Symmetrization and Sharp Sobolev Inequalities in Metric Spaces

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ABSTRACT

We derive sharp Sobolev inequalities for Sobolev spaces on metric spaces. In particular, we obtain new sharp Sobolev embeddings and Faber-Krahn estimates for Hörmander vector fields.

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1. Introduction

Recently, a rich theory of Sobolev spaces on metric spaces has been developed (see [6,8], and the references therein). In particular, this has led to the unification of some aspects of the classical theory of Sobolev spaces with the theory of Sobolev spaces of vector fields satisfying Hörmander's condition. At the root of these developments are suitable forms of Poincaré inequalities which, in fact, can be used to provide a natural method to define the notion of a gradient in the setting of metric spaces. In the theory of Hörmander vector fields, the relevant Poincaré inequalities had been obtained much earlier by Jerison [9]:

$$\left(\frac{1}{|B|}\int_B |f-f_B|^2\,dx\right)^{1/2} \leq Cr(B)\left(\frac{1}{|B|}\int_B |Xf|^2\,dx\right)^{1/2},$$

where $X=(X_1,\dots,X_m)$ is a family of $\mathcal{C}^\infty$ Hörmander vector fields,

$$|Xf| = \left(\sum |X_i f|^2\right)^{1/2},$$

dx is the Lebesgue measure, B is a ball of radius $r(B)$ with respect to the Carnot-Carathéodory metric. For more on the connection between the theory of Sobolev spaces on metric spaces and Sobolev spaces on Carnot groups we refer to the Appendix below and [6].

The purpose of this paper is to prove sharp forms of the classical Sobolev inequalities in the context of metric spaces. In fact, we develop an approach to symmetrization in the metric setting which has applications to other problems as well. In particular, we will show some functional forms of the Faber-Krahn inequalities which are new even in the classical setting.

Let us briefly describe our plan of attack. A well-known, and very natural, approach to the Sobolev inequalities is through the use of the isoperimetric inequality and related rearrangement inequalities (for an account see [24]). For example, a good deal of the classical inequalities can be in fact derived from (see [2, 11], and also [15])

$$\frac{1}{t}\int_0^t [f^*(s) - f^*(t)]ds \le ct^{1/n}\left(\frac{1}{t}\int_0^t \left|\nabla f\right|^*(s)ds\right), \tag{1}$$

where $t > 0$ and $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. For example, in [2] and [19] it is shown how, starting from (1), one can derive Sobolev inequalities which are sharp, including the borderline cases, within the class of Sobolev spaces based on rearrangement invariant spaces. Therefore, it seemed natural to us to try to extend (1) to the metric setting. At the outset one obstacle is that the usual methods to prove (1) are not available for metric spaces (see [2, 15]) . However, we noticed that, in the Euclidean setting, (1) is the rearranged version of a Poincaré inequality. More specifically, suppose that f and g are functions such that, for any cube $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, we have

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx \le c\frac{|Q|^{1/n}}{|Q|}\int_Q g(x)dx. \tag{2}$$

Then the following version of (1) holds,

$$\frac{1}{t}\int_0^t [f^*(s) - f^*(t)]ds \le ct^{1/n}\left(\frac{1}{t}\int_0^t g^*(s)ds\right). \tag{3}$$

By Poincaré's inequality, (2) holds with $g = |\nabla f|$ and therefore the implication (2) $\Rightarrow$ (3) provides us with a proof of (1). This is somewhat surprising since the usual proofs of (1) depend on a suitable representation of f in terms of ∇f. Since, in the context of metric spaces, the gradient is defined through the validity of (2), this is a crucial point for our development of the symmetrization method in this setting.

Since the mechanism involved in transforming (2) into (3) plays an important role in our approach, it is instructive to present it here in the somewhat simpler, but central, Euclidean case. The first step is to reformulate (2) as an inequality between maximal operators

$$
\begin{aligned}
f^{\#}_{1/n}(x) &:= \sup_{Q \ni x} \frac{1}{|Q|^{1+1/n}} \int_Q |f(x) - f_Q|\, dx \\
&\leq c \sup_{Q \ni x} \frac{1}{|Q|} \int_Q g(x) dx = cMg(x),
\end{aligned} \tag{4}
$$

where M is the non-centered maximal operator of Hardy-Littlewood. The expression on the left-hand side is a modification of the well-known sharp maximal operator of Fefferman-Stein (see [4]). At this point, taking rearrangements on both sides of (4) leads to

$$
(f^{\#}_{1/n})^*(t) \leq cMg^*(t) \leq cg^{**}(t). \tag{5}
$$

Here the estimate for the maximal operator of Hardy-Littlewood is a well-known, and easy, consequence of the fact that M is weak type $(1,1)$ and strong type (∞,∞). Moreover, by a simple variant of an inequality of Bennett-DeVore-Sharpley [3, Theorem V.7.3], we have

$$
(f^{**}(t) - f^*(t))\, t^{-1/n} \leq c(f^{\#}_{1/n})^*(t). \tag{6}
$$

Combining (5) and (6), we see that if (2) holds then (3) holds.

The method of proof outlined above can be developed in more general settings as long as suitable variants of the classical covering lemmas, which are needed to estimate the underlying maximal operators, are available. In the context of metric spaces the covering lemmas we need were obtained in [13] (see also [10]). Once the rearrangement inequalities are at hand we can use standard machinery to derive suitable Sobolev inequalities (see Section 3).

To give a more precise description of the contents of this paper we now recall the definition of a (p,q)-Poincaré inequality. In what follows (X,μ) is a homogenous metric space with a doubling Borel measure μ of dimension s (see Definition 2.1 below).

Definition 1.1. (see [6, 8]) Let Ω be a measurable subset of X, and let f and g be measurable functions defined on Ω, with $g \geq 0$. Let $p, q \geq 1$. We shall say that f and g satisfy a (p,q)-Poincaré inequality, if for some constants $c_P > 0$, $\sigma \geq 1$,

$$
\left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p d\mu(x)\right)^{1/p} \leq c_P r(B) \left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} g^q(x) d\mu(x)\right)^{1/q}
$$

holds for every ball B such that $\sigma B \subset \Omega$, where $f_B = (\mu(B))^{-1} \int_B f(x) d\mu(x)$. We may then refer to f as a $(p-q-)$ Sobolev function and to g as its gradient.

We can now state our main results. We start with the following extension of (1).

Theorem 1.2. *(see Theorem 2.9 below) Let $B_0 \subset X$ be a ball, and suppose that f and g satisfy a (p,q)-Poincaré inequality on $4\sigma B_0$. Then there exist constants $c_1 > 0$, $0 < c_2 \leq 1$, independent of B_0, f and g, such that*

$$t^{-\frac{1}{s}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p \, ds \right)^{1/p} \leq c_1 [(g^q)^{**}(t)]^{1/q}, \tag{7}$$

for $0 < t < c_2 \mu(B_0)$.

Following [19], given a rearrangement invariant space Y, we introduce the spaces $Y^p(\infty, s)$ (see Section 2 below) that contain all the functions for which the Y-norm of the expression on the left-hand side of (7) is finite. The following sharp Sobolev embedding theorem then follows immediately.

Theorem 1.3. *(see Theorem 3.1 below) Let $B_0 \subset X$ be a ball, and let $Y(X)$ be an r.i. space. Suppose that the operator $P_{\max\{p,q\}}$ (see (10) below) is bounded on $Y(X)$. Then, if f and g satisfy a (p,q)-Poincaré inequality on $4\sigma B_0$ with constant c_P, there exists a constant $c = c(B_0, c_P, p, q, Y) > 0$ such that*

$$\|f\chi_{B_0}\|_{Y^p(\infty,s)} \leq c(\|g\|_Y + \|f\|_Y).$$

We also provide a new application of our rearrangement inequality (7) to the study of the so called functional forms of the Faber-Krahn inequalities in metric spaces (see Section 4 below). We now illustrate these ideas in the classical Euclidean case. For example, using $\int_{f^*(t)}^{\infty} \lambda_f(u) = t(f^{**}(t) - f^*(t))$, where λ_f denotes the distribution function of f, and Hölder's inequality, we see that (1) implies

$$\begin{aligned} \int_{f^*(t)}^{\infty} \lambda_f(u) &\preceq t^{1/n} \int_0^t |\nabla f|^* (u) du \\ &\preceq t^{1/n} \left(\int_0^t |\nabla f|^* (u)^p du \right)^{1/p} t^{1/p'}, \end{aligned}$$

where $\approx$ denotes equivalence modulo constants, and $\preceq$ denotes smaller or equal modulo constants. Now let $t = \|f\|_0 = |\mathrm{supp}(f)|$, and observe that then $f^*(t) = 0$, and $\int_{f^*(t)}^{\infty} \lambda_f(u) = \|f\|_1$. We have thus obtained the following Faber-Krahn inequality

$$\|f\|_1 \preceq \|f\|_0^{1/n+1-1/p} \|\nabla f\|_p \,.$$

More generally, if a (p,q)-Poincaré inequality holds then we can use (7) and a similar argument to prove

Theorem 1.4 (see Theorem 3.4 below)**.** *Let $B_0 \subset X$ be a ball, and let f be a function with supp$(f) \subset B_0$. Let $Z(X)$ be an r.i. space and let $\phi_{Z'}$ denote the fundamental function of its associate space $Z'(X)$.*

(i) *Let f be a $p-q-$Sobolev function and let g be a gradient of f. If $\|f\|_0 < c_2\mu(B_0)$, where c_2 is the constant of Theorem* 2.9, *then*

$$\|f\|_{L^p(X)} \leq c\left[\|g^q\|_{Z(X)}\phi_{Z'}(\|f\|_0)\right]^{1/q}\|f\|_0^{\frac{s+p}{sp}-\frac{1}{q}}.$$

(ii) *Let f be a $1-q-$Sobolev function, $q > s$, and let g be a gradient of f. If $\|f\|_0 < c_2\mu(B_0)$, where c_2 is the constant of Theorem* 2.9, *then*

$$\|f\|_{L^\infty(X)} \leq c\left[\|g^q\|_{Z(X)}\phi_{Z'}(\|f\|_0)\right]^{1/q}\|f\|_0^{1/s-1/q}.$$

2. The basic symmetrization inequality in metric spaces

We start with some definitions.

Definition 2.1. A homogeneous space consists of a metric space X and a Borel measure μ on X, such that $0 < \mu(B(x,r)) < \infty$, for all $x \in X, r > 0$, and, moreover, the measure μ satisfies a doubling condition:

$$\mu(B(x,2r)) \leq c_d\mu(B(x,r)), \tag{8}$$

for all $x \in X$ and $r > 0$. If c_d is the smallest constant in (8) then the number $s = \log_2 c_d$ is called the doubling order, the dimension or homogeneous dimension of μ.

Remark 2.2. Note that if we fix a ball $\widetilde{B} \subset X$, then by iterating (8) (see Lemma 14.6 in [8]) we can find a positive constant c (possibly depending on $\widetilde{B}$) such that for every ball $B \subset \widetilde{B}$ we have

$$\mu(B) \geq cr(B)^s. \tag{9}$$

In what follows, given a ball $B = B(x,r)$, ϱB will denote the ball concentric with B, whose radius is ϱr.

A rearrangement invariant (r.i.) space $Y = Y(X)$ is a Banach function space of $\mu-$measurable functions on X endowed with a norm $\|\cdot\|_Y$ such that if $f \in Y$, g is a $\mu-$measurable function and for distribution functions of f and g we have $\lambda_f = \lambda_g$ then $g \in Y$ and $\|g\|_Y = \|f\|_Y$. The fundamental function ϕ_Y of Y is defined for t in the range of μ by

$$\phi_Y(t) = \|\chi_E\|_Y,$$

where E is any subset of X with $\mu(E) = t$. The associate space Y' is defined as the space of all $\mu-$measurable functions on X endowed with a norm

$$\|f\|_{Y'} = \sup_{g\in Y, \|g\|_Y\leq 1}\int_X |fg|.$$

For any $f \in Y$, its non-increasing rearrangement is defined as

$$f^*(t) = \inf\{u > 0 : \lambda_f(u) \leq t\},$$

for $t > 0$. Recall that any resonant r.i. space Y has a representation as a function space $Y\hat{}(0,\infty)$ such that (see [4, Theorem II.4.10])

$$\|f\|_{Y(X)} = \|f^*\|_{Y\hat{}(0,\infty)}.$$

Since the measure space will be always clear from the context, it is convenient to "drop the hat" and use the same letter Y to indicate the different versions of the space Y that we use.

Let P denote the usual Hardy operator $P : f(t) \mapsto t^{-1}\int_0^t f^*(s)ds$. The operators P_p, $p \geq 1$, are defined by

$$(P_p f)(t) = [P((f^*)^p)(t)]^{1/p}. \tag{10}$$

The following space will play a crucial role in our theory.

Definition 2.3. Let Y be a r.i. space, and let $p \geq 1$ and $r > 0$. Then space $Y^p(\infty, r)(X)$ is defined as a set of all $f \in Y(X)$ such that

$$\|f\|_{Y^p(\infty,r)(X)} = \left\| t^{-1/r} \left(\frac{1}{t} \int_0^t [f^*(s) - f^*(t)]^p ds \right)^{1/p} \right\|_Y < \infty.$$

Remark 2.4. When it is clear from the context, we will simply write $Y^p(\infty, r)$ instead of $Y^p(\infty, r)(X)$.

Remark 2.5. Under suitable assumptions the expression defining the "norm" of the $Y^p(\infty, r)$ spaces can be simplified (see [22]). For example, suppose that p and r are such that

$$1 \leq p < \frac{p_Y r}{p_Y - r}, \tag{11}$$

where p_Y is the lower Boyd index p_Y of Y, and suppose, moreover (see [19]),

$$\int_1^\infty s^{1/r} d_Y\left(\frac{1}{s}\right) \frac{ds}{s} < \infty, \tag{12}$$

where $d_Y(s)$ is the norm of the dilation operator $D_s : f(\cdot) \mapsto f(\cdot s)$. Then, for f with $f^{**}(\infty) = 0$, we have

$$\|f\|_{Y^p(\infty,r)} \approx \|f\|_{Y^1(\infty,r)}.$$

Proof. From (12) it follows that, for $f^{**}(\infty) = 0$, we have (see [19, Lemma 2.6]),

$$\|t^{-1/r}(f^{**}(t) - f^*(t))\|_Y \approx \|t^{-1/r} f^{**}(t)\|_Y. \tag{13}$$

On the other hand, since $\|D_a[f(t)t^{-1/r}]\|_Y = a^{-1/r}\|[D_a f(t)]t^{-1/r}\|_Y$, we have

$$\|D_a[f(t)t^{-1/r}]\|_Y \leq ca^{-1/p}\|f(t)t^{-1/r}\|_Y,$$

if and only if

$$\|D_a[f(t)]t^{-1/r}\|_Y \leq ca^{-1/p+1/r}\|f(t)t^{-1/r}\|_Y.$$

Consequently, if we let $Y(t^{-1/r})$ be the space defined by the norm $\|f(t)t^{-1/r}\|_Y$ then the lower Boyd index of $Y(t^{-1/r})$ is equal to $p_Y r/(p_Y - r)$, where p_Y is the lower Boyd index of Y. Now, in view of (11) it follows from [20, Theorem 2 (i)], that the operator P_p is continuous on $Y(t^{-1/r})$. Thus,

$$\left\| \left(\frac{1}{t}\int_0^t [f^*(s)]^p ds\right)^{1/p} t^{-1/r}\|_Y \leq c\|f^*(t)t^{-1/r} \right\|_Y \leq \|f^{**}(t)t^{-1/r}\|_Y.$$

Combining the last inequality with (13) we obtain,

$$\|f\|_{Y^p(\infty,r)} \lesssim \|f\|_{Y^1(\infty,r)}.$$

The reverse inequality follows readily from Hölder's inequality. □

In our approach, the following operator will naturally come up (see the left-hand side of (4)). It is a modification of the well-known sharp maximal operator of Fefferman-Stein (see [4]) and is defined for $f \in L^1_{\mathrm{loc}}(B_0)$, $B_0 \subset X$ a ball, and $p, q \geq 1$ by

$$f^{\#}_{B_0,p,q}(x) = \sup_{x \in B \subset B_0 \text{ a ball}} \left(\frac{1}{\mu(B)^q}\int_B |f(y) - f_B|^p d\mu(y)\right)^{1/p},$$

for $x \in B_0$.

For the proof of our symmetrization inequality we need the following version of a covering lemma from [13].

Lemma 2.6. *There exist positive constants c, λ, with $\lambda < 1$, such that for any ball B, and any open set $E \subset B$ with $\mu(E) \leq \lambda\mu(B)$, there is a countable family of balls $\{B_i\}_{i=1}^{\infty}$ such that*

(i) $B_i \subset 4B$, *for* $i = 1, 2, \dots$

(ii) $E \subset \bigcup_{i=1}^{\infty} B_i$.

(iii) $\sum_i \mu(B_i) \leq c\mu(E)$.

(iv) $0 < \mu(B_i \cap E) \leq (1/2)\mu(B_i \cap B)$, *for* $i = 1, 2, \dots$

Proof. Follows readily from the proof of [13, Lemma 3.1]. □

Theorem 2.7. *There exist positive constants c_1, c_2, such that, for any ball $B_0 \subset X$, $p, q \geq 1$, and for all $f \in L^1_{loc}$, $0 < t < c_2\mu(B_0)$, we have*

$$t^{-q/p}\left(\int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p ds\right)^{1/p} \leq c_1(f^{\#}_{4B_0,p,q})^*(t). \tag{14}$$

Proof. Follows along the lines of the corresponding proof in [4, Theorem V.7.3].

It suffices to establish (14) for nonnegative functions. Let λ be given by Lemma 2.6 and fix $0 < t < \frac{\lambda}{3}\mu(B_0)$. Let

$$E = \{x \in B_0 : f(x) > [f\chi_{B_0}]^*(t)\}$$

and

$$F = \{x \in B_0 : f^{\#}_{4B_0,p,q}(x) > [f^{\#}_{4B_0,p,q}\chi_{B_0}]^*(t)\}.$$

There exists an open set $\Omega \supset E \cup F$ with measure at most $3t \leq \lambda\mu(B_0)$. Consequently, we can apply Lemma 2.6 to obtain a family of balls, $\{B_j\}_j$, such that all the conditions of this Lemma are verified. Define disjoint sets by letting $M_1 = B_1$ and $M_k = B_k \setminus \bigcup_{i=1}^{k-1} B_i$ for $k = 2, \ldots$ We have

$$\begin{aligned}
\int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p ds &= \int_E \{f(x) - [f\chi_{B_0}]^*(t)\}^p d\mu(x) \\
&= \sum_{j=1}^{\infty} \int_{E\cap M_j} \{f(x) - [f\chi_{B_0}]^*(t)\}^p d\mu(x) \\
&\leq c\sum_{j=1}^{\infty} \int_{B_j} |f - f_{B_j}|^p d\mu(x) \\
&\quad + c\sum_{j=1}^{\infty} \mu(E \cap M_j)\{f_{B_j} - [f\chi_{B_0}]^*(t)\}_+^p \\
&= c(\alpha + \beta), \quad \text{say.}
\end{aligned}$$

Now

$$\begin{aligned}
\beta &\leq \sum_{\{j: f_{B_j} > [f\chi_{B_0}]^*(t)\}} \mu(E \cap B_j)\{f_{B_j} - [f\chi_{B_0}]^*(t)\}^p \\
&\leq \sum_{\{j: f_{B_j} > [f\chi_{B_0}]^*(t)\}} \mu(B_0 \cap B_j \setminus E)\{f_{B_j} - [f\chi_{B_0}]^*(t)\}^p \\
&\leq \sum_{\{j: f_{B_j} > [f\chi_{B_0}]^*(t)\}} \int_{B_0 \cap B_j \setminus E} \{f_{B_j} - f(x)\}^p d\mu(x) \\
&\leq \sum_{j=1}^{\infty} \int_{B_j} |f - f_{B_j}|^p d\mu(x) = \alpha,
\end{aligned}$$

with the second inequality by (iv) of Lemma 2.6.

Combining the previous estimates, we obtain

$$\int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p ds \leq 2c\alpha = 2c \sum_j \mu(B_j)^q \frac{1}{\mu(B_j)^q} \int_{B_j} |f - f_{B_j}|^p d\mu(x).$$

By (iv) of Lemma 2.6, the set $B_0 \cap B_j \setminus F$ is nonempty and, therefore, we can find a point $x_j \in B_0 \cap B_j \setminus F$. It follows that

$$\begin{aligned} \int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p ds &\leq 2c \sum_j \mu(B_j)^q [f^{\#}_{4B_0,p,q} \chi_{B_0}(x_j)]^p \\ &\leq ct^q [(f^{\#}_{4B_0,p,q})^*(t)]^p. \end{aligned}$$

□

Corollary 2.8. *(see* [23, page 228]*) Suppose that* $\mu(X) = \infty$ *and let* c_1 *be the constant of Theorem* 2.7*. Then*

$$t^{-q/p} \left(\int_0^t (f^*(s) - f^*(t))^p ds \right)^{1/p} \leq c_1 (f^{\#}_{X,p,q})^*(t), \tag{15}$$

for all $f \in L^1_{loc}(X)$, $t > 0$.

Proof. Let $t > 0$ and let c_2 be as in Theorem 2.7. Fix an arbitrary $x_0 \in X$, since $\mu(X) = \infty$, we can find a positive integer n_0 such that, for $n \geq n_0$, and $B_n := B(x_0, n)$, we have $t < c_2\mu(B_n)$. Therefore, by (14),

$$t^{-q/p} \left(\int_0^t ([f\chi_{B_n}]^*(s) - [f\chi_{B_n}]^*(t))^p ds \right)^{1/p} \leq c_1 (f^{\#}_{4B_n,p,q})^*(t),$$

and, consequently,

$$t^{-q/p} \left(\int_0^t ([f\chi_{B_n}]^*(s) - f^*(t))_+^p ds \right)^{1/p} \leq c_1 (f^{\#}_{X,p,q})^*(t).$$

Letting $n \to \infty$ and using Fatou's lemma, we see that

$$t^{-q/p} \left(\int_0^t (f^*(s) - f^*(t))^p ds \right)^{1/p} \leq c_1 (f^{\#}_{X,p,q})^*(t).$$

□

Once the inequality (14) is available then it can be combined with the Poincaré inequality, as described in the introduction, to obtain the symmetrization inequality.

Theorem 2.9. *Let $B_0 \subset X$ be a ball, and suppose that f and g satisfy a (p,q)-Poincaré inequality on $4\sigma B_0$ (with constant c_P). Then there exist positive constants $c_1 = c_1(B_0, c_P)$ and $1 \geq c_2 = c_2(X)$, such that, for $0 < t < c_2\mu(B_0)$,*

$$t^{-\frac{1}{s}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p ds \right)^{1/p} \leq c_1 [(g^q)^{**}(t)]^{1/q}. \tag{16}$$

Proof. From the underlying Poincaré inequality and (9) (with $\widetilde{B} = 4B_0$), we get

$$\left(\frac{1}{(\mu(B))^{1+p/s}} \int_B |f - f_B|^p d\mu \right)^{1/p} \leq c \left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} g^q d\mu \right)^{1/q},$$

for every ball B with $B \subset 4B_0$.

Fix an arbitrary point $x \in B_0$. Taking a supremum over all balls containing x on the right hand side, and over all balls $B \subset 4B_0$ containing x on the left hand side, we arrive at

$$f^{\#}_{4B_0,p,1+p/s}(x) \leq c \left(Mg^q(x) \right)^{1/q},$$

where M is the maximal operator of Hardy-Littlewood. After passing to rearrangements, and, using (recall that the underlying measure is doubling)

$$(Mh)^*(t) \leq ch^{**}(t),$$

combined with Theorem 2.7, we obtain positive constants c_1, c_2 such that

$$t^{-\frac{1}{s}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p ds \right)^{1/p} \leq c_1 [(g^q)^{**}(t)]^{1/q},$$

for $0 < t < c_2\mu(B_0)$. □

Remark 2.10. It may not be possible to extend the inequality (16) to all $0 < t < \mu(B_0)$. This can be seen from the following counterexample for $p = q = 1$.

Let $f_k(x) = 1$ if $x \in [-1,1]^2 \setminus B_{1/k}(0)$ and $f_k(x) = k|x|$ for $x \in B_{1/k}(0)$. Now, $|\nabla f_k|^*(t) = k\chi_{(0,\frac{\pi}{k^2}]}(t)$ and $|\nabla f_k|^{**}(t) = k\chi_{(0,\frac{\pi}{k^2}]}(t) + \frac{\pi}{k}\frac{1}{t}\chi_{[\frac{\pi}{k^2},\infty)}(t)$. If (16) were true for $0 < t < 4$, then taking the limit as $t \to 4$ in (16) would give us

$$\|f_k\|_{L^1(Q)} \leq c\frac{\pi}{4}\frac{1}{k}.$$

But while the right-hand side $\to 0$ as $k \to \infty$ the left-hand side ≈ 4 . One possible way to overcome this problem is to consider functions with zero average, by means of replacing f by $f - f_Q$ (see [15]).

Remark 2.11. Suppose that the following global growth condition holds for every ball $B \subset X$,

$$\mu(B) \geq cr(B)^s. \tag{17}$$

Then using the proof of Theorem 2.9 together with (15) yields

$$t^{-\frac{1}{s}} \left(\frac{1}{t} \int_0^t (f^*(s) - f^*(t))^p ds \right)^{1/p} \leq c_1[(g^q)^{**}(t)]^{1/q},$$

for $t > 0$.

3. Applications

First we consider the Sobolev embedding theorem for metric spaces.

Theorem 3.1. *Let $B_0 \subset X$ be a ball and let $Y = Y(X)$ be an r.i. space. Suppose that the operator $P_{\max\{p,q\}}$ is bounded on Y, and let f and g satisfy a (p,q)-Poincaré inequality on $4\sigma B_0$. Then there exists a constant $c > 0$, independent of f and g, such that*

$$\|f\chi_{B_0}\|_{Y^p(\infty,s)} \leq c(\|g\|_Y + \|f\chi_{B_0}\|_Y).$$

Proof. By Theorem 2.9 there are constants c_1, c_2 such that, for $0 < t < c_2\mu(B_0)$,

$$t^{-\frac{1}{s}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p ds \right)^{1/p} \leq c_1[(g^q)^{**}(t)]^{1/q}.$$

If $t \geq c_2\mu(B_0)$, we have

$$t^{-\frac{1}{s}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p ds \right)^{1/p} \leq (c_2\mu B_0))^{-\frac{1}{s}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^*(s))^p ds \right)^{1/p}.$$

Therefore,

$$t^{-\frac{1}{s}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p ds \right)^{1/p} \leq c\left[P_q(g)(t) + P_p(f\chi_{B_0})(t)\right],$$

for all $t > 0$. In view of our assumption on $P_{\max\{p,q\}}$ it follows, upon applying the Y norm to both sides of the previous inequality, that

$$\|t^{-\frac{1}{s}} \left(\frac{1}{t} \int_0^t ([f\chi_{B_0}]^*(s) - [f\chi_{B_0}]^*(t))^p ds \right)^{1/p} \|_Y \leq c\left[\|g\|_Y + \|f\chi_{B_0}\|_Y\right],$$

as we wished to show. □

Remark 3.2. Arguing as in Remark 2.11, we conclude that, if the global growth condition (17) holds for all balls $B \subset X$, then, for all f and g as in Theorem 3.1, we have

$$\|f\|_{Y^p(\infty,s)} \le c\|g\|_Y.$$

Remark 3.3. The assumption that the modified Hardy operator $P_{max\{p,q\}}$ is bounded on the space Y, excludes the space L^1 from our theory. But as we now rather briefly indicate, in some cases, it is possible to remove this restriction using a variant of the truncation method originally due to Maz'ya [18] (see also [7] for further references), and futher refined in [17]. Moreover, the results of [17] were extended to the metric setting in [10]. The only price we pay is an additional assumption: the Poincaré inequality needs to hold also for any truncated pair (note that this condition is automatically satisfied in the Euclidean space). Let us recall that for a positive function f and $t_1 < t_2$, we let

$$f_{t_1}^{t_2}(x) = \begin{cases} t_2 - t_1 & \text{if } f(x) \ge t_2, \\ f(x) - t_1 & \text{if } t_1 < f(x) < t_2, \\ 0 & \text{if } f(x) \le t_1. \end{cases}$$

Then (see [10]):

Let f, g be measurable functions defined on X, $f^{**}(\infty) = 0$, $g \in L^1(X)$, $f, g \ge 0$ and $\sigma > 1$. Suppose that Y is a r.i. space with lower Boyd index $i_Y > 0$. Consider the following statements:

(A) For any $t_1 < t_2$

$$\sup_{t>0} t\mu(\{x \in X : |f_{t_1}^{t_2}(x)| > t\})^{1-1/\sigma} \le C \int_{\{x\in X: t_1 < f(x) \le t_2\}} g(x)d\mu(x).$$

(B) For all $t > 0$,

$$\int_0^t s^{-1/\sigma}[f^{**}(s) - f^*(s)]ds \le c \int_0^t g^*(s)ds.$$

(C)

$$\|s^{-1/\sigma}(f^{**}(s) - f^*(s))\|_Y \le C\|g\|_Y.$$

Then **(A)** $\Rightarrow$ **(B)** $\Rightarrow$ **(C)**. We refer to [10] where the analysis follows the one given in [17].

We now apply our symmetrization inequality to derive functional forms of Faber-Krahn inequalities (see [1] for a brief introduction to inequalities of this type). In the following we denote

$$\|f\|_0 := \mu(\text{supp}(f)).$$

We will also assume that μ is nonatomic.

Theorem 3.4. *Let $B_0 \subset X$ be a ball and let f be a function with $\{f \neq 0\} \subset B_0$. Let $Z(X)$ be an r.i. space and let $\phi_{Z'}$ denote the fundamental function of its associate space $Z'(X)$.*

(i) *Let f be a $p-q-$Sobolev function and let g be a gradient of f. If $\|f\|_0 < c_2\mu(B_0)$, where c_2 is the constant of Theorem 2.9, then*

$$\|f\|_{L^p(X)} \leq c\left[\|g^q\|_{Z(X)}\phi_{Z'}(\|f\|_0)\right]^{1/q}\|f\|_0^{\frac{s+p}{sp}-\frac{1}{q}}.$$

(ii) *Let f be a $1-q-$Sobolev function, $q > s$, and let g be a gradient of f. If $\|f\|_0 < c_2\mu(B_0)$, where c_2 is the constant of Theorem 2.9, then*

$$\|f\|_{L^\infty(X)} \leq c\left[\|g^q\|_{Z(X)}\phi_{Z'}(\|f\|_0)\right]^{1/q}\|f\|_0^{1/s-1/q}.$$

Proof. (i) By Theorem 2.9 we have, for $0 < t < c_2\mu(B_0)$,

$$t^{-\frac{s+p}{sp}}\left(\int_0^t [f^*(s) - f^*(t)]^p ds\right)^{1/p} \leq c_1[(g^q)^{**}(t)]^{1/q}.$$

Now, since $\|f\|_0 < c_2\mu(B_0)$, using right-continuity of the non-increasing rearrangement we can substitute $t = \|f\|_0$ in (3). Thus,

$$\|f\|_0^{-\frac{s+p}{sp}}\|f\|_{L^p} \leq c_1\left[\frac{1}{\|f\|_0}\int_0^{\|f\|_0}(g^q)^*(s)ds\right]^{1/q}.$$

Applying Hölder's inequality we finally obtain

$$\|f\|_{L^p} \leq c_1\left[\|g^q\|_Z\phi_{Z'}(\|f\|_0)\right]^{1/q}\|f\|_0^{\frac{s+p}{sp}-\frac{1}{q}}.$$

(ii) We first observe that $-\frac{d}{dt}f^{**}(t) = [f^{**}(t) - f^*(t)]/t$. Thus, by Theorem 2.9, we have

$$-\frac{d}{dt}f^{**}(t) \leq c_1 t^{1/s-1}\left(\frac{1}{t}\int_0^t (g^q)^*(s)ds\right)^{1/q}.$$

Integrating over $(0, \|f\|_0)$ yields

$$\|f\|_{L^\infty(X)} - f^{**}(\|f\|_0) \leq c_1\int_0^{\|f\|_0} t^{1/s-1-1/q}\left(\int_0^t (g^q)^*(s)ds\right)^{1/q}dt.$$

Thus, estimating the inner integral using Hölder's inequality,

$$\begin{aligned}\|f\|_{L^\infty(X)} &\leq c_1\left[\|g^q\|_Z\phi_{Z'}(\|f\|_0)\right]^{1/q}\int_0^{\|f\|_0} t^{1/s-1-1/q}dt + f^{**}(\|f\|_0)\\ &= c_1\left[\|g^q\|_Z\phi_{Z'}(\|f\|_0)\right]^{1/q}\frac{1}{1/s-1/q}\|f\|_0^{1/s-1/q} + \frac{\|f\|_{L^1(X)}}{\|f\|_0}\\ &\leq c\left[\|g^q\|_Z\phi_{Z'}(\|f\|_0)\right]^{1/q}\|f\|_0^{1/s-1/q},\end{aligned}$$

where in the last line we used the result obtained in the first half of the theorem. □

Finally, for a different connection between Poincaré inequalities and symmetrization, with other interesting applications, we refer to [16] (see also [12] for the relevant family of Poincaré inequalities). It would be of interest to extend the results of these papers to the metric setting.

4. Appendix: Vector fields satisfying Hörmander's condition

We present a concrete application of our embedding theorem. But first let us briefly review some relevant definitions and facts.

Let $X_1, \dots, X_m$ be a collection of $\mathcal{C}^\infty$ vector fields defined in a neighborhood Ω of the closure of the unit ball in $\mathbb{R}^n$. For a multiindex $\alpha = (i_1, \dots, i_k)$, denote by X_α the commutator $[X_{i_1}, [X_{i_2}, \dots, [X_{i_{k-1}}, X_{i_k}]] \dots]$ of length $|\alpha| = k$. We shall assume that $X_1, \dots, X_m$ satisfy Hörmander's condition: there exists an integer d such that the family of commutators, up to order d, $\{X_\alpha\}_{|\alpha|\le d}$, spans the tangent space $\mathbb{R}^n$ at each point of Ω. A metric on Ω is defined using the following construction. A Lipschitz curve $\gamma : [a,b] \to \Omega$ is called an admissible path, if there exist functions $c_i(t)$, $a \le t \le b$, satisfying $\sum_{i=1}^m c_i^2(t) \le 1$, and

$$\gamma'(t) = \sum_{i=1}^m c_i(t) X_i(\gamma(t)),$$

for a.e. $t \in [a,b]$. A natural metric (the so-called Carnot-Carathéodory metric) on Ω associated to $X_1, \dots, X_m$, is defined by

$$\begin{aligned}\varrho(\xi,\nu) = \min\{b \ge 0 : \text{there is an admissible path } \gamma : [0,b] \to \Omega \\ \text{such that } \gamma(0) = \xi \text{ and } \gamma(b) = \nu\}.\end{aligned}$$

By Nagel et al. [21, §3 and Theorem 4], there exist $C > 0$ and $R_0 > 0$ such that for all $x \in B(0,1)$ and $0 < R \le R_0$ we have

$$|B(x,2R)| \le C|B(x,R)|,$$

where $|\cdot|$ indicates Lebesgue measure. In the following $s = \log_2 C$ is the homogeneous dimension of $|\cdot|$ with respect to Carnot-Carathéodory metric ϱ (see Definition 2.1).

In this setting Capogna et al. [5] proved the following theorem.

Theorem 4.1 ([5, Theorem 1]). *There exist $C > 0$ and $R_0 > 0$ such that for any $x \in B(0,1)$, $B_R = B(x,R)$, with $0 < R \le R_0$, and every $f \in \mathcal{C}_0^\infty(\overline{B_R})$ one has*

$$\left(\frac{1}{|B_R|}\int_{B_R} |f|^k dx\right)^{1/k} \le CR\left(\frac{1}{|B_R|}\int_{B_R}\left[\sum_{i=1}^m |X_i f(x)|\right] dx\right),$$

for any $1 \le k \le s/(s-1)$.

Using our theory we can extend this result to the setting of r.i. spaces. We start by recalling the Poincaré inequality by Jerison [9]. Jerison's original proof was done for the exponent $p = 2$ but using the same method we can easily obtain a $1-1-$Poincaré inequality (see notes in Hajłasz [8], pages 70, 71).

Theorem 4.2 ([9, Theorem 2.1])**.** *There exist a constant $C > 0$, and a radius R_0, such that, for every x from the unit ball and every R, $0 < R \le R_0$, for which $B_R = B(x, R) \subset \Omega$, we have*

$$\int_{B_R} |f(x) - f_{B_R}| dx \le CR \int_{B_R} \sum_{i=1}^{m} |X_i f(x)| dx,$$

for all $f \in \mathcal{C}^\infty(\overline{B_R})$, where the integration is with respect to Lebesgue measure.

Now we are ready to prove our embedding theorem for vector fields satisfying Hörmander condition:

Theorem 4.3. *There exist a constant $C > 0$, and a radius R_0, such that, for every x from the unit ball and every R, $0 < R \le R_0$, for which $B_R = B(x, R) \subset \Omega$, we have*

$$\|f\|_{Y^1(\infty,s)(B_R)} \le C \| \sum_{i=1}^{m} |X_i f| \|_{Y(B_R)}, \tag{18}$$

for all $f \in \mathcal{C}_0^\infty(\overline{B_R})$, where the integration is with respect to the Lebesgue measure.

Proof. We will show that

$$f^{**}(t) - f^*(t) \le Ct^{1/s} |Xf|^{**}(t), \tag{19}$$

$|Xf| = \sum_{i=1}^m |X_i f|$, for any $t > 0$. Then (18) follows readily by applying the Y-norm to both sides of the inequality.

Using Theorem 2.9 we obtain a constant $0 < \lambda \le 1$ such that, for $0 < t < \lambda |B_R|$,

$$f^{**}(t) - f^*(t) \le Ct^{1/s} |Xf|^{**}(t),$$

where $|Xf| = \sum_{i=1}^m |X_i f|$. For $\lambda |B_R| \le t \le |B_R|$ we have

$$f^{**}(t) - f^*(t) \le f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds \le \frac{1}{\lambda |B_R|} \|f\|_{L^1(B_R)}.$$

By Theorem 4.1 with $k = 1$

$$\begin{aligned} \frac{1}{\lambda |B_R|} \|f\|_{L^1(B_R)} &\le \frac{1}{\lambda} CR \frac{1}{|B_R|} \int_{B_R} |Xf| dx \\ &\le \frac{1}{\lambda} CR |Xf|^{**}(|B_R|) \\ &\le \frac{1}{\lambda} C |B_R|^{1/s} |Xf|^{**}(|B_R|) \\ &\lesssim t^{1/s} |Xf|^{**}(t), \end{aligned}$$

we used (9) and in the last inequality the fact that $|Xf|^{**}$ is non-increasing and the assumption $|B_R| \lesssim t$. Finally, for $t > |B_R|$, inequality (19) is a mere reformulation of

$$\|f\|_{L^1} \leq C\||Xf|\|_{L^1},$$

the validity of which follows again from Theorem 4.1 with $k = 1$. $\square$

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