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The Cauchy-Riemann Operator in Infinite Dimensional Spaces

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ABSTRACT. In this survey article we present a sketch of the techniques which allowed to advance into the problem of solving the $\bar{\partial}$-problem in infinite dimensional spaces. As an example we show how the $\bar{\partial}$-problem for a holomorphic (0,2)-form on a D.F.N. space can be solved by using the solution of the $\bar{\partial}$-problem for a holomorphic (0,2)-form on a separable Hilbert space. For further details we refer to SORAGGI [16].

The solution of the $\bar{\partial}$—Neumann problem by J. J. Kohn in 1963 and the publication in 1966 of L. Hörmander's book on several complex variables were the beginning of a reapproachment between several complex variables and analysis. The interaction between several complex variables and the theory of partial differential equations during the 1960s, and the study of $\bar{\partial}$ through integral representations during the 1970s, have proved to be a fruitful source of ideas for the development of Mathematics. So the relevance of the $\bar{\partial}$—operator to the theory of several complex variables is a good reason for studying it on infinite dimensional spaces. On the other hand, the existence of a nice theory of holomorphic functions on nuclear spaces (due to the work of Boland, Dineen, Meise and Vogt) justifies the study of $\bar{\partial}$ in such spaces.

The study of partial differential operators with an infinite number of independent variables has encountered formidable difficulties. Apart from the absence of local compactness, there are three basic difficulties in studying partial differential operators on infinite dimensional spaces: the non-existence of Lebesgue measure for infinite dimensional spaces, the non-existence of $\mathscr{C}^\infty$ partitions of unity and the non-existence of a powerful generalization of Schwartz' distribution theory to infinite dimensional spaces.

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We refer to the article of Thompson [17] for further details and references on the literature on partial differential operators in infinite dimensional spaces.

In order to develop the theory of partial differential operators in infinite dimensional spaces we must have an appropriate definition of such an operator as well as an appropiate notion of differentiability. The standard notions of differentiable mappings coincide on the strong duals of Fréchet nuclear spaces (DFN spaces). We refer to Colombeau ([2], [5]) and Colombeau-Meise [4] for a systematic discussion on this direction.

Concerning the $\bar{\partial}$—operator, the counterexamples of Coeuré [1] and Dineen [8] showed that we have to impose some restrictions on the spaces where the forms are defined, as well as, on the forms themselves. Henrich [9], in 1972, was the first to obtain a positive result in infinite dimensional spaces. Henrich's solution produced a phenomena, particular to infinite dimensions, the solution is only defined on a dense subspace. Afterwards, Raboin [11], [12] and Colombeau-Perrot [3], [6] studied the problem $\bar{\partial} u = \omega$ when ω is a $\mathscr{C}^\infty$ differential (0,1) form on a Hilbert space and on DFN spaces. In 1981, Mazet [10], in a very elegant way, using a generalized Cauchy-integral representation and standard Functional Analysis arguments, showed that the solutions which were constructed by Raboin and Colombeau-Perrot were $\mathscr{C}^\infty$. We refer to Soraggi [16] for further comments on the $\bar{\partial}$-operator in infinite dimensional spaces.

We have studied the $\bar{\partial}$-operator in infinite dimensional spaces when ω is a $(0, q)$ form, $q > 1$. The aim of this note is to present the results we have obtained. We refer to Soraggi [14], [15], [16] for terminology and notation, as well as for details and proofs of the results.

Complete results we only obtained when ω was a holomorphic (0,2) form and we restrict ourselves to this case.

First of all we make some comments on the definition of the $\bar{\partial}$-operator.

Definition 1

Let E be a complex locally convex space. Let $q \geq 1$. Let $\Lambda^{(0,q)}(E)$ be the space of continuous alternating q anti-linear forms on E, endowed with the topology of uniform convergence on the bounded subsets of E. A $\mathscr{C}^\infty$ (differential) $(0, q)$ form is a mapping $\omega: \Omega \to \Lambda^{(0,q)}$ —Ω an open subset of E— such that for every $a \in \Omega$, there exist a continuous semi-norm α on E and an open subset Ω' of E_α (the semi-normed space (E, α)) with $a \in \Omega' \subset \Omega$ such that $\omega: \Omega' \to \Lambda^{(0,q)}(E_\alpha)$ is a $\mathscr{C}^\infty$ mapping in Ω'.

It is easily seen that a $\mathscr{E}^\infty$ differential $(0, q)$ form in the sense of definition 1 is a $\mathscr{E}^\infty$ mapping in the sense of Sebastião e Silva[13].

Consider the underlying space $E_{\mathbf{R}}$ and differentiability in the sense of Sebastião e Silva. Let $k \geq 1$ and let F be a complex locally convex space. Denote by $\mathscr{E}^k(\Omega, F)$ the vector space of k-times differentiable mappings from Ω into F. As usual, $\mathscr{E}(\Omega; F)$ denotes the vector space of continuous mappings from Ω into F. Suppose E and F are complex normed spaces. Let $u \in \mathscr{E}^1(\Omega; F)$ and let $u' : \Omega \to \mathscr{L}_{\mathbf{R}}(E, F)$ be its derivative. ($\mathscr{L}_{\mathbf{K}}(E; F)$ denotes the vector space of continuous $\mathbf{K}$-linear mappings from E into F). For $x \in \Omega$, $y \in E$ we have.

Definition 2

$$[\bar{\partial}]u(x)(y) = \frac{1}{2}[u'(x)(y) + i u'(x)(iy)].$$

If $\bar{E}$ denotes the conjugate space of E, $[\bar{\partial}]\, u(x) \in \mathscr{L}(\bar{E}; F) = \mathscr{L}_{\mathbf{C}}(\bar{E}; F) = \overline{\mathscr{L}_{\mathbf{C}}}(E; F)$ is the anti-linear component of $u'(x) \in \mathscr{L}_{\mathbf{R}}(E; F)$. So we have defined an operator $[\bar{\partial}] : \mathscr{E}^1(\Omega; F) \to \mathscr{E}(\Omega; \mathscr{L}_{\mathbf{C}}(\bar{E}; F))$.

Definition 3 (Mazet [10])

Let $E; F$ be complex Banach spaces. Given $\omega : \Omega \to \mathscr{L}_{\mathbf{C}}(\bar{E}, F)$. We say that $u : \Omega \to F$ is a weak solution of $[\bar{\partial}]u = \omega$ if for every fixed $z \in \Omega$ and $x \in E$ the mapping $g : \lambda \to u(z + \lambda x)$ is continuous on a disc $\Delta = \Delta(0, r) \subset \mathbf{C}$ and satisfies in the sense of distribution.

$$\frac{\partial g}{\partial \bar{\lambda}}(\lambda) = \frac{\partial}{\partial \bar{\lambda}} u(z + \lambda x) = \omega(z + \lambda z)(x).$$

Hence for every $\psi \in \mathfrak{D}(\Delta)$ we have

$$\int_\Delta \frac{\partial \psi}{\partial \bar{\lambda}}(\lambda)\, u(z + \lambda x)\, d\lambda = -\int_\Delta \psi(\lambda)\, \omega(z + \lambda x)\,(x)\, d\lambda,$$

where the F-vector valued integral is the Bochner integral.

We note that

$$(1) \qquad \frac{\partial}{\partial \bar{\lambda}} u(z + \lambda x) = [\bar{\partial}]\, u(z + \lambda x)(x).$$

A result of Mazet [10] states that every locally bounded weak solution of $[\bar{\partial}]u=\omega$ has the same regularity properties as ω, provided ω and its derivaties are locally bounded.

Definition 4

Let E be a complex Banach space. Let $\omega:\Omega\to\Lambda^{(0,q)}(E)$ *be a* $\mathscr{C}^{\infty}(0,q)$ *form on an open subset* Ω *of E. Define for each* $x\in\Omega$ *and for* $y_1,\dots,y_{q+1}\in E$, $q\geq 1$

$$(\bar{\partial}\omega)(x)(y_1,\dots,y_{q+1})=\frac{1}{q+1}\sum_{k=1}^{q+1}(-1)^{k+1}[\bar{\partial}]\,\omega(x)(y_k)(y_1,\dots,\hat{y}_k,\dots,y_{q+1})$$

where $\hat{y}_k$ *indicates that* y_k *has been omitted.*

If we denote by $\xi^{(0,q)}(\Omega)$ *the vector space of* $\mathscr{C}^{\infty}(0,q)$ *forms on* Ω, *an operator* $\bar{\partial}:\xi^{(0,q)}(\Omega)\to\xi^{(0,q+1)}(\Omega)$ *has been defined. We note that* $\bar{\partial}\omega(x)$ *is the alternating component of* $[\bar{\partial}]\,\omega(x)\in\mathscr{L}_{\mathbf{C}}(E,\Lambda^{(0,q)}(E))$. *When dealing with a function* $u\colon E\to\mathbf{C}$ *we set* $[\bar{\partial}]u=\bar{\partial}u$.

When $u:\Omega\to\mathscr{L}(\bar{E})=\mathscr{L}_{\mathbf{C}}(\bar{E},\mathbf{C})=\mathscr{L}_{\bar{\mathbf{C}}}(E)$ is a $\mathscr{C}^{\infty}(0,1)$ form on Ω, we observe that $[\bar{\partial}]\,u(x)\in\mathscr{L}(^2\bar{E})$ and $\bar{\partial}w(x)\in\Lambda^{(0,2)}(E)$ is the anty-symmetric component of $[\bar{\partial}]u(x)$. Hence, we may write.

$$[\bar{\partial}]u=\bar{\partial}u+G(u) \text{ where } G(u)(x) \text{ is the symmetric component of } [\bar{\partial}]\,u(x). \tag{2}$$

The expression (2) leads to two basic difficulties when trying to solve $\bar{\partial}u=\omega$ on the whole space for a (0,2) form. The first difficulty concerns the integral representations. On the whole space we only have Cauchy integral representations and it is easily seen that such a representation envolves $[\bar{\partial}]$ and not $\bar{\partial}$. (See expression (1)). The second difficulty concerns Mazet's result which involves again $[\bar{\partial}]$ and not $\bar{\partial}$. So the following question arises: when is $[\bar{\partial}]u=\bar{\partial}u$? In Soraggi [15], we showed that even if u is the canonical solution of $\bar{\partial}u=\omega$, ω a holomorphic (0, 2) form on $\mathbf{C}^n$, it may happen that $[\bar{\partial}]\,u\neq\bar{\partial}u$ and hence it is necessary to provide a correction to the solution u. We obtained the following fundamental Lemma.

Lemma 5

Let ω: $\mathbf{C}^n\to\Lambda^{(0,2)}(\mathbf{C}^n)$ *be a (0,2) form on* $\mathbf{C}^n$ *with holomorphic coefficients. Suppose the coefficient functions are in* $L^2(\mathbf{C}^n,\varphi)$ *[the vector space of square integrable functions with respect to the* $\mathscr{C}^{\infty}$ *strictly* *plurisubharmonic (with constant $h=1/2$) weight φ]. Let u be the canonical solution of $\bar{\partial}u=\omega$. There exists a $\mathscr{C}^\infty$ (0, 1) form u_2 on $\mathbf{C}^n$ satisfying $\bar{\partial}u_2=0$ such that $u_1=u-u_2$ is a $\mathscr{C}^\infty$ (0, 1) form satisfying*

$$[\bar{\partial}]u_1=\bar{\partial}u_1=\omega \text{ and } ||u_1||_\varphi^2\leq 8\,||\omega||_\varphi^2.$$

Here for $u(z)=\sum_{j=1}^{n} a_j(z)d\bar{z}_j$ and $\omega(z)=\sum_{j<k} a_{jk}(z)\,d\bar{z}_j \wedge d\bar{z}_k$ forms with coefficient functions in $L^2(\mathbf{C}^n, \varphi)$, we let

$$||u||_\varphi^2=\sum_{j=1}^{n}\int_{\mathbf{C}^n}|a_j(z)|^2 e^{-\varphi(z)}\frac{dV}{(2\pi)^n}$$

$$||\omega||_\varphi^2=\sum_{j<k}\int_{\mathbf{C}^n}|a_{jk}(z)|^2 e^{-\varphi(z)}\frac{dV}{(2\pi)^n}$$

We first construct a solution of $\bar{\partial}u=\omega$ when ω is a holomorphic G – analytic and continuous) (0,2) – form which is bounded on the bounded subsets of a separable complex Hilbert space H. Our construction of a solution u of bounded type on a dense subspace H_o follows Raboin's construction. This construction is obtained in three steps. The first step consists of projecting the $\bar{\partial}$ – problem onto finite dimensional subspaces and choosing a good solution $\tilde{u}_n$ on these subspaces by applying Lemma 5. The second step consists of constructing cylindrical solutions u_n on H from the solutions $\tilde{u}_n$. The last step consists of constructing a weak solution on H_o and then applying Mazet's result to show that u is a $\mathscr{C}^\infty$ solution.

We obtained the following result.

Theorem 6

Let ω: $H\to\Lambda^{0,2)}(H)$ be a holomorphic (G – analytic and continuous) (0,2) form which is bounded on the bounded subsets of H. There exists $u: H_o\to\mathscr{L}(\bar{H}_o)$, where H_o is a dense subspace of H, such that u is a $\mathscr{C}^\infty$ (0, 1) form bounded on the bounded subsets of H_o, moreover the derivatives of u are locally bounded and $\bar{\partial}u=\omega$.

We observe that the subspace H_o of H is defined by $H_o=TH$, where T is an injective, self-adjoint Hilbert-Schmidt operator and the inner product on H_o is defined by

$(x;y)_{H_o}=(T^{-1}x;\ T^{-1}y)_H$ where $(\cdot;\cdot)_H$ is the inner product of H.

Now, the $\bar{\partial}$ – problem can be studied on a scale of Hilbert spaces with nuclear injections by using Theorem 6 and following an argument of Colombeau-Perrot [6]. (See also Colombeau [5]).

Theorem 7

Let $H_o \subset H_l$ be separable, complex Hilbert spaces with a nuclear injection. Let $\tilde{\omega}_l: H_l \to \Lambda^{(0,2)}(H_l)$ be a holomorphic (0,2) form on H_l. There exists $u: H_o \to \mathscr{L}(\bar{H}_o)$ such that u is a $\mathscr{C}^\infty$ (0, 1) form, bounded on the bounded subsets of H_o, its derivatives are locally bounded and $\bar{\partial} u = \tilde{\omega}$. ($\tilde{\omega}$: $H_o \to \Lambda^{(0,2)}(H_o)$) is the restriction of $\tilde{\omega}_l$ to H_o).

Now, we can study the $\bar{\partial}$-problem on a D.F.N. space. Let us observe that, since E is a D.F. space, the space $\mathscr{L}(^q\bar{E})$ of continuous q anti-linear forms- endowed with the topology of uniform convergence on the bounded subsets of E-- is a metrizable locally convex space. Since E is a nuclear and dual nuclear space, $\mathscr{L}(^q\bar{E})$ is also nuclear. Hence $\Lambda^{(0,q)}(E)$ is a Fréchet-Schwartz space. Let ω: $\Omega \to \Lambda^{(0,q)}(E)$ be a holomorphic (G-analytic and continuous) mapping. Since E is the strong dual of a Fréchet-Schwartz space, ω is locally bounded. By applying a result of Colombeau-Mujica[7] we can show that there exists a convex, balanced, open subset V of E such that ω factors as in the following diagram,

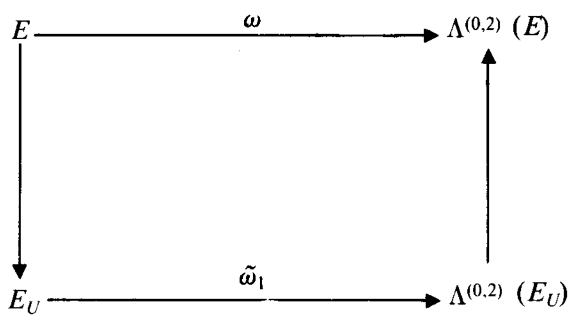

Diagram 1

where $\tilde{\omega}_1$ is a holomorphic mapping.

Now, since E is nuclear, there exist convex, balanced, open subsets W and V, $W \subset V \subset U$ such that $\tilde{E}_V \cong \ell^2 \cong \tilde{E}_W$, $\tilde{E}_W \xrightarrow{i} \tilde{E}_V$, $E \to \tilde{E}_W$ are nuclear mappings. By extending the mappings from E_U to $\tilde{E}_U$ and observing that $\Lambda^{(0,2)}(\tilde{E}_U) = \Lambda^{(0,2)}(E_U)$, we have the following diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\quad\omega\quad} & \Lambda^{(0,2)}(E) \\
\downarrow & & \uparrow \\
\tilde{E}_W & \xrightarrow{\quad\tilde{\omega}_3\quad} & \Lambda^{(0,2)}(\tilde{E}_W) \\
\hat{\imath}\ \downarrow & & \uparrow \\
\tilde{E}_V & \xrightarrow{\quad\tilde{\omega}_2\quad} & \Lambda^{(0,2)}(\tilde{E}_V) \\
\downarrow & & \uparrow \\
\tilde{E}_U & \xrightarrow{\quad\tilde{\omega}_1\quad} & \Lambda^{(0,2)}(\tilde{E}_U)
\end{array}
$$

Diagram 2

Now, by considering quotient mappings and spaces, if necessary, we can suppose without loss of generality that the following diagram commutes.

$$
\begin{array}{ccc}
E & \xrightarrow{\quad\omega\quad} & \Lambda^{(0,2)}(E) \\
\downarrow & & \uparrow \\
\tilde{E}_W & \xrightarrow{\quad\tilde{\omega}\quad} & \Lambda^{(0,2)}(\tilde{E}_W) \\
\hat{\imath}\ \downarrow & & \uparrow \\
\tilde{E}_V & \xrightarrow{\quad\tilde{\omega}_1\quad} & \Lambda^{(0,2)}(\tilde{E}_V)
\end{array}
$$

Diagram 3

Where $\tilde{E}_W$, $\tilde{E}_V$ are separable Hilbert spaces, $\hat{\imath}$ is an injective nuclear mapping and $\tilde{\omega}_1$ and $\tilde{\omega}$ are holomorphic mappings. By applying Theorem 7, there exists $\tilde{u}: \tilde{E}_W \to \mathscr{L}_{\bar{\mathbb{C}}}(\tilde{E}_W)$ such that u is a $\mathscr{C}^\infty$ (0,1)—form, bounded on the bounded subsets of $\tilde{E}_W$, its derivatives are locally bounded and $\bar{\partial}\tilde{u} = \tilde{\omega}$. By taking a convex balanced open subset $W_1 \subset W$ such that $\tilde{E}_{W_1} \to \tilde{E}_W$ is nuclear, we can define $\tilde{u}_1 : \tilde{E}_{W_1} \to \mathscr{L}_{\bar{\mathbb{C}}}(\tilde{E}_{W_1})$ such that $\tilde{u}_1$ is a $\mathscr{C}^\infty$ (0,1) form of uniform bounded type ($\tilde{u}_1$ and all its derivatives are bounded on bounded subsets) and $\bar{\partial}\tilde{u}_1 = \tilde{\omega}^*$ ($\tilde{\omega}^*: \tilde{E}_{W_1} \to \Lambda^{(0,2)}(\tilde{E}_{W_1})$). Hence $u: E \to \mathscr{L}_{\mathbb{C}}(\bar{E})$ can be defined such that u factors through $\tilde{u}_1$. This implies that u is a $\mathscr{C}^\infty$ (0,1) form of uniform bounded type and we write $\bar{\partial}u = \omega$.

So we have obtained the following.

Theorem 8

Let E be a D.F.N. space and let $\omega: E \to \Lambda^{(0,2)}(E)$ be a holomorphic (0,2) form. There exists a $\mathscr{C}^\infty$ (0,1) form u of uniform bounded type on E such that $\bar{\partial}u = \omega$.

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