Asymptotic Uniform Moduli and Kottman Constant of Orlicz Sequence Spaces

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ABSTRACT

We give lower and upper bounds, involving moduli of asymptotic uniform convexity and smoothness, for the Kottman separation constant of Orlicz sequence spaces equipped with the Luxemburg norm.

Key words: Modulus of asymptotic uniform convexity, modulus of asymptotic uniform smoothness, Kottman constant, Orlicz sequence space.

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1. Introduction

The Kottman separation constant of a normed space $(X, \|\,.\,\|)$ is defined by

$$K(X) = \sup_{(x_n)\subset S_X} \left\{ \inf_{m\neq p} \{\|x_m - x_p\|\} \right\},$$

where S_X denotes the unit sphere of X. In this paper we are interested in the study of $K(X)$ when X is an Orlicz sequence spaces (see Section 3 for definitions). We refer to [11] and to [13], and references therein, for detailed works in this topic. Our main result is Theorem 4.1, which provides an estimation in terms of moduli of asymptotic uniform convexity and smoothness of the space (see Section 2 for definitions and for two lemmas which may be of independent interest). Section 3 is devoted to the characterization of asymptotically uniformly convex (respectively smooth) Orlicz sequence spaces in connection with the condition Δ_2 (respectively ∇_2). Moreover, we give here some quantitative estimates of the moduli. In Section 4 we give the statement and the proof or our main result. In Section 5, we discuss the accuracy of our estimate. Moreover we prove the analogue of our main theorem in the setting of Musielak-Orlicz sequence spaces.

2. The moduli of asymptotic uniform smoothness and convexity

Milman in [9] introduced two moduli for the study of an infinite-dimensional Banach space X. Johnson, Lindenstrauss, Preiss and Schechtman investigated these moduli in [7] and called them modulus of asymptotic uniform convexity, given for $t > 0$ by

$$\overline{\delta}_X(t) = \inf_{\|x\|=1} \sup_{\substack{Z\subset X \\ \text{co-dim } Z<\infty}} \inf_{\substack{z\in Z \\ \|z\|\geq t}} \|x+z\| - 1$$

and modulus of asymptotic uniform smoothness, given for $t > 0$ by

$$\overline{\rho}_X(t) = \sup_{\|x\|=1} \inf_{\substack{Z\subset X \\ \text{co-dim } Z<\infty}} \sup_{\substack{z\in Z \\ \|z\|\leq t}} \|x+z\| - 1.$$

The Banach space X is said to be asymptotically uniformly convex if $\overline{\delta}_X(t) > 0$ for every $0 < t < 1$, and asymptotically uniformly smooth if $\overline{\rho}_X(t)/t \to 0$ as $t \to 0$. For example, if X is a subspace of ℓ_p, with $1 \leq p < \infty$, then, for every $t \in [0, 1]$, $\overline{\rho}_X(t) = \overline{\delta}_X(t) = (1+t^p)^{1/p} - 1$. In particular, ℓ_1 is asymptotically uniformly convex. If X is a subspace of c_0, then, for every $t \in [0, 1]$, $\overline{\rho}_X(t) = \overline{\delta}_X(t) = 0$. In particular, c_0 is asymptotically uniformly smooth.

Lemma 2.1. *Let X and Y be infinite-dimensional Banach spaces such that X contains almost isometric copies of Y. Then, for every $0 < t < 1$*

$$\overline{\delta}_X(t) \leq \overline{\delta}_Y(t) \leq \overline{\rho}_Y(t) \leq \overline{\rho}_X(t).$$

Proof. Fix $0 < t < 1$. The inequality $\overline{\delta}_Y(t) \leq \overline{\rho}_Y(t)$ is given in [7, Proposition 2.3.(1)]. Let us show the inequality $\overline{\delta}_X(t) \leq \overline{\delta}_Y(t)$. Fix $\varepsilon > 0$. There exists an isomorphism $\varphi : Y \to \varphi(Y) \subseteq X$ such that for every $y \in Y$

$$\|y\| \leq \|\varphi(y)\| \leq (1+\varepsilon)\|y\|.$$

For every $y \in Y$ such that $\|y\| = 1$ and for every $z \in Y$

$$\begin{aligned} \|y+z\| &\geq \frac{1}{1+\varepsilon}\|\varphi(y)+\varphi(z)\| \\ &\geq \frac{\|\varphi(y)\|}{1+\varepsilon}\left\|\frac{\varphi(y)}{\|\varphi(y)\|} + \frac{\varphi(z)}{\|\varphi(y)\|}\right\| \\ &\geq \frac{1}{1+\varepsilon}\left\|\frac{\varphi(y)}{\|\varphi(y)\|} + \frac{\varphi(z)}{\|\varphi(y)\|}\right\|. \end{aligned} \tag{1}$$

There exists a finite co-dimensional subspace $Z \subset \varphi(Y)$ which depends on y, ε and t such that for every $z \in \varphi^{-1}(Z)$ with $\|z\| \geq t$

$$\begin{aligned} \left\|\frac{\varphi(y)}{\|\varphi(y)\|} + \frac{\varphi(z)}{\|\varphi(y)\|}\right\| &\geq \frac{1}{1+\varepsilon} \sup_{\substack{U\subset\varphi(Y) \\ \text{co-dim } U<\infty}} \inf_{\substack{u\in U \\ \|u\|\geq\frac{t}{1+\varepsilon}}} \left\|\frac{\varphi(y)}{\|\varphi(y)\|} + u\right\| \\ &\geq \frac{1}{1+\varepsilon}\left(\overline{\delta}_{\varphi(Y)}\left(\frac{t}{1+\varepsilon}\right)\right). \end{aligned}$$

Using (1) and the fact that $\varphi^{-1}(Z)$ is a finite co-dimensional subspace of Y, we obtain

$$\overline{\delta}_Y(t) \geq \frac{1}{(1+\varepsilon)^2}\left(\overline{\delta}_{\varphi(Y)}\left(\frac{t}{1+\varepsilon}\right)\right).$$

Recall that $\varphi(Y) \subseteq X$. According to [7, Proposition 2.3.(2)], for every $0 < s < 1$, $\overline{\delta}_{\varphi(Y)}(s) \geq \overline{\delta}_X(s)$. So

$$\overline{\delta}_Y(t) \geq \frac{1}{(1+\varepsilon)^2}\left(\overline{\delta}_X\left(\frac{t}{1+\varepsilon}\right)\right)$$

and letting ε tends to 0 the inequality $\overline{\delta}_Y(t) \geq \overline{\delta}_X(t)$ is done. The last inequality $\overline{\rho}_Y(t) \leq \overline{\rho}_X(t)$ is given by similar arguments. □

Remark 2.2. According to the James theorem, see for example [8, Proposition 2.e.3], if a Banach space X contains an isomorphic copy of c_0 (respectively of ℓ_1) then X contains in fact almost isometric copies of c_0 (respectively ℓ_1). Using this result and Lemma 2.1, we can state that a Banach space that contains an isomorphic copy of c_0 (respectively of ℓ_1) admits no asymptotically uniformly convex (respectively smooth) equivalent renorming.

Next lemma is connected to Lemma 2.1 and Remark 2.2 in [3]. Following the terminology of [3], we denote by X a Banach space with a finite-dimensional decomposition (E_n) and, for all $n \geq 1$, we consider the subspaces $H_n = \oplus_{i=1}^n E_i$ and $H^n = \overline{\oplus_{i=n+1}^\infty E_i}$, the closure of $\oplus_{i=n+1}^\infty E_i$. We denote by S_X the unit sphere of X and we consider D the dense subset of S_X defined by $D = \cup_{n=1}^\infty H_n \cap S_X$. We say that $x^* \in X^*$ has finite support if there exists $n \geq 1$ such that $H^n \subseteq \ker x^*$.

Lemma 2.3. *Let X be a Banach space with a finite-dimensional decomposition such that elements with finite support are dense in X^*. Then, for every $t > 0$*

$$\overline{\rho}_X(t) = \sup_{x\in D} \inf_{n\geq 1} \sup_{\substack{z\in H^n\\ \|z\|=t}} \|x+z\| - 1$$

and

$$\overline{\delta}_X(t) = \inf_{x\in D} \sup_{n\geq 1} \inf_{\substack{z\in H^n\\ \|z\|=t}} \|x+z\| - 1.$$

Proof. Fix $t > 0$. Our starting point is Lemma 2.1 in [3] which states that

$$\overline{\rho}_X(t) = \sup_{x\in D} \inf_{\substack{Z\subset X\\ \text{co-dim } Z<\infty}} \sup_{\substack{z\in Z\\ \|z\|=t}} \|x+z\| - 1 \tag{2}$$

and

$$\overline{\delta}_X(t) = \inf_{x\in D} \sup_{\substack{Z\subset X\\ \text{co-dim } Z<\infty}} \inf_{\substack{z\in Z\\ \|z\|=t}} \|x+z\| - 1. \tag{3}$$

Let us consider

$$\widetilde{\rho}_X(t) = \sup_{x\in D} \inf_{n\geq 1} \sup_{\substack{z\in H^n\\ \|z\|=t}} \|x+z\|-1$$

and

$$\widetilde{\delta}_X(t) = \inf_{x\in D} \sup_{n\geq 1} \inf_{\substack{z\in H^n\\ \|z\|=t}} \|x+z\|-1.$$

For all $n\geq 1$, H^n is a finite co-dimensional subspace of X, so by (2) we have $\overline{\rho}_X(t)\leq \widetilde{\rho}_X(t)$ and by (3) we have $\overline{\delta}_X(t)\geq \widetilde{\delta}_X(t)$. To show reverse inequalities, we fix $\varepsilon>0$, $x\in D$ and Z a finite co-dimensional subspace of X. There exist $z_1,\dots,z_m$ in S_X and $z_1^*,\dots,z_m^*$ in X^* such that $Z=\cap_{i=1}^m \ker z_i^*$, $z_i^*(z_i)=1$ and $z_i^*(z_k)=0$ for all $i=1,\dots,m$ and $k\neq i$. By the density assumption, there exist $\widetilde{z_1^*},\dots,\widetilde{z_m^*}$ in X^*, each with finite support, such that for all $i=1,\dots,m$ and for every $x\in X$, $\|z_i^*(x)-\widetilde{z_i^*}(x)\|\leq \frac{\varepsilon}{m}\|x\|$. Consider $n\geq 1$ such that $H^n\subseteq \cap_{i=1}^m \ker \widetilde{z_i^*}$. For every $z\in H^n$ with $\|z\|=t$ we can write

$$z = z-\sum_{i=1}^m z_i^*(z)z_i+\sum_{i=1}^m z_i^*(z)z_i,$$

with $\sum_{i=1}^m z_i^*(z)z_i\in X$ such that

$$\left\|\sum_{i=1}^m z_i^*(z)z_i\right\|\leq \varepsilon t$$

and $z-\sum_{i=1}^m z_i^*(z)z_i\in Z$ such that

$$(1-\varepsilon)t\leq \left\|z-\sum_{i=1}^m z_i^*(z)z_i\right\|\leq (1+\varepsilon)t.$$

To summarize, for every $x\in D$, for every finite co-dimensional subspace $Z\subset X$ and for every $\varepsilon>0$, there exists $n\geq 1$ such that

$$\sup_{\substack{z\in H^n\\ \|z\|=t}} \|x+z\|-1\leq \sup_{\substack{z\in Z\\ \|z\|\leq(1+\varepsilon)t}} \|x+z\|-1+\varepsilon t. \tag{4}$$

Taking successively the infinimum over n, the infinimum over Z, the supremum over x and letting ε tend to 0 we obtain $\widetilde{\rho}_X(t)\leq \overline{\rho}_X(t)$. Concerning the modulus of asymptotic uniform convexity, (4) is replaced by

$$\inf_{\substack{z\in H^n\\ \|z\|=t}} \|x+z\|-1\geq \inf_{\substack{z\in Z\\ \|z\|\geq(1-\varepsilon)t}} \|x+z\|-1-\varepsilon t.$$

We conclude taking successively the supremum over n, the supremum over Z, the infinimum over x and letting ε tend to 0. □

3. The setting of Orlicz sequence spaces

We refer to [8] for all the background about Orlicz sequence spaces. Let us recall that a continuous function $M : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* whenever M is convex and satisfies $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. Moreover we suppose that $M(u) = 0$ if and only if $u = 0$. We denote by p the right-derivative of an Orlicz function M so that, for every $u > 0$, $M(u) = \int_0^u p(t)dt$. The *Orlicz sequence space* ℓ_M is defined as the space of all real sequences $x = (x_i)$ such that there exists $\lambda > 0$ satisfying $\sum_{i=1}^{\infty} M(|x_i|/\lambda) < \infty$. Equipped with the Luxemburg norm

$$\|x\| = \inf\left\{\lambda > 0 \,:\, \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\lambda}\right) \leq 1\right\},$$

ℓ_M is a Banach space. We denote by h_M the closed subspace of ℓ_M consisting of all real sequences $x = (x_i)$ such that for every $\lambda > 0$, $\sum_{i=1}^{\infty} M(|x_i|/\lambda) < \infty$.

Recall that an Orlicz function M is said to satisfy the Δ_2 condition at zero, we write $M \in \Delta_2$, if there exist $K > 0$ and $t_0 > 0$ such that for every $t \in [0, t_0]$, $M(2t) \leq KM(t)$. If $M \in \Delta_2$, then $\ell_M = h_M$. The complementary function of M is the Orlicz function defined for $t \in [0,\infty)$ by $M^*(t) = \sup\{tu - M(u) : u \in [0,\infty)\}$. The condition $M^* \in \Delta_2$ is denoted by $M \in \nabla_2$.

The set D defined above is dense in the unit sphere of h_M. If $M \in \nabla_2$, then elements with finite support are dense in the dual space of h_M. So, if $M \in \Delta_2 \cap \nabla_2$, then Lemma 2.3 may be used in $h_M = \ell_M$. We begin with the following lemma. These inequalities can be found, for example, in [4]. We give a proof for the sake of completeness.

Lemma 3.1. *Let M be an Orlicz function.*

(i) *If $M \in \Delta_2$, then there exists $0 < \beta < +\infty$ such that for every $\lambda \in [0, 1]$ and for every $u \in \left[0, M^{-1}(1)\right]$, $M(\lambda u) \geq \lambda^\beta M(u)$.*

(ii) *If $M \in \nabla_2$, then there exists $1 < \alpha < +\infty$ such that for every $\lambda \in [0, 1]$ and for every $u \in \left[0, M^{-1}(1)\right]$, $M(\lambda u) \leq \lambda^\alpha M(u)$.*

Proof. Observe that by the case of equality in Young's inequality, we have for every $t > 0$

$$\frac{tp(t)}{M(t)} = 1 + \underbrace{\frac{M^*(p(t))}{M(t)}}_{f(t)}.$$

First, assume $M \in \Delta_2$. Following [8, page 140], we obtain that there exist $K > 0$ and $t_0 > 0$ such that for every $t \in (0, t_0]$, $\dfrac{tp(t)}{M(t)} \leq K$. If $t_0 \geq M^{-1}(1)$, take $\beta = K$.

Otherwise, if $t_0 < M^{-1}(1)$, by the boundedness of f on $\big[t_0, M^{-1}(1)\big]$, we obtain the existence of $0 < \beta < +\infty$ such that for every $t \in \big(0, M^{-1}(1)\big]$,

$$\frac{p(t)}{M(t)} \leq \frac{\beta}{t}. \tag{5}$$

Fix $\lambda \in (0,1]$ and $u \in \big(0, M^{-1}(1)\big]$ (when $\lambda = 0$ or $u = 0$ the desired inequalities are obvious). We conclude by integrating (5) between λu and u.

Second, assume $M \in \nabla_2$. This implies that there exist $\varepsilon > 0$ and $t_0 > 0$ such that for every $t \in (0, t_0]$,

$$1 + \varepsilon \leq \frac{tp(t)}{M(t)}. \tag{6}$$

If $t_0 \geq M^{-1}(1)$, take $\alpha = 1 + \varepsilon$. Otherwise, if $t_0 < M^{-1}(1)$, by the fact that $f(t) \geq M^*(p(t_0)) > 0$ for every $t \in \big[t_0, M^{-1}(1)\big]$, we can suppose that (6) holds for every $t \in \big(0, M^{-1}(1)\big]$,with $1 + \varepsilon$ replaced by some $1 < \alpha < +\infty$. We conclude by integrating as above. □

Proposition 3.2. *Let M be an Orlicz function. The following are equivalent*

(i) $M \in \Delta_2$.

(ii) *h_M is asymptotically uniformly convex.*

Proof. We write $X = h_M$ for readability. First, assume $M \in \Delta_2$. Consider $0 < \beta < +\infty$ given by Lemma 3.1. Fix $x = (x_i) \in D$. There exists $n \in \mathbb{N}$ such that $x \in H_n \cap S_X$. Fix $h = (h_i)$ in $H^n \cap S_X$, $t \in [0,1]$ and $\lambda > 0$. By the disjointness of supports of x and h we have

$$\sum_{i=1}^{\infty} M\left(\frac{|x_i + th_i|}{1+\lambda}\right) = \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{1+\lambda}\right) + \sum_{i=1}^{\infty} M\left(\frac{t|h_i|}{1+\lambda}\right).$$

Note that for every $i \in \mathbb{N}$, $|x_i|, |h_i| \in \big(0, M^{-1}(1)\big]$.

$$\sum_{i=1}^{\infty} M\left(\frac{|x_i + th_i|}{1+\lambda}\right) \geq \frac{1}{(1+\lambda)^{\beta}} \sum_{i=1}^{\infty} M(|x_i|) + \frac{t^{\beta}}{(1+\lambda)^{\beta}} \sum_{i=1}^{\infty} M(|h_i|).$$

As $M \in \Delta_2$, for every $z = (z_i) \in S_X$, we have $\sum_{i=1}^{\infty} M(|z_i|) = 1$. This gives

$$\sum_{i=1}^{\infty} M\left(\frac{|x_i + th_i|}{1+\lambda}\right) \geq \frac{1 + t^{\beta}}{(1+\lambda)^{\beta}}.$$

Take $\lambda = (1 + t^{\beta})^{1/\beta} - 1$ to obtain that for every $x \in D$, there exists a finite co-dimensional subspace of h_M such that for every z in the unit sphere of this subspace and for every $t \in [0,1]$ we have

$$\|x + th\| - 1 \geq (1 + t^{\beta})^{1/\beta} - 1.$$

By the equality (3), in the proof of Lemma 2.3, we have $\overline{\delta}_X(t) \geq (1+t^\beta)^{1/\beta} - 1$, and the asymptotic uniform convexity of h_M follows.

Second, assume $M \notin \Delta_2$. Then, according to [8, Theorem 4.a.9], h_M contains an isomorphic copy of c_0. Using Remark 2.2 we conclude that h_M is not asymptotically uniformly convex. □

Proposition 3.3. *Let M be an Orlicz function. The following are equivalent*

(i) $M \in \nabla_2$,

(ii) h_M *is asymptotically uniformly smooth.*

Proof. We write again $X = h_M$. First, suppose that $M \in \nabla_2$. Consider $1 < \alpha < +\infty$ given by Lemma 3.1. Proceed exactly as above, with the fact that for every $z = (z_i) \in S_X$, we have $\sum_{i=1}^\infty M(|z_i|) \leq 1$, to obtain that for every $x \in D$, there exists a finite co-dimensional subspace of h_M such that for every z in the unit sphere of this subspace and for every $t \in [0,1]$ we have

$$\|x + th\| - 1 \leq (1+t^\alpha)^{1/\alpha} - 1.$$

By (2), in the proof of Lemma 2.3, we have for every $t \in [0,1]$, $\overline{\rho}_X(t) \leq (1+t^\alpha)^{1/\alpha} - 1$. As $\alpha > 1$, we have $\lim_{t\to 0} \overline{\rho}_X(t)/t = 0$. Thus h_M is asymptotically uniformly smooth.

Second, assume $M \notin \nabla_2$. Then h_M contains an isomorphic copy of ℓ_1. Using Remark 2.2 we conclude that h_M is not asymptotically uniformly smooth. □

Remark 3.4. Following [4, page 81] or [11, page 239], we consider two indices of Simonenko type:

$$\widetilde{A}_M = \inf\left\{\frac{tp(t)}{M(t)} : 0 < t < M^{-1}(1)\right\}$$

and

$$\widetilde{B}_M = \sup\left\{\frac{tp(t)}{M(t)} : 0 < t < M^{-1}(1)\right\}.$$

Quantitatively, we have proved that

(i) If $1 < \widetilde{A}_M < +\infty$, then $X = h_M$ is asymptotically uniformly smooth for every $t \in [0,1]$
$$\overline{\rho}_X(t) \leq (1+t^{\widetilde{A}_M})^{1/\widetilde{A}_M} - 1.$$

(ii) If $0 < \widetilde{B}_M < +\infty$, then $X = h_M$ (which equals ℓ_M here) is asymptotically uniformly convex for every $t \in [0,1]$
$$\overline{\delta}_X(t) \geq (1+t^{\widetilde{B}_M})^{1/\widetilde{B}_M} - 1.$$

4. Separated sequences in the unit sphere of Orlicz sequence spaces

The modulus of asymptotic uniform convexity of an infinite-dimensional Banach space X is connected with $K(X)$ by

$$1+\overline{\delta}_X(1)\le K(X). \tag{7}$$

The complete proof is given in [1]. We sketch it for the sake of completeness. Fix $\alpha<1+\overline{\delta}_X(1)$ and $x_1\in S_X$. By definition, there exists a finite co-dimensional subspace $Z_1\subset X$ such that for every $-z\in S_{Z_1}$, $\|x_1-z\|\ge\alpha$. Starting from x_1, a sequence (x_n) such that, for every $m\neq p$, $\|x_m-x_p\|\ge\alpha$, is constructed by induction in S_X. For every $n\in\mathbb{N}$, x_{n+1} is taken in the unit sphere of Z_n, with (Z_n) a non-increasing sequence of finite co-dimensional subspaces of X, starting from Z_1, obtained by induction, using the fact that for every $Z\subset X$, $1+\overline{\delta}_Z(1)\ge 1+\overline{\delta}_X(1)>\alpha$ (see Proposition 2.3.(2) in [7]). As this construction is done for every $\alpha<1+\overline{\delta}_X(1)$, the inequality (7) is proved. More can be said in the setting of Orlicz sequence spaces.

Theorem 4.1. *Let $M\in\Delta_2$ be an Orlicz function and $X=\ell_M$ the associated Orlicz sequence space. Then*

$$1+\overline{\delta}_X(1)\le K(X)\le 1+\overline{\rho}_X(1).$$

To prove this theorem we need the following technical lemma.

Lemma 4.2. *Let X be a Banach space and $x,y\in X$ be such that $0<\|x\|\le 1$, $0<\|y\|\le 1$ and $\|x-y\|\ge 1$. Then*

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\ge\|x-y\|.$$

Proof. We can suppose that $\dfrac{\|x\|}{\|y\|}\ge 1$. Let us consider $\phi: t\mapsto\|x-ty\|-\|x\|$, which is a convex function of $t\in[0,\infty)$ such that $\phi(0)=0$. We have

$$\begin{aligned}
\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|-\|x-y\| &= \frac{1}{\|x\|}\left(\phi\left(\frac{\|x\|}{\|y\|}\right)+\|x\|\right)-\|x-y\|\\
&\ge \frac{1}{\|y\|}\phi(1)+1-\|x-y\| \quad \text{by convexity of } \phi,\\
&= \left(\frac{1}{\|y\|}-1\right)\|x-y\|+1-\frac{\|x\|}{\|y\|}\\
&\ge \frac{1}{\|y\|}-\frac{\|x\|}{\|y\|} \quad (\text{because } \|y\|\le 1 \text{ and } \|x-y\|\ge 1)\\
&\ge 0 \quad (\text{because } \|x\|\le 1).
\end{aligned}$$

□

Proof of Theorem 4.1. The left-hand side inequality has been discussed before (see [1]).

First, suppose that $M \notin \nabla_2$. As explained above, this implies that $\ell_M = h_M$ contains almost isometric copies of ℓ_1. According to Lemma 2.1, we have $\overline{\rho}_{\ell_1}(1) = 1 \leq \overline{\rho}_{\ell_M}(1)$. So, in this case, the right-hand side inequality is trivial because $K(X) \leq 2$.

Second, suppose that $M \in \nabla_2$. This allows us to use Lemma 2.3 in the sequel. We proceed by contradiction to prove the right-hand side inequality. Suppose that $K(X) > 1 + \overline{\rho}_X(1)$. There exists β such that $1 + \overline{\rho}_X(1) < \beta < K(X)$. According to [14] (see also [12] or [5] or [6]) if $M \in \Delta_2$,

$$K(X) = \sup_{(x_i)=x\in S_X} \left\{ \lambda_x > 0 \,:\, \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\lambda_x}\right) = \frac{1}{2} \right\}. \tag{8}$$

So there exists $(x_i) = x \in S_X$ such that

$$\sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\beta}\right) > \frac{1}{2}. \tag{9}$$

Claim 4.3. *We can suppose that there exists $p \geq 1$ such that $x \in H_p$ and $\|x\| = 1$.*

Indeed, fix $\varepsilon > 0$ and $p \geq 1$ such that

$$\frac{1}{2} + \varepsilon < \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\beta}\right) \quad \text{and} \quad \sum_{i=p+1}^{\infty} M\left(\frac{|x_i|}{\beta}\right) < \varepsilon.$$

Consider $\tilde{x} \in H_p$ given by $\tilde{x} = (x_1, \ldots, x_p, 0, 0, \ldots)$. Then $\|\tilde{x}\| \leq \|x\| = 1$ and, as M is non-decreasing, x above can be replaced by $\dfrac{\tilde{x}}{\|\tilde{x}\|} \in S_{H_p}$. The claim is proved.

Now, as $1 + \overline{\rho}_X(1) < \beta$, according to Lemma 2.3, there exists $n \geq 1$ such that for every $z \in S_{H^n}$, $\|x - z\| < \beta$. Let us consider $Z = H^n \cap H^p$. For every $z \in S_Z$ we have

$$\sum_{i=1}^{\infty} M\left(\frac{|z_i|}{\beta}\right) + \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\beta}\right) = \sum_{i=1}^{\infty} M\left(\frac{|z_i - x_i|}{\beta}\right) \leq 1.$$

Then, using inequality (9), we obtain that for every $z \in S_Z$

$$\sum_{i=1}^{\infty} M\left(\frac{|z_i|}{\beta}\right) \leq \frac{1}{2}.$$

The characterization (8) for Z, seen itself as an Orlicz sequence space, implies

$$K(Z) \leq \beta < K(X).$$

As Z is a one complemented finite co-dimensional subspace of X, this contradicts the following

Claim 4.4. *Let X be a Banach space and Z be a one complemented finite co-dimensional subspace of X. Then $K(Z) = K(X)$.*

Indeed, denote by $\Pi : X \to Z$ the canonical projection. Consider $1 < \alpha < K(X)$. There exists $\varepsilon > 0$ and a sequence $(x_n) \in S_X$ such that for every $m \neq p$

$$\begin{aligned} 1 < \varepsilon + \alpha &\leq \|x_m - x_p\| \\ &\leq \|(x_m - \Pi(x_m)) - (x_p - \Pi(x_p))\| + \|\Pi(x_m) - \Pi(x_p)\|. \end{aligned}$$

The projection Π is a continuous linear operator with a norm less than 2 because Z is one complemented in X. Thus the sequence $(x_n - \Pi(x_n))$ takes values in a compact subset of X. After relabeling, as Z is one complemented in X, we can suppose that for every $n \geq 1$, $\Pi(x_n) > 0$, $\Pi(x_n) \leq 1$ and for every $m \neq p$, $\|(x_m - \Pi(x_m)) - (x_p - \Pi(x_p))\| \leq \varepsilon$. This implies that for every $m \neq p$,

$$1 < \alpha \leq \|\Pi(x_m) - \Pi(x_p)\|.$$

Let us consider the sequence $\left(\dfrac{\Pi(x_n)}{\|\Pi(x_n)\|}\right) \subset S_Z$. Using Lemma 4.2, for every $m \neq p$ we have

$$\left\| \frac{\Pi(x_m)}{\|\Pi(x_m)\|} - \frac{\Pi(x_p)}{\|\Pi(x_p)\|} \right\| \geq \|\Pi(x_m) - \Pi(x_p)\| \geq \alpha.$$

This implies that $\alpha \leq K(Z)$ for every $\alpha < K(X)$ and so $K(X) \leq K(Z)$. As the reverse inequality is clear, the claim is proved and Theorem 4.1 too. □

5. Comments

Recall that for every $1 \leq p < +\infty$ and for every $t > 0$, $\overline{\delta}_{\ell_p}(t) = \overline{\rho}_{\ell_p}(t) = (1+t^p)^{1/p} - 1$. If $X = \ell_p$, then inequalities in Theorem 4.1 are accurate:

$$K(X) = 1 + \overline{\delta}_X(1) = 1 + \overline{\rho}_X(1) = 2^{1/p}.$$

It is not often true as explained below.

By Remark 3.4 and Theorem 4.1, we have proved that if $X = \ell_M$ is an Orlicz sequence space built on an Orlicz function $M \in \Delta_2 \cap \nabla_2$, then

$$2^{1/\widetilde{B}_M} \leq 1 + \overline{\delta}_X(1) \leq K(X) \leq 1 + \overline{\rho}_X(1) \leq 2^{1/\widetilde{A}_M}. \tag{10}$$

Left and right inequalities in (10), involving $\widetilde{B}_M$ and $\widetilde{A}_M$, are in connection with the better estimates of $K(X)$ given in [11, Theorem 2.3; 13, Theorem 2.5]. Moreover, it must be said that there is often a gap between $1 + \overline{\delta}_X(1)$ and $1 + \overline{\rho}_X(1)$. Indeed, let us consider an Orlicz function $M \in \Delta_2 \cap \nabla_2$. Following [8, page 143] we set

$$\alpha_M = \sup \left\{ p : \sup_{0<\lambda, t \leq 1} \frac{M(\lambda t)}{t^p M(t)} < \infty \right\}$$

and

$$\beta_M \quad = \quad \inf\left\{q \,:\, \inf_{0<\lambda,t\leq 1}\frac{M(\lambda t)}{t^q M(t)} > 0\right\}.$$

According to Theorem 4.a.9 in [8] and comments after its proof (page 144), $X = \ell_M$ (which coincide with h_M here because $M \in \Delta_2$) contains almost isometric copies of ℓ_p for every $p \in [\alpha_M, \beta_M]$. Thus, according to Lemma 2.1,

$$1 + \overline{\delta}_X(1) \leq 2^{1/\beta_M} \leq 2^{1/\alpha_M} \leq 1 + \overline{\rho}_X(1).$$

Moreover, using Proposition 1 in [5] and the fact that $\Lambda(X) = \dfrac{K(X)}{2+K(X)}$, we obtain $K(\ell_{\alpha_M}) = 2^{1/\alpha_M} \leq K(X)$. Thus if $\alpha_M \neq \beta_M$, we have

$$1 + \overline{\delta}_X(1) \leq 2^{1/\beta_M} < 2^{1/\alpha_M} \leq K(X) \leq 1 + \overline{\rho}_X(1),$$

so that the left-hand side of (10) is not accurate.

The equality (8) plays an essential role in the proof of Theorem 4.1. Hudzik, Wu and Ye, in [6], give an analogue of (8) in the setting of Musielak-Orlicz sequence spaces. We refer to [10], [8], [6] and [3] for some background about modular sequence spaces. Let us recall here that a sequence of Orlicz functions (M_i) is called *a Musielak-Orlicz function*. The *Musielak-Orlicz sequence space* $\ell_{(M_i)}$ is defined as the modular sequence space of all real sequences $x = (x_i)$ such that there exists $\lambda > 0$ satisfying $\sum_{i=1}^{\infty} M_i\left(\frac{|x_i|}{\lambda}\right) < \infty$. Equipped with the Luxemburg norm

$$\|x\| = \inf\left\{\lambda > 0 \,:\, \sum_{i=1}^{\infty} M_i\left(\frac{|x_i|}{\lambda}\right) \leq 1\right\},$$

$\ell_{(M_i)}$ is a Banach space. The modulus of asymptotic uniform smoothness of the corresponding space $h_{(M_i)}$ is studied in [3]. Let $X = \ell_{(M_i)}$ be a Musielak-Orlicz sequence space built on (M_i) satisfying the assumptions of Theorem 1 in [6], and satisfying the assumptions of our Lemma 2.3, then we have

$$1 + \overline{\delta}_X(1) \leq K(X) \leq 1 + \overline{\rho}_X(1).$$

Indeed, as explained before, the left-hand side inequality is valid for all infinite-dimensional Banach spaces. To prove the right-hand side inequality, our starting point is Theorem 1 in [6] which gives the analogue of (8). Namely

$$K(X) = \inf_{n\in\mathbb{N}} \sup_{(x_i)=x\in S_X} \sup_{m\in\mathbb{N}} c(x,m,n), \tag{11}$$

where, for every $(x_i) = x \in S_X$ and $m, n \in \mathbb{N}$

$$c(x,m,n) = \inf\left\{c > 0 \,:\, \sum_{i=n}^{n+m} M_i\left(\frac{x_i}{c}\right) \leq \frac{1}{2}\right\}.$$

We proceed as in the proof of Theorem 4.1. Toward a contradiction, assume that there exists $0 < \beta < +\infty$ such that $1 + \overline{\rho}_X(1) < \beta < K(X)$. With (11) this gives in particular

$$\beta < \sup_{(x_i)=x\in S_X} \sup_{m\in\mathbb{N}} c(x,m,1).$$

So there exists $(x_i) = x \in S_X$ and there exists $m \in \mathbb{N}$ such that

$$\sum_{i=1}^{\infty} M_i\left(\frac{x_i}{\beta}\right) \geq \sum_{i=1}^{m+1} M_i\left(\frac{x_i}{\beta}\right) > \frac{1}{2}.$$

Claim 4.3 remains true. We use the inequality $1+\overline{\rho}_X(1) < \beta$ as before to obtain a one complemented finite co-dimensional subspace Z of X such that for every $(z_i) = z \in S_Z$ and for every $m \in \mathbb{N}$,

$$\sum_{i=1}^{m+1} M_i\left(\frac{z_i}{\beta}\right) \leq \sum_{i=1}^{\infty} M_i\left(\frac{z_i}{\beta}\right) \leq \frac{1}{2}.$$

So for every $z \in S_Z$ and for every $m \in \mathbb{N}$, $c(z,m,1) \leq \beta$. The equality (11) applied with Z gives $K(Z) \leq \beta < K(X)$, which contradicts Claim 4.4.

According to [2], $1 < K(X)$ for every infinite-dimensional normed space X. So, the inequality $K(X) \leq 1 + \overline{\rho}_X(1)$ is false for infinite-dimensional Banach spaces X such that $\overline{\rho}_X(1) = 0$, like for example $X = c_0$. We conclude with the following question: is the inequality $K(X) \leq 1+\overline{\rho}_X(1)$ true for every infinite-dimensional Banach spaces X such that $0 < \overline{\rho}_X(1)$?

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