The SL(2, C)-Character Varieties of Torus Knots

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ABSTRACT

Let G be the fundamental group of the complement of the torus knot of type (m, n). This has a presentation G = <x, y|xm = yn>. We find the geometric description of the character variety X(G) of characters of representations of G into SL(2, C).

Key words: Torus knot, characters, representations.


Introduction

Since the foundational work of Culler and Shalen [1], the varieties of SL(2, C)-characters have been extensively studied. Given a manifold M, the variety of representations of π1(M) into SL(2, C) and the variety of characters of such representations both contain information of the topology of M. This is specially interesting for 3-dimensional manifolds, where the fundamental group and the geometrical properties of the manifold are strongly related.

This can be used to study knots K ⊂ S³, by analysing the SL(2, C)-character variety of the fundamental group of the knot complement S³ − K. In this paper, we study the case of the torus knots Km,n of any type (m, n). The case (m, n) = (m, 2) was analysed in [3] and the general case was recently determined in [2] by a method different from ours.

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1. Character varieties

A representation of a group $G$ in $\text{SL}(2, \mathbb{C})$ is a homomorphism $\rho : G \to \text{SL}(2, \mathbb{C})$. Consider a finitely presented group $G = \langle x_1, \ldots, x_k | r_1, \ldots, r_s \rangle$, and let $\rho : G \to \text{SL}(2, \mathbb{C})$ be a representation. Then $\rho$ is completely determined by the $k$-tuple $(A_1, \ldots, A_k) = (\rho(x_1), \ldots, \rho(x_k))$ subject to the relations $r_j(A_1, \ldots, A_k) = 0$, $1 \leq j \leq s$. Using the natural embedding $\text{SL}(2, \mathbb{C}) \subset \mathbb{C}^4$, we can identify the space of representations as

$$R(G) = \text{Hom}(G, \text{SL}(2, \mathbb{C})) = \{(A_1, \ldots, A_k) \in \mathbb{C}^4 \mid r_j(A_1, \ldots, A_k) = 0, 1 \leq j \leq s \} \subset \mathbb{C}^{4k}.$$ 

Therefore $R(G)$ is an affine algebraic set.

We say that two representations $\rho$ and $\rho'$ are equivalent if there exists $P \in \text{SL}(2, \mathbb{C})$ such that $\rho'(g) = P^{-1} \rho(g) P$, for every $g \in G$. This produces an action of $\text{SL}(2, \mathbb{C})$ in $R(G)$. The moduli space of representations is the GIT quotient

$$M(G) = \text{Hom}(G, \text{SL}(2, \mathbb{C})) / \text{SL}(2, \mathbb{C}).$$

A representation $\rho$ is reducible if the elements of $\rho(G)$ all share a common eigenvector, otherwise $\rho$ is irreducible.

Given a representation $\rho : G \to \text{SL}(2, \mathbb{C})$, we define its character as the map $\chi_\rho : G \to \mathbb{C}$, $\chi_\rho(g) = \text{tr} \rho(g)$. Note that two equivalent representations $\rho$ and $\rho'$ have the same character, and the converse is also true if $\rho$ or $\rho'$ is irreducible [1, Proposition 1.5.2].

There is a character map $\chi : R(G) \to \mathbb{C}^G$, $\rho \mapsto \chi_\rho$, whose image

$$X(G) = \chi(R(G))$$

is called the character variety of $G$. Let us give $X(G)$ the structure of an algebraic variety. By the results of [1], there exists a collection $g_1, \ldots, g_a$ of elements of $G$ such that $\chi_\rho$ is determined by $\chi_\rho(g_1), \ldots, \chi_\rho(g_a)$, for any $\rho$. Such collection gives a map

$$\Psi : R(G) \to \mathbb{C}^a, \quad \Psi(\rho) = (\chi_\rho(g_1), \ldots, \chi_\rho(g_a)).$$

We have a bijection $X(G) \cong \Psi(R(G))$. This endows $X(G)$ with the structure of an algebraic variety. Moreover, this is independent of the chosen collection as proved in [1].

**Lemma 1.1.** The natural algebraic map $M(G) \to X(G)$ is a bijection.

**Proof.** The map $R(G) \to X(G)$ is algebraic and $\text{SL}(2, \mathbb{C})$-invariant, hence it descends to an algebraic map $\varphi : M(G) \to X(G)$. Let us see that $\varphi$ is a bijection.

For $\rho$ an irreducible representation, if $\varphi(\rho) = \varphi(\rho')$ then $\rho$ and $\rho'$ are equivalent representations; so they represent the same point in $M(G)$.

Now suppose that $\rho$ is reducible. Consider $e_1 \in \mathbb{C}^2$ the common eigenvector of all $\rho(g)$. This gives a sub-representation $\rho' : G \to \mathbb{C}^*$ of $G$. We have a quotient
representation \( \rho'' = \rho/\rho' : G \rightarrow \mathbb{C}^* \), defined as the representation induced by \( \rho \) in the quotient space \( \mathbb{C}^2/\langle e_1 \rangle \). As characters, \( \rho'' = \rho^{-1} \). The representation \( \rho' \oplus \rho'' \) is the semisimplification of \( \rho \). It is in the closure of the \( \text{SL}(2, \mathbb{C}) \)-orbit through \( \rho \). Clearly, \( \chi_\rho(g) = \rho'(g) + \rho'(g)^{-1} \). Now if \( \rho \) and \( \tilde{\rho} \) are two reducible representations and \( \varphi(\rho) = \varphi(\tilde{\rho}) \), then their semisimplifications have the same character, that is
\[
\chi_\rho(g) = \chi_{\tilde{\rho}}(g) \Rightarrow \rho'(g) + \rho'(g)^{-1} = \tilde{\rho}'(g) + \tilde{\rho}'(g)^{-1}.
\]
Therefore \( \rho' = \tilde{\rho}' \) or \( \rho' = \tilde{\rho}'^{-1} \). In either case \( \rho \) and \( \tilde{\rho} \) represent the same point in \( M(G) \), which is actually the point represented by \( \rho' \oplus \rho'^{-1} \).

2. Character varieties of torus knots

Let \( T^2 = S^1 \times S^1 \) be the 2-torus and consider the standard embedding \( T^2 \subset S^3 \). Let \( m, n \) be a pair of coprime positive integers. Identifying \( T^2 \) with the quotient \( \mathbb{R}^2/\mathbb{Z}^2 \), the image of the straight line \( y = \frac{m}{n}x \) in \( T^2 \) defines the torus knot of type \((m,n)\), which we shall denote as \( K_{m,n} \subset S^3 \) (see [4, Chapter 3]).

For any knot \( K \subset S^3 \), we denote by \( G(K) \) the fundamental group of the exterior \( S^3 - K \) of the knot. It is known that

\[
G_{m,n} = G(K_{m,n}) \cong \langle x, y \mid x^m = y^n \rangle.
\]

The purpose of this paper is to describe the character variety \( X(G_{m,n}) \).

In [3], the character variety \( X(G_{m,2}) \) is computed. We want to extend the result to arbitrary \( m, n \), and give a simpler argument than that of [3].

After the completion of this work, we became aware of the paper [2] where the character varieties of \( X(G_{m,n}) \) are determined (even without the assumption of \( m, n \) being coprime). However, our method is more direct than the one presented in [2].

To start with, note that

\[
R(G_{m,n}) = \{(A, B) \in \text{SL}(2, \mathbb{C}) \mid A^m = B^n \}.
\]

Therefore we shall identify a representation \( \rho \) with a pair of matrices \( (A, B) \) satisfying the required relation \( A^m = B^n \).

We decompose the character variety

\[
X(G_{m,n}) = X_{\text{red}} \cup X_{\text{irr}},
\]

where \( X_{\text{red}} \) is the subset consisting of the characters of reducible representations (which is a closed subset by [1]), and \( X_{\text{irr}} \) is the closure of the subset consisting of the characters of irreducible representations.

**Proposition 2.1.** There is an isomorphism \( X_{\text{red}} \cong \mathbb{C} \). The correspondence is defined by

\[
\rho = \left( A = \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}, B = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix} \right) \mapsto s = t + t^{-1} \in \mathbb{C}.
\]
Proof. By the discussion in Lemma 1.1, an element in $X_{\text{red}}$ is described as the character of a split representations $\rho = \rho' \oplus \rho'^{-1}$. This means that in a suitable basis, $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$.

The equality $A^m = B^n$ implies $\lambda^m = \mu^n$. Therefore there is a unique $t \in \mathbb{C}$ with $t \neq 0$ such that

\[
\begin{align*}
\lambda &= t^n, \\
\mu &= t^m.
\end{align*}
\]

(Here we use the coprimality of $(m, n)$). Note that the pair $(A, B)$ is well-defined up to permuting the two vectors in the basis. This corresponds to the change $(\lambda, \mu) \mapsto (\lambda^{-1}, \mu^{-1})$, which in turn corresponds to $t \mapsto t^{-1}$. So $(A, B)$ is parametrized by $s = t + t^{-1} \in \mathbb{C}$. 

**Lemma 2.2.** Suppose that $\rho = (A, B) \in R(G_{m,n})$. In any of the following cases:

(a) $A^m = B^n \neq \pm \text{Id}$,

(b) $A = \pm \text{Id}$ or $B = \pm \text{Id}$,

(c) $A$ or $B$ is non-diagonalizable,

the representation $\rho$ is reducible.

Proof. First suppose that $A$ is diagonalizable with eigenvalues $\lambda, \lambda^{-1}$, and suppose that $\lambda^m \neq \pm 1$. Then there is a basis $e_1, e_2$ in which $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, which is well-determined up to multiplication of the basis vectors by non-zero scalars. Then

$B^n = A^m = \begin{pmatrix} \lambda^m & 0 \\ 0 & \lambda^{-m} \end{pmatrix}$

is a diagonal matrix, different from $\pm \text{Id}$. Therefore $B$ must be diagonal in the same basis, $B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, with $\lambda^m = \mu^n$. This proves the reducibility in case (a).

Now suppose that $A = \lambda \text{Id}$, $\lambda = \pm 1$. Then $B^n = \lambda^m \text{Id}$, so it must be that $B$ is diagonalizable. Using a basis in which $B$ is diagonal, we get the reducibility in case (b).

Finally, suppose that $A$ is not diagonalizable. Then there is a suitable basis on which $A$ takes the form $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, with $\lambda = \pm 1$. Clearly

$B^n = A^m = \lambda^m \begin{pmatrix} 1 & m\lambda \\ 0 & 1 \end{pmatrix}$

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Then and so

\[ B = \begin{pmatrix} \mu & x \\ 0 & \mu \end{pmatrix}, \]

with \( \mu = \pm 1 \), \( \mu^n = \lambda^m \) and \( \mu nx = \lambda m \). In this basis, the vector \( e_1 \) is an eigenvector for both \( A \) and \( B \). Hence the representation \((A, B)\) is reducible, completing the case (c). \( \Box \)

**Proposition 2.3.** Let \( X^\rho_{\text{irr}} \) be the set of irreducible characters, and \( X_{\text{irr}} \) its closure. Then

\[ X^\rho_{\text{irr}} \cong \{ (\lambda, \mu, r) \mid \lambda^m = \mu^n = \pm 1, \lambda \neq \pm 1, \mu \neq \pm 1, r \in \mathbb{C} - \{0, 1\} \}/\mathbb{Z}_2 \times \mathbb{Z}_2, \]

\[ X_{\text{irr}} \cong \{ (\lambda, \mu, r) \mid \lambda^m = \mu^n = \pm 1, \lambda \neq \pm 1, \mu \neq \pm 1, r \in \mathbb{C} \}/\mathbb{Z}_2 \times \mathbb{Z}_2. \]

where \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) acts as \((\lambda, \mu, r) \sim (\lambda^{-1}, \mu, 1 - r) \sim (\lambda, \mu^{-1}, 1 - r) \sim (\lambda^{-1}, \mu^{-1}, r)\).

**Proof.** Let \( \rho = (A, B) \) be an element of \( R(G_{m,n}) \) which is an irreducible representation. By Lemma 2.2, \( A \) is diagonalizable but not equal to \( \pm \text{Id} \), and \( A^n = \pm \text{Id} \). So the eigenvalues \( \lambda, \lambda^{-1} \) of \( A \) satisfy \( \lambda^m = \pm 1 \) and \( \lambda \neq \pm 1 \). Analogously, \( B \) is diagonalizable but not equal to \( \pm \text{Id} \), with eigenvalues \( \mu, \mu^{-1} \), with \( \mu^n = \pm 1, \mu \neq \pm 1 \). Moreover,

\[ \lambda^m = \mu^n. \]

We may choose a basis \( \{e_1, e_2\} \) under which \( A \) diagonalizes. This is well-defined up to multiplication of \( e_1 \) and \( e_2 \) by two non-zero scalars. Let \( \{f_1, f_2\} \) be a basis under which \( B \) diagonalizes, which is well-defined up to multiplication of \( f_1, f_2 \) by non-zero scalars. Then \( \{e_1, e_2, [f_1], [f_2]\} \) are four points of the projective line \( \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2) \). Note that the pair \((A, B)\) is irreducible if and only if the four points are different.

The only invariant of four points in \( \mathbb{P}^1 \) is the double ratio

\[ r = ([e_1] : [e_2] : [f_1] : [f_2]) = ([e_1] : [e_2] : [f_1] : [f_2]) \in \mathbb{P}^1 - \{0, 1, \infty\} = \mathbb{C} - \{0, 1\}. \]

So \((A, B)\) is parametrized, up to the action of \( \text{SL}(2,\mathbb{C}) \), by \((\lambda, \mu, r)\). Permuting the two basis vectors \( e_1, e_2 \) corresponds to \((\lambda, \mu, r) \mapsto (\lambda^{-1}, \mu, 1 - r)\), since

\[ ([e_2] : [e_1] : [f_1] : [f_2]) = 1 - ([e_1] : [e_2] : [f_1] : [f_2]). \]

Analogously, permuting the two basis vectors \( f_1, f_2 \) corresponds to \((\lambda, \mu, r) \mapsto (\lambda, \mu^{-1}, 1 - r)\).

Note that this gives an action of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( X^\rho_{\text{irr}} \) is the quotient of the set of \((\lambda, \mu, r)\) as above by this action.

To describe the closure of \( X^\rho_{\text{irr}} \), we have to allow \( f_1 \) to coincide with \( e_1 \). This corresponds to \( r = 1 \) (the same happens if \( f_2 \) coincides with \( e_2 \)). In this case, \( e_1 \) is
an eigenvector of both $A$ and $B$, so the representation $(A, B)$ has the same character as its semisimplification $(A', B')$ given by
\[
A' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B' = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}.
\]
This means that the point $(\lambda, \mu, 1)$ corresponds under the identification $X_{red} \cong \mathbb{C}$ given by Proposition 2.1 to $s_1 = t_1 + t_1^{-1}$, where $t_1 \in \mathbb{C}$ satisfies
\[
\begin{cases}
\lambda = t_1^n, \\
\mu = t_1^m.
\end{cases}
\]
(1)

Also, we have to allow $f_1$ to coincide with $e_2$ (or $f_2$ to coincide with $e_1$). This corresponds to $r = 0$. The representation $(A, B)$ has semisimplification $(A', B')$ where
\[
A' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B' = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}.
\]
So the point $(\lambda, \mu, 1)$ corresponds to $s_0 = t_0 + t_0^{-1} \in X_{red} \cong \mathbb{C}$, where $t_0 \in \mathbb{C}$ satisfies
\[
\begin{cases}
\lambda = t_0^n, \\
\mu^{-1} = t_0^m.
\end{cases}
\]
(2)

Proposition 2.3 says that $X_{irr}$ is a collection of $\frac{(m-1)(n-1)}{2}$ lines. A pair $(\lambda, \mu)$ with $\lambda^m = \pm 1$ and $\mu^n = \pm 1$ is given as
\[
\lambda = e^{\pi i k/m}, \quad \mu = e^{\pi i k'/n},
\]
where $0 \leq k < 2m$, $0 \leq k' < 2n$. The condition $\lambda \neq \pm 1$, $\mu \neq \pm 1$ gives $k \neq 0, m$, $k' \neq 0, n$. Finally, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action allows us to restrict to $0 < k < m$, $0 < k' < n$. The condition $\lambda^m = \mu^n$ means that
\[
k \equiv k' \pmod{2}.
\]

Denote by $X_{irr}^{k,k'}$ the line of $X_{irr}$ corresponding to the values of $k, k'$. Then
\[
X_{irr} = \bigsqcup_{0<k<m, 0<k'<n \atop k \equiv k' \pmod{2}} X_{irr}^{k,k'}.
\]

The line $X_{irr}^{k,k'}$ intersects $X_{red}$ in two points. This gives a collection of $(m-1)(n-1)$ points in $X_{red}$, which are defined as follows: under the identification $X_{red} \cong \mathbb{C}$, these are the points $s_l = t_l + t_l^{-1}$, where
\[
t_l = e^{\pi il/nm},
\]
and $0 < l < mn$, $m \not| l$, $n \not| l$. Assume that $n$ is odd (note that either $m$ or $n$ should be odd). Then from (1) and (2), the line $X^k_{irr}$ intersects at the points $s_{l_0}, s_{l_1} \in X_{red}$ where

\begin{align*}
nl_0 &\equiv k \pmod{m}, & ml_0 &\equiv n - k' \pmod{n}, \\
nl_1 &\equiv k \pmod{m}, & ml_1 &\equiv k' \pmod{n}.
\end{align*}

These two points are different since $k' \not\equiv n - k' \pmod{n}$, as $n$ is odd.

In the case $(m, n) = (2, n)$, this result coincides with [3, Corollary 4.2].

3. The algebraic structure of $X(G_{m,n})$

We want to give a geometric realization of $X(G_{m,n})$ which shows that the algebraic structure of this variety is that of a collection of rational lines as in Figure 1 intersecting with nodal curve singularities.

The map $R(G_{m,n}) \to \mathbb{C}^3$, $\rho = (A, B) \mapsto (\text{tr}(A), \text{tr}(B), \text{tr}(AB))$, defines a map

$$\Psi : X(G_{m,n}) \to \mathbb{C}^3.$$ 

**Theorem 3.1.** The map $\Psi$ is an isomorphism with its image $C = \Psi(X(G_{m,n}))$. $C$ is a curve consisting of $\frac{(n-1)(m-1)}{2} + 1$ irreducible components, all of them smooth and isomorphic to $\mathbb{C}$. They intersect with nodal normal crossing singularities following the pattern in Figure 1.
Proof. Let us look first at $\Psi_0 = \Psi|_{X_{\text{red}}}: X_{\text{red}} \to \mathbb{C}^3$. For a given $\rho = (A, B) \in X_{\text{red}}$, with the shape given in Proposition 2.1, we have that

$$
\Psi_0: s = t + t^{-1} \mapsto (t^n + t^{-n}, t^m + t^{-m}, t^{n+m} + t^{-(n+m)}).
$$

This map is clearly injective: the image recovers

$$\{t^n, t^{-n}\}, \{t^m, t^{-m}\}, \{t^{n+m}, t^{-(n+m)}\}.$$

From this, we recover $\{(t^n, t^m), (t^{-n}, t^{-m})\}$ and hence the pair $t, t^{-1}$ (since $n, m$ are coprime).

Let us see that $\Psi_0$ is an immersion. The differential is

$$
d\Psi_0\left(\frac{dt}{ds}\right) = (nt^{n-1}(t^2 - 1), mt^{m-1}(t^2 - 1), (n + m)t^{n-m-1}(t^2 - 1)).
$$

This is non-zero at all $t \neq \pm 1$. As $\frac{ds}{dt} \neq 0$, we have $\frac{d\Psi_0}{ds} \neq (0, 0, 0)$. For $t = \pm 1$, we note that $\frac{ds}{dt} = t^{-2}(t^2 - 1)$, so

$$
d\Psi_0\left(\frac{ds}{dt}\right) = \left(\frac{nt^{n+1}t^{2n-1} - 1}{t^2 - 1}, \frac{mt^{m+1}t^{2m-1} - 1}{t^2 - 1}, (n + m)t^{n-m+1}(t^{2n+2m-2} - 1)ight),
$$

which is non-zero again.

Now, consider a component of $X_{\text{irr}}$ corresponding to a pair $(\lambda, \mu)$. Take $r \in \mathbb{C}$. Fix the basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ which is given as the eigenbasis of $A$. Let $\{f_1, f_2\}$ be the eigenbasis of $B$. As the double ratio $(0 : \infty : 1 : r/(r-1)) = r$, we can take $f_1 = (1, 1)$ and $f_2 = (r - 1, r)$. This corresponds to the matrices:

$$
A = \begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
1 & r - 1 \\
1 & r
\end{pmatrix}
\begin{pmatrix}
\mu & 0 \\
0 & \mu^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & r - 1 \\
1 & r
\end{pmatrix}^{-1}
\begin{pmatrix}
r(\mu - \mu^{-1}) + \mu^{-1} \\
r(\mu - \mu^{-1})
\end{pmatrix}
\begin{pmatrix}
1 - r(\mu - \mu^{-1}) \\
\mu - r(\mu - \mu^{-1})
\end{pmatrix}.
$$

Therefore:

$$
\Psi(A, B) = (\text{tr}(A), \text{tr}(B), \text{tr}(AB))
= (\lambda + \lambda^{-1}, \mu + \mu^{-1}, (\lambda\mu^{-1} + \lambda^{-1}\mu) + r(\lambda - \lambda^{-1})(\mu - \mu^{-1})).
$$

The image of this component is a line in $\mathbb{C}^3$. Its direction vector is $(0, 0, 1)$. At an intersection point with $\Psi_0(X_{\text{red}})$, the tangent vector to $\Psi_0(X_{\text{red}})$, given in (3), has non-zero first and second component, since $\lambda = t^n$, $\mu = t^m$ and $t \neq 0, \lambda^2 \neq 1$; $\mu^2 \neq 1$. So the intersection of these components is a transverse nodal singularity.

Finally, note that the map $\Psi: X(G_{m,n}) \to \mathbb{C}$ is an algebraic map, it is a bijection, and $C$ is a nodal curve (the mildest possible type of singularities). Therefore $\Psi$ must be an isomorphism. \qed
Corollary 3.2. $M(G) \cong X(G)$, for $G = G_{m,n}$.

Proof. By Lemma 1.1, $\varphi : M(G) \to X(G)$ is an algebraic map which is a bijection. As the singularities of $X(G)$ are just transverse nodes, $\varphi$ must be an isomorphism. \qed

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References


