, provided that $\{\infty\} \cup \overline{\Omega}^c$ is connected ([25], [26], [16]), or holomorphically extendable outside Ω, provided that $\partial\Omega \neq \partial\overline{\Omega}$ ([31]).

In the sequel we shall need some of the properties proved in the papers mentioned in Remark 1.3. We will present these at the beginning of Section 3. In Section 2, first we give the definition of Ostrowski gaps. Then, we prove a basic proposition, concerning subsequences of the Cesàro means of the sequence of the partial sums of a series possessing Ostrowski gaps, required for the proof of our main results. In Section 3 we prove our main results. We notice that in order to obtain these results, we shall impose some restrictions on the domain Ω. More precisely, we define:

Definition 1.4. Let Ω be a simply connected domain. We shall say that Ω is *admissible*, if Ω is contained in the complement of an angle with strictly positive opening and it satisfies the following geometrical and topological restrictions:

For any compact set $K \subset \mathbb{C}$ with K^c connected and such that $K \cap \Omega = \emptyset$ for the case of Definition 1.1 and $K \cap \overline{\Omega} = \emptyset$ for the case of Definition 1.2, there exist two compact subsets L and M of $\mathbb{C}$ and a positive real number r, such that $K \subset L \subset M$ and

- $M \cap \Omega = \emptyset$ for the case of Definition 1.1 and $M \cap \overline{\Omega} = \emptyset$ for the case of Definition 1.2.

- M^c is connected.

- For any $z \in L$, there exists $w \in L$ such that $z \in \overline{D}(w, r)$ and $\overline{D}(w, r) \subset L$.

Remark 1.5. In [2, Section 4], F. Bayart proved that circles, squares and so on, are admissible domains. Of course, for the case of Definition 1.2, any simply connected domain Ω is admissible provided that Ω is contained in the complement of an angle with strictly positive opening.

Now, the main result, which we shall prove in Section 3, is the following:

Theorem 1.6. *If Ω is an admissible simply connected domain, then all the classes of universal functions in Definition 1.1 coincide and the same is true for the classes of Definition 1.2. In particular, for the classes of Definition 1.2, this is true for any simply connected domain Ω such that Ω is contained in the complement of an angle with strictly positive opening.*

2. Ostrowski gaps and Cesàro means

By [33, Chapter III, Theorem 1.21] we know that

> if a series is summable (C, j), $j > -1$, to sum s, it is also summable
> $(C, j + \lambda)$ to s, for every $\lambda > 0$.

Moreover, if we assume that the series possesses gaps (p_m, q_m) with $\frac{q_m}{p_m} \geq q > 1$, which means that for $p_m < \nu \leq q_m$ its terms a_ν equal zero, then ([33, Chapter III, Theorem 1.27])

> if it is summable $(C, 1)$ to sum s, the subsequence S_{p_m} (and so S_{q_m} too)
> of its partial sums tends to s.

In this section we shall prove that under some further restrictions, a stronger result of this kind is true for subsequences of Cesàro means of complex series possessing Ostrowski gaps (see Proposition 2.3).

We start by recalling the definition of Ostrowski gaps. Ostrowski gaps were successfully used, for example in [2, 4, 6, 8, 21, 22, 24, 27], to obtain certain properties of universal Taylor series with respect to overconvergence. Here, in order to prove our results, Ostrowski gaps will also be our main tool.

Definition 2.1. Let $\sum_{\nu=0}^{\infty} a_\nu$ be a series of complex numbers, or, more generally, let $\sum_{\nu=0}^{\infty} a_\nu (z - \xi)^\nu$ be a power series with positive radius of convergence. We say that each of them has *Ostrowski gaps* (p_m, q_m), if (p_m) and (q_m) are two sequences of natural numbers such that:

(i) $p_1 < q_1 \leq p_2 < q_2 \leq p_3 < q_3 \leq \cdots$ and $\lim_{m \to \infty} \frac{q_m}{p_m} = \infty$.

(ii) For $I = \bigcup_{m=1}^{\infty} \{p_m + 1, ..., q_m\}$, we have $\lim_{\nu \in I} |a_\nu|^{\frac{1}{\nu}} = 0$.

Remark 2.2. By condition (ii) of Definition 2.1 we can subtract a constant from $\sum_{\nu=0}^{\infty} a_\nu$, or an entire function from $\sum_{\nu=0}^{\infty} a_\nu (z - \xi)^\nu$, in order to have

$$a_{p_m+1} = a_{p_m+2} = \cdots = a_{q_m} = 0.$$

Thus, without loss of generality, we shall assume this in the sequel.

Now, our basic proposition of this section is the following:

Proposition 2.3. *Let $\sum_{\nu=0}^{\infty} a_\nu$ be a complex series with Ostrowski gaps (p_m, q_m). If for some $j > -1$ we have*

$$\sigma_{q_m}^j \to s, \quad as \ \ m \to \infty \tag{1}$$

and there are constants $B \geq 2$ and $C_1 > 0$ such that

$$\forall \nu \in \mathbb{N}, \quad |a_\nu| \leq B^\nu \tag{2}$$

and

$$\left| \sum_{\nu=k}^{p_m} \nu(\nu-1)\cdots(\nu-k+1) \frac{A_{q_m-\nu}^j}{A_{q_m}^j} a_\nu \right| \leq C_1(Bp_m)^k, \quad k=1,2,...,p_m, \qquad (3)$$

then

- *for $j > -1$, $\sigma_{q_m}^{j+1} \to s$ and also*

- *for $j \geq 0$, $\sigma_{q_m}^{j+\lambda} \to s$, for every λ, $-1 < \lambda < 1$.*

In order to prove Proposition 2.3 we shall need two lemmas, which we state and prove immediately after the following remarks. First, as we notice in Remark 2.2 we can assume that

$$a_{p_m+1} = a_{p_m+2} = \cdots = a_{q_m} = 0.$$

Then, if we set $a_n^m = \frac{A_{q_m-n}^j}{A_{q_m}^j} a_n$ and $S_\nu(a_n^m) = \sum_{n=1}^{\nu} a_n^m$, we have that

$$\sigma_{q_m}^j = a_0 + S_{q_m}(a_n^m) = \cdots = a_0 + S_{p_m}(a_n^m).$$

By (1) we conclude that there exists a constant $C_2 > 0$, such that $|S_{p_m+1}(a_n^m)| = |S_{p_m}(a_n^m)| \leq C_2$. Let $C = \max\{1, C_1, C_2\}$. Having now the sequences (p_m) and (q_m), the constants B and C and the fact that $\frac{p_m}{q_m} \to 0$ as $m \to \infty$, we obtain that for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$

$$C\frac{(B+1)p_m}{q_m-1-(B+1)p_m} < \frac{\varepsilon}{4}. \qquad (4)$$

If in (4) we take $\varepsilon = 1$, then the corresponding m_0 is independent of ε and we denote this by m_1. In the following lemma, we shall use this m_1.

Lemma 2.4. *For $m \in \mathbb{N}$ we denote by (a_n^m) a sequence of complex numbers (see for example the sequence (a_n^m) defined above). Let $S_\nu(a_n^m) = \sum_{n=1}^{\nu} a_n^m$ and let (p_m), (q_m) be two sequences of natural numbers, satisfying (i) of Definition 2.1. Assume also that there exist two constants $B \geq 2$ and $C \geq 1$ such that:*

$$\forall m \geq m_1, \quad |S_{p_m}(a_n^m)| \leq C \quad and \quad |S_{p_m+1}(a_n^m)| \leq C. \qquad (5)$$

$$\forall m \geq m_1, \quad |a_\nu^m| \leq B^\nu \quad for \quad \nu \leq p_m. \qquad (6)$$

$$\forall m \geq m_1, \quad \left| \sum_{\nu=k}^{p_m} \nu(\nu-1)\cdots(\nu-k+1)a_\nu^m \right| \leq C(Bp_m)^k, \quad k=1,2,...,p_m. \qquad (7)$$

Then we have:

$$\forall m \geq m_1, |S_\nu(a_n^m)| \leq B^{\nu+1} \, for \, \nu < p_m, \, and \, \left| \sum_{\nu=1}^{p_m-1} S_\nu(a_n^m) \right| \leq C(B+1)p_m. \qquad (8)$$

$$\forall m \geq m_1, \left| \sum_{\nu=k}^{\mu} \nu(\nu-1)\cdots(\nu-k+1)S_\nu(a_n^m) \right| \leq C(Bp_m)^{k+1}, \ \mu = p_m, p_m+1, \quad (9)$$

and if (x_m) is a sequence of real numbers with $x_m > q_m - 1$, then

$$\forall m \geq m_1, \ \left| \sum_{\nu=1}^{p_m-1} \frac{1}{x_m - \nu} S_\nu(a_n^m) \right| \leq C \frac{(B+1)p_m}{x_m - (B+1)p_m}. \quad (10)$$

Moreover, for $m \in \mathbb{N}$ and $\nu \leq p_m + 1$ we define

$$b_\nu^m = \frac{1}{x_m - \nu} S_\nu(a_n^m).$$

Then, (5), (6) and (7) are valid if we replace (a_n^m) by (b_n^m) and thus the same is true for (8), (9) and (10) with (b_n^m) instead of (a_n^m).

Proof. For the proof we shall need the following well known formula of summation by parts

$$\sum_{\nu=1}^{n} c_\nu u_\nu = \sum_{\nu=1}^{n-1} (c_\nu - c_{\nu+1})S_\nu + c_n S_n,$$

where $S_k = u_1 + \cdots + u_k$, $k = 1, 2, ..., n$. Appropriately modified, it yields ($c_k = 0$)

$$\sum_{\nu=k+1}^{n} c_\nu u_\nu = \sum_{\nu=k}^{n-1} (c_\nu - c_{\nu+1})S_\nu + c_n S_n,$$

for every $k \geq 1$.

Now, in order to prove (8), we have by summation by parts that

$$\sum_{\nu=1}^{p_m} \nu a_\nu^m = -\sum_{\nu=1}^{p_m-1} S_\nu(a_n^m) + p_m S_{p_m}(a_n^m).$$

Then, by (5) and (7) we obtain that for every $m \geq m_1$,

$$\left| \sum_{\nu=1}^{p_m-1} S_\nu(a_n^m) \right| \leq CBp_m + Cp_m = C(B+1)p_m.$$

The first part of (8) is trivial:

$$|S_\nu(a_n^m)| \leq \sum_{n=1}^{\nu} |a_n^m| \leq B + B^2 + \cdots + B^\nu = B\frac{B^\nu - 1}{B - 1} \leq B^{\nu+1},$$

since $B \geq 2$. For (9), we have by summation by parts that

$$\sum_{\nu=k+1}^{p_m} \nu(\nu-1)\cdots(\nu-k)a_\nu^m$$

$$= -(k+1)\sum_{\nu=k}^{p_m-1} \nu(\nu-1)\cdots(\nu-k+1)S_\nu(a_n^m) + p_m(p_m-1)\cdots(p_m-k)S_{p_m}(a_n^m).$$

Then, again by (5) and (7) we obtain that for every $m \geq m_1$,

$$\left|\sum_{\nu=k}^{p_m-1} \nu(\nu-1)\cdots(\nu-k+1)S_\nu(a_n^m)\right| \leq \frac{C(B^{k+1}+1)}{k+1}p_m^{k+1} \leq C(Bp_m)^{k+1}.$$

Moreover, for every $m \geq m_1$, it is evident that

$$\left|\sum_{\nu=k}^{p_m} \nu(\nu-1)\cdots(\nu-k+1)S_\nu(a_n^m)\right| \leq \frac{C(B^{k+1}+1)}{k+1}p_m^{k+1} + Cp_m^k \leq C(Bp_m)^{k+1}.$$

To prove (10) consider a sequence of real numbers (x_m) with $x_m > q_m - 1$. Then, by the definition of m_1 (see (4)), we have that for every $m \geq m_1$,

$$C\frac{(B+1)p_m}{x_m-(B+1)p_m} \leq C\frac{(B+1)p_m}{q_m-1-(B+1)p_m} < \frac{1}{4}.$$

Besides this, we shall use frequently below the fact that

$$\frac{B}{x_m-p_m}, \quad \frac{C}{x_m-p_m}, \quad \frac{C}{x_m-p_m-1} \quad \text{and} \quad \frac{Bp_m}{x_m-p_m}$$

are less than $C\frac{(B+1)p_m}{x_m-(B+1)p_m}$ and so less than $\frac{1}{4}$ for every $m \geq m_1$. Now we have

$$\sum_{\nu=1}^{p_m-1}\frac{1}{x_m-\nu}S_\nu(a_n^m) = \frac{1}{x_m}\sum_{\nu=1}^{p_m-1}\frac{x_m}{x_m-\nu}S_\nu(a_n^m)$$

$$= \frac{1}{x_m}\sum_{\nu=1}^{p_m-1}S_\nu(a_n^m) + \frac{1}{x_m}\sum_{\nu=1}^{p_m-1}\frac{\nu}{x_m-\nu}S_\nu(a_n^m)$$

$$= \frac{1}{x_m}\sum_{\nu=1}^{p_m-1}S_\nu(a_n^m) + \frac{1}{x_m(x_m-1)}\sum_{\nu=1}^{p_m-1}\nu S_\nu(a_n^m)$$

$$+ \frac{1}{x_m(x_m-1)}\sum_{\nu=1}^{p_m-1}\frac{\nu(\nu-1)}{x_m-\nu}S_\nu(a_n^m) = \cdots$$

$$= \frac{1}{x_m}\sum_{\nu=1}^{p_m-1}S_\nu(a_n^m) + \frac{1}{x_m(x_m-1)}\sum_{\nu=1}^{p_m-1}\nu S_\nu(a_n^m)$$

$$+ \frac{1}{x_m(x_m-1)(x_m-2)}\sum_{\nu=2}^{p_m-1}\nu(\nu-1)S_\nu(a_n^m)$$

$$+ \frac{1}{x_m(x_m-1)\cdots(x_m-3)}\sum_{\nu=3}^{p_m-1}\nu(\nu-1)(\nu-2)S_\nu(a_n^m) + \ldots$$

$$+ \frac{(p_m-1)!}{x_m(x_m-1)\cdots(x_m-p_m+1)}S_{p_m-1}(a_n^m).$$

Thus, by (8) and (9) we easily obtain that

$$\left|\sum_{\nu=1}^{p_m-1}\frac{1}{x_m-\nu}S_\nu(a_n^m)\right| \leq C\frac{(B+1)p_m}{x_m-p_m}\left(1 + \frac{Bp_m}{x_m-p_m} + \left(\frac{Bp_m}{x_m-p_m}\right)^2 + \cdots\right)$$

$$= C\frac{(B+1)p_m}{x_m-(B+1)p_m}.$$

It remains to prove (5), (6) and (7) for the sequence (b_n^m). Since

$$b_\nu^m = \frac{1}{x_m-\nu}S_\nu(a_n^m),$$

we have by (5) and (10) that for every $m \geq m_1$

$$|S_{p_m}(b_n^m)| = \left|\sum_{\nu=1}^{p_m}b_\nu^m\right| \leq \left|\sum_{\nu=1}^{p_m-1}\frac{1}{x_m-\nu}S_\nu(a_n^m)\right| + \left|\frac{1}{x_m-p_m}S_{p_m}(a_n^m)\right|$$

$$\leq C\frac{(B+1)p_m}{x_m-(B+1)p_m} + \frac{C}{x_m-p_m} \leq C,$$

and similarly

$$|S_{p_m+1}(b_n^m)| \leq C \frac{(B+1)p_m}{x_m - (B+1)p_m} + \frac{C}{x_m - p_m} + \frac{C}{x_m - p_m - 1} \leq C.$$

Furthermore, by (8)

$$|b_\nu^m| = \left| \frac{1}{x_m - \nu} S_\nu(a_n^m) \right| \leq \frac{B}{x_m - p_m} B^\nu \leq B^\nu.$$

Finally,

$$\sum_{\nu=k}^{p_m} \nu(\nu-1)\cdots(\nu-k+1)b_\nu^m = \sum_{\nu=k}^{p_m} \nu(\nu-1)\cdots(\nu-k+1)\frac{1}{x_m - \nu}S_\nu(a_n^m)$$

$$= \frac{1}{x_m - k}\sum_{\nu=k}^{p_m} \nu(\nu-1)\cdots(\nu-k+1)\frac{x_m - k}{x_m - \nu}S_\nu(a_n^m),$$

and continuing as in the proof of (10), we arrive at the following formula:

$$\sum_{\nu=k}^{p_m} \nu(\nu-1)\cdots(\nu-k+1)b_\nu^m = \frac{1}{x_m - k}\sum_{\nu=k}^{p_m} \nu(\nu-1)\cdots(\nu-k+1)S_\nu(a_n^m)$$

$$+ \frac{1}{(x_m - k)(x_m - k - 1)}\sum_{\nu=k+1}^{p_m} \nu(\nu-1)\cdots(\nu-k)S_\nu(a_n^m) + \ldots$$

$$+ \frac{p_m!}{(x_m - k)(x_m - k - 1)\cdots(x_m - p_m)}S_{p_m}(a_n^m).$$

Then, by (5) and (9) we conclude that for every $m \geq m_1$ we have

$$\left| \sum_{\nu=k}^{p_m} \nu(\nu-1)\cdots(\nu-k+1)b_\nu^m \right| \leq C(Bp_m)^k \left(\frac{Bp_m}{x_m - p_m} + \left(\frac{Bp_m}{x_m - p_m} \right)^2 + \cdots \right)$$

$$= C(Bp_m)^k \frac{Bp_m}{x_m - (B+1)p_m} \leq C(Bp_m)^k,$$

and the proof is complete.   □

From Lemma 2.4 we conclude that everything we can prove for the sequence (a_n^m), using (5) to (10), is automatically true for the sequence (b_n^m). For example, by summation by parts we have:

$$\sum_{\nu=1}^{p_m} \left(\frac{1}{x_m} + \frac{1}{x_m - 1} + \cdots + \frac{1}{x_m - \nu + 1} \right) a_\nu^m = -\sum_{\nu=1}^{p_m-1} \frac{1}{x_m - \nu}S_\nu(a_n^m)$$

$$+ \left(\frac{1}{x_m} + \frac{1}{x_m - 1} + \cdots + \frac{1}{x_m - p_m + 1} \right) S_{p_m}(a_n^m).$$

So, by (5) and (10) we have

$$\left| \sum_{\nu=1}^{p_m} \left(\frac{1}{x_m} + \frac{1}{x_m - 1} + \cdots + \frac{1}{x_m - \nu + 1} \right) a_\nu^m \right| \leq 2C \frac{(B+1)p_m}{x_m - (B+1)p_m},$$

for $m \geq m_1$. Since $x_m > q_m - 1$, we have by (4) that for every $\varepsilon \in (0,1)$ the corresponding m_0 is greater than m_1 and so for every $m \geq m_0$ we have

$$\left| \sum_{\nu=1}^{p_m} \left(\frac{1}{x_m} + \frac{1}{x_m - 1} + \cdots + \frac{1}{x_m - \nu + 1} \right) a_\nu^m \right| < \frac{\varepsilon}{2}.$$

Then, we automatically conclude that for the same $m \geq m_0$ it is also true that

$$\left| \sum_{\nu=1}^{p_m} \left(\frac{1}{x_m} + \frac{1}{x_m - 1} + \cdots + \frac{1}{x_m - \nu + 1} \right) b_\nu^m \right| < \frac{\varepsilon}{2}. \tag{11}$$

Let us suppose now that for the sequence (a_n^m) of Lemma 2.4 we have that $a_{p_m+1}^m$ tends to zero and more precisely

$$|a_{p_m+1}^m| \leq \frac{C}{x_m - p_m - 1},$$

in accordance to $b_{p_m+1}^m$ (in fact we shall take later $a_{p_m+1}^m = 0$). Next, we denote by

$$s_k \left(\frac{1}{x_m}, \frac{1}{x_m - 1}, \cdots, \frac{1}{x_m - n} \right) = \sum_{0 \leq j_1 < \cdots < j_k \leq n} \frac{1}{(x_m - j_1) \cdots (x_m - j_k)},$$

$k = 2, 3, ..., n$, the kth symmetric polynomial in the variables $\frac{1}{x_m}, \frac{1}{x_m - 1}, \cdots, \frac{1}{x_m - n}$. Then, by summation by parts we have

$$\sum_{\nu=2}^{p_m+1} s_2 \left(\frac{1}{x_m}, \frac{1}{x_m - 1}, \cdots, \frac{1}{x_m - \nu + 1} \right) a_\nu^m$$

$$= - \sum_{\nu=1}^{p_m} \left(\frac{1}{x_m} + \frac{1}{x_m - 1} + \cdots + \frac{1}{x_m - \nu + 1} \right) \frac{1}{x_m - \nu} S_\nu(a_n^m)$$

$$+ s_2 \left(\frac{1}{x_m}, \frac{1}{x_m - 1}, \cdots, \frac{1}{x_m - p_m} \right) S_{p_m+1}(a_n^m)$$

$$= - \sum_{\nu=1}^{p_m} \left(\frac{1}{x_m} + \frac{1}{x_m - 1} + \cdots + \frac{1}{x_m - \nu + 1} \right) b_\nu^m$$

$$+ s_2 \left(\frac{1}{x_m}, \frac{1}{x_m - 1}, \cdots, \frac{1}{x_m - p_m} \right) S_{p_m+1}(a_n^m).$$

which, by (5) and (11), implies that

$$\left| \sum_{\nu=2}^{p_m+1} s_2 \left(\frac{1}{x_m}, \frac{1}{x_m-1}, \cdots, \frac{1}{x_m-\nu+1} \right) a_\nu^m \right| \le \frac{\varepsilon}{2} + \frac{C}{2} \left(\frac{p_m+1}{x_m-p_m} \right)^2$$

and so

$$\left| \sum_{\nu=2}^{p_m} s_2 \left(\frac{1}{x_m}, \frac{1}{x_m-1}, \cdots, \frac{1}{x_m-\nu+1} \right) a_\nu^m \right| \le \frac{\varepsilon}{2} + \frac{C}{2} \left(\frac{p_m+1}{x_m-p_m} \right)^2 + \frac{Cp_m(p_m+1)}{2(x_m-p_m)^3}$$

$$\le \frac{\varepsilon}{2} + \left(\frac{\varepsilon}{2} \right)^2,$$

since

$$\frac{C}{2} \left(\frac{p_m+1}{x_m-p_m} \right)^2 + \frac{Cp_m(p_m+1)}{2(x_m-p_m)^3} \le \frac{C}{2} \left(\frac{p_m+1}{x_m-p_m} \right)^2 \left(1 + \frac{1}{x_m-p_m} \right)$$

$$\le \left(2C \frac{(B+1)p_m}{x_m-(B+1)p_m} \right)^2 \le \left(\frac{\varepsilon}{2} \right)^2$$

for every $m \ge m_0$. Then, we automatically conclude that for the same $m \ge m_0$ it is also true that

$$\left| \sum_{\nu=2}^{p_m} s_2 \left(\frac{1}{x_m}, \frac{1}{x_m-1}, \cdots, \frac{1}{x_m-\nu+1} \right) b_\nu^m \right| \le \frac{\varepsilon}{2} + \left(\frac{\varepsilon}{2} \right)^2,$$

which we shall need in order to continue one more step further. Thus, for every $m \ge m_0$, an easy induction argument on k implies that the following holds:

$$\left| \sum_{\nu=k}^{p_m} s_k \left(\frac{1}{x_m}, \frac{1}{x_m-1}, \cdots, \frac{1}{x_m-\nu+1} \right) a_\nu^m \right| \le \sum_{n=1}^{k} \left(\frac{\varepsilon}{2} \right)^n.$$

Since for every $\varepsilon < 1$, $\sum_{n=1}^{k} \left(\frac{\varepsilon}{2} \right)^n < \varepsilon$, we have proved the following:

Lemma 2.5. *If for the sequence (a_n^m) of Lemma 2.4 we have that $a_{p_m+1}^m = 0$, then for every $0 < \varepsilon < 1$ there exists $m_0 \in \mathbb{N}$ such that for all $m \ge m_0$*

$$\left| \sum_{\nu=k}^{p_m} s_k \left(\frac{1}{x_m}, \frac{1}{x_m-1}, \cdots, \frac{1}{x_m-\nu+1} \right) a_\nu^m \right| < \varepsilon,$$

$k = 1, 2, ..., p_m.$

We are now ready to prove our basic Proposition 2.3.

Proof. Without loss of generality, we assume that $a_{p_m+1} = \cdots = a_{q_m} = 0$. Thus,

$$
\sigma_{q_m}^{j+\lambda} = \sum_{\nu=0}^{q_m} \frac{A_{q_m-\nu}^{j+\lambda}}{A_{q_m}^{j+\lambda}} a_\nu = \sum_{\nu=0}^{p_m} \frac{A_{q_m-\nu}^{j+\lambda}}{A_{q_m}^{j+\lambda}} a_\nu =
$$

$$
= a_0 + \sum_{\nu=1}^{p_m} \frac{(q_m+j)(q_m+j-1)\cdots(q_m+j-\nu+1)}{(q_m+j+\lambda)\cdots(q_m+j+\lambda-\nu+1)} \frac{A_{q_m-\nu}^{j}}{A_{q_m}^{j}} a_\nu.
$$

For $\lambda = 1$ this yields that

$$
\sigma_{q_m}^{j+1} = a_0 + \sum_{\nu=1}^{p_m} \frac{q_m+j-\nu+1}{q_m+j+1} \frac{A_{q_m-\nu}^{j}}{A_{q_m}^{j}} a_\nu = \sigma_{q_m}^{j} - \frac{1}{q_m+j+1} \sum_{\nu=1}^{p_m} \nu \frac{A_{q_m-\nu}^{j}}{A_{q_m}^{j}} a_\nu,
$$

which implies the desired result for any $j > -1$, i.e.

$$
|\sigma_{q_m}^{j+1} - s| \leq |\sigma_{q_m}^{j} - s| + \left| \frac{1}{q_m+j+1} \sum_{\nu=1}^{p_m} \nu \frac{A_{q_m-\nu}^{j}}{A_{q_m}^{j}} a_\nu \right| \to 0,
$$

by (1), (3) and the fact that $\frac{p_m}{q_m} \to 0$.

For $j \geq 0$ and $-1 < \lambda < 1$, we set $a_\nu^m = \frac{A_{q_m-\nu}^{j}}{A_{q_m}^{j}} a_\nu$, $x_m = q_m+j+\lambda > q_m - 1$ and we have:

$$
\sigma_{q_m}^{j+\lambda} = a_0 + \sum_{\nu=1}^{p_m} \left(1 - \frac{\lambda}{x_m}\right) \cdots \left(1 - \frac{\lambda}{x_m-\nu+1}\right) a_\nu^m
$$

$$
= \sigma_{q_m}^{j} - \lambda \sum_{\nu=1}^{p_m} \left(\frac{1}{x_m} + \frac{1}{x_m-1} + \cdots + \frac{1}{x_m-\nu+1}\right) a_\nu^m
$$

$$
+ \lambda^2 \sum_{\nu=2}^{p_m} s_2 \left(\frac{1}{x_m}, \frac{1}{x_m-1}, \cdots, \frac{1}{x_m-\nu+1}\right) a_\nu^m
$$

$$
- \lambda^3 \sum_{\nu=3}^{p_m} s_3 \left(\frac{1}{x_m}, \frac{1}{x_m-1}, \cdots, \frac{1}{x_m-\nu+1}\right) a_\nu^m + \cdots
$$

$$
+ (-1)^{p_m} \lambda^{p_m} \frac{1}{x_m \cdots (x_m-p_m+1)} a_{p_m}^m.
$$

By [33, Chapter III, Theorem 1.17], we know that

> A_n^j is positive for $j > -1$, increasing (as a function of n) for $j \geq 0$ and decreasing for $-1 < j < 0$,

so that, in our case ($j \geq 0$), $\forall m \in \mathbb{N}$, $|a_\nu^m| \leq |a_\nu| \leq B^\nu$ for $\nu \leq p_m$, by (2). Moreover, as we already saw, there exists, by (1), a constant $C_2 > 0$, such that

$|S_{p_m+1}(a_n^m)| = |S_{p_m}(a_n^m)| \le C_2$. If $C = \max\{1, C_1, C_2\}$, we see that (a_n^m) satisfies the hypotheses of Lemma 2.5 and so for every $\varepsilon \in (0,1)$, there exists $m_0 \in \mathbb{N}$ such that, for every $m \ge m_0$ we have that

$$|\sigma_{q_m}^{j+\lambda} - \sigma_{q_m}^j| < (|\lambda| + |\lambda|^2 + \cdots + |\lambda|^{p_m})\varepsilon \le \frac{|\lambda|}{1 - |\lambda|}\varepsilon,$$

which implies the desired result and the proof is finished. $\qquad\qquad\qquad\square$

3. Main Results

Now we are ready to prove our main results. First, as we mentioned in the Introduction, we recall some known properties of universal functions. In [12] it has been proved that if Ω is a bounded simply connected domain, $\xi \in \Omega$ and $f \in U(\xi, \overline{\Omega}^c)$, then the Taylor series of f at ξ has Ostrowski gaps and $U(\xi, \overline{\Omega}^c) = U(\Omega, \overline{\Omega}^c)$. This in turn is a corollary of Theorem 9.1 of [24], where, moreover, it has been proved that $U(\xi, \Omega^c) = U(\Omega, \Omega^c)$ provided that Ω is contained in the complement of an angle with strictly positive opening. Finally, the general case has been proved in [27] and we state this in the following lemma.

Lemma 3.1. *Let Ω be any simply connected domain, $\xi \in \Omega$ and $f \in U(\xi, \Omega^c)$. Then*

- *The Taylor series of f at ξ has Ostrowski gaps.*

- $U(\zeta, \Omega^c) = U(\Omega, \Omega^c)$, *for any $\zeta \in \Omega$.*

However, for the case of $U(\xi, \overline{\Omega}^c)$ we have not such a result for all simply connected domains, but we have for example the following lemma ([24, 27]).

Lemma 3.2. *Let Ω be a simply connected domain such that $\overline{\Omega}^c$ has an unbounded component, $\xi \in \Omega$ and $f \in U(\xi, \overline{\Omega}^c)$. Then*

- *The Taylor series of f at ξ has Ostrowski gaps.*

- $U(\zeta, \overline{\Omega}^c) = U(\Omega, \overline{\Omega}^c)$, *for any $\zeta \in \Omega$.*

Next, the following property for (C, j)-means, $j \ge 0$ an integer, has been proved in Lemma 2.6 of [2], which we state here for admissible simply connected domains Ω and $j > -1$. Thus, in our case, j is no longer an integer (in general), but the proof remains unchanged, except in its last step, where instead of j it suffices to use the integral part of j and more precisely the integer $[j] + 1$.

Lemma 3.3. *Let Ω be an admissible simply connected domain, $\xi \in \Omega$, $j > -1$ and $f \in U_{Ces(j)}(\xi, \Omega^c)$. If K and h are as in Definition 1.1, then we have*

- *The Taylor series of f at ξ has Ostrowski gaps (p_m, q_m).*

- $\sup_{z \in K} |\sigma^j_{q_m}(f, \xi)(z) - h(z)| \to 0, \quad as \ m \to \infty.$

Finally, we will need a classical lemma which comes from Harmonic Analysis.

Lemma 3.4 ([2, Lemma 4.2]). *Let $p(z)$ be a holomorphic polynomial of degree N, $w \in \mathbb{C}$, $r > 0$ and $k \geq 1$. Then we have*

$$\sup_{z \in \overline{D}(w,r)} |p^{(k)}(z)| \leq \frac{N(N-1)\cdots(N-k+1)}{r^k} \sup_{z \in \overline{D}(w,r)} |p(z)|.$$

For $w = 0$ and $r = 1$ this is just Bernstein's inequality. The general case follows easily.

We are ready now to state and prove our main Theorem.

Theorem 3.5. *Let Ω be an admissible simply connected domain, $\xi \in \Omega$, $j > -1$ and $f \in U_{Ces(j)}(\xi, \Omega^c)$. Then, for every compact set K as in Definition 1.1 and any holomorphic polynomial $P(z)$, there exists a sequence (q_m) of natural numbers such that, for every $\alpha > -1$,*

$$\sup_{z \in K} |\sigma^\alpha_{q_m}(f, \xi)(z) - P(z)| \to 0, \quad as \ m \to \infty.$$

Proof. Let K be a compact set such that $K \cap \Omega = \emptyset$, K^c connected and let $P(z)$ be a holomorphic polynomial. Since Ω is an admissible simply connected domain there exist compact sets L, M and a positive number r such that $K \subset L \subset M$, $M \cap \Omega = \emptyset$, M^c connected and for any $z \in L$, there exists $w \in L$ such that $z \in \overline{D}(w, r)$ and $\overline{D}(w, r) \subset L$. By Lemma 3.3, $f \in U_{Ces(j)}(\xi, \Omega^c)$ implies that the Taylor series $\sum_{\nu=0}^{\infty} a_\nu (z - \xi)^\nu$ of f at ξ has Ostrowski gaps (p_m, q_m) and

$$\sup_{z \in M} |\sigma^j_{q_m}(f, \xi)(z) - P(z)| \to 0, \quad as \ m \to \infty.$$

Hence, there exists a constant $C_1 > 0$ such that, for any $m \geq 0$, we have

$$\sup_{w \in L} |\sigma^j_{q_m}(f, \xi)(w)| \leq C_1.$$

By Remark 2.2 we assume, without loss of generality, that

$$a_{p_m+1} = a_{p_m+2} = \cdots = a_{q_m} = 0.$$

Then, for every $m \geq 0$, we obtain that

$$\sup_{w \in L} |\sigma^j_{q_m}(f, \xi)(w)| = \sup_{w \in L} \left| \sum_{\nu=0}^{p_m} a^m_\nu \right| = \sup_{w \in L} \left| \sum_{\nu=0}^{p_m+1} a^m_\nu \right| \leq C_1,$$

where $a_\nu^m = \frac{A_{q_m-\nu}^j}{A_{q_m}^j} a_\nu (z - \xi)^\nu$. Since for any $z \in K$ there exists $w \in L$ such that $z \in \overline{D}(w, r)$ and $\overline{D}(w, r) \subset L$, one obtains, by Lemma 3.4, that

$$|\sigma_{q_m}^j (f^{(k)}, \xi)(z)| \leq \frac{p_m \cdots (p_m - k + 1)}{r^k} C_1, \quad 1 \leq k \leq p_m.$$

This in turn implies that for all $m \geq 0$

$$\sup_{z \in K} \left| \sum_{\nu=k}^{p_m} \nu(\nu - 1) \cdots (\nu - k + 1) a_\nu^m \right| = \sup_{z \in K} |z - \xi|^k |\sigma_{q_m}^j (f^{(k)}, \xi)(z)| \leq C_1 \left(\frac{Ap_m}{r} \right)^k,$$

where $A = \sup_{z \in K} |z - \xi|$. Moreover, if R_ξ is the radius of convergence of the Taylor series of f at ξ and $0 < R_1 < R_\xi$, then there exists $n_0 \in \mathbb{N}$ such that $|a_\nu| \leq \left(\frac{1}{R_1} \right)^\nu$, for every $\nu \geq n_0$. So, we can find a positive number r_1 such that $|a_\nu| \leq \left(\frac{1}{r_1} \right)^\nu$, for all $\nu \in \mathbb{N}$. Hence, for every $\nu \in \mathbb{N}$ we have that $|a_\nu (z - \xi)^\nu| \leq \left(\frac{A}{r_1} \right)^\nu$. Taking $B = \max \left\{ 2, \frac{A}{r}, \frac{A}{r_1} \right\}$ all the hypotheses of Proposition 2.3 are fulfilled and since by (4) it is obvious that m_0 is independent from z, we conclude that

(i) if $j \geq 0$, then for every $\lambda \in (-1, 1]$

$$\sup_{z \in K} |\sigma_{q_m}^{j+\lambda}(f, \xi)(z) - P(z)| \to 0, \quad \text{as } m \to \infty,$$

and continuing in the same way we obtain that for every $\alpha > -1$

$$\sup_{z \in K} |\sigma_{q_m}^\alpha (f, \xi)(z) - P(z)| \to 0, \quad \text{as } m \to \infty.$$

(ii) If $-1 < j < 0$, then we know (by Proposition 2.3 again) that

$$\sup_{z \in K} |\sigma_{q_m}^{j+1}(f, \xi)(z) - P(z)| \to 0, \quad \text{as } m \to \infty,$$

and since $j + 1 > 0$ we have the required result by case (i) and the proof is finished.

$\square$

Now, let Ω be an admissible simply connected domain and $f \in U_{Ces(j)}(\xi, \Omega^c)$. Let K be a compact subset of $\mathbb{C}$ with $K \cap \Omega = \emptyset$ and K^c connected, and let $h : K \to \mathbb{C}$ be a continuous function on K and holomorphic in K^o. By Mergelyan's Theorem, there exists a sequence (P_n) of holomorphic polynomials such that $P_n \to h$ uniformly on K. By Theorem 3.5, for any $n \in \mathbb{N}$, there exists a sequence $(q_m(n))_{m \geq 0}$ such that, for any $\alpha > -1$, one has

$$\sup_{z \in K} |\sigma_{q_m(n)}^\alpha (f, \xi)(z) - P_n(z)| \to 0, \quad \text{as } m \to \infty.$$

Using a diagonal argument, we easily deduce that $f \in U_{Ces}(\xi, \Omega^c)$. In particular $f \in U(\xi, \Omega^c)$ and hence, by Lemma 3.1, $f \in U(\Omega, \Omega^c)$. Moreover, if N is any compact subset of Ω, we can take the positive number R_1, appearing in the proof of Theorem 3.5, so that $0 < R_1 < dist(N, \Omega^c)$. Then, we can repeat the proof of Theorem 3.5 and the above argument in order to deduce that, for any $\alpha > -1$ and any compact set $N \subset \Omega$, we have

$$\sup_{\xi \in N} \sup_{z \in K} |\sigma_{r_m}^\alpha(f, \xi)(z) - h(z)| \to 0, \quad \text{as } m \to \infty,$$

for some sequence (r_m) of natural numbers i.e. $f \in U_{Ces}(\Omega, \Omega^c)$. Thus,

$$U_{Ces(j)}(\xi, \Omega^c) \subset U_{Ces}(\Omega, \Omega^c)$$

and since

$$U_{Ces}(\Omega, \Omega^c) \subset U_{Ces(j)}(\Omega, \Omega^c) \subset U_{Ces(j)}(\xi, \Omega^c)$$

and

$$U_{Ces}(\Omega, \Omega^c) \subset U_{Ces}(\xi, \Omega^c) \subset U_{Ces(j)}(\xi, \Omega^c),$$

we have proved the following.

Corollary 3.6. *If Ω is an admissible simply connected domain, then all the classes of universal functions in Definition 1.1 coincide.*

For the classes of Definition 1.2 we know, by Remark 1.5, that any simply connected domain Ω, which is contained in the complement of an angle with strictly positive opening, is an admissible domain and thus, essentially the same argument as above yields the following.

Corollary 3.7. *If Ω is a simply connected domain and Ω is contained in the complement of an angle with strictly positive opening, then all the classes of universal functions in Definition 1.2 coincide.*

Remark 3.8. We notice that for a function f holomorphic in a simply connected domain Ω, there is another natural way to obtain an expansion in series: the Faber series expansion. For the existence of Universal Faber series and their properties we refer to [4, 18, 22]. In summary, let Γ be a compact, connected subset of Ω, with more than one point and $\mathbb{C}\backslash\Gamma$ connected. Let Y be the set of all such subsets Γ of Ω. If $\Gamma \in Y$ and K, h are as in Definition 1.1, then we say that a holomorphic function $f \in H(\Omega)$ belongs to the class $U(\Omega, \Gamma)$, if there exists a sequence (λ_n) of natural numbers such that the subsequence $S_{\lambda_n}(f, \Gamma)(z)$ of the partial sums of the Faber series of f, converges to $h(z)$ uniformly on K. On the other hand we say that f belongs to the class $U_{Fab}(\Omega)$, if for any compact set $M \subset Y$, where Y is endowed with the metric topology introduced in [3], section B.5 (see also [4]), we have that

$$\sup_{\Gamma \in M} \sup_{z \in K} |S_{\lambda_n}(f, \Gamma)(z) - h(z)| \to 0, \quad \text{as } n \to \infty.$$

In [22], Mayenberger and Müller have proved that for any $\Gamma \in Y$, $U(\Omega, \Gamma) = U(\Omega, \Omega^c)$. Moreover, Theorem 1.5 in [4] extends this result to the following: For any $\Gamma \in Y$, $U(\Omega, \Gamma) = U(\Omega, \Omega^c) = U_{Fab}(\Omega)$. Thus, for the case of an admissible simply connected domain Ω we have that all the classes of Universal functions of Definition 1.1 coincide with the above classes of Universal Faber series. However, we do not know what happens for non-admissible domains. As pointed out to us by V. Vlachou, we can prove Lemma 3.3 using the methods of [27], in order to avoid the restriction that Ω is contained in the complement of an angle with strictly positive opening, but the general case remains still open.

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