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On a Nonlinear Wave Equation with Damping

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Dedicated to Jacques Louis Lions on the occasion of his 60th birthday.

ABSTRACT. In this work the authors prove the existence of global solutions for damping nonlinear wave equation $u''+M\left(|A^{\frac{1}{2}}u|^2\right)Au+A^{\alpha}u'=f$, $0<\alpha\leq 1$. Uniqueness is obtained for $\frac{1}{2}\leq\alpha\leq 1$. They also prove the exponential decay for the energy, when $0<\alpha\leq 1$.

INTRODUCTION

In this work we are concerned with global existence and exponential decay for solutions of the mixed problem:

$$\frac{\partial^2 u}{\partial t^2}-M\left(\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u+(-\Delta)^\alpha\frac{\partial u}{\partial t}=f, \text{ in } Q, \text{ with } 0<\alpha\leq 1$$

$$u(x,t)=0 \text{ for } (x,t)\in\Sigma \tag{1}$$

$$u(x,0)=u_o(x), \frac{\partial u}{\partial t}(x,0)=u_1(x) \text{ in } \Omega,$$

where $M(s)$ is a positive continuous function on $[0,\infty[$; Ω is a bounded open set of $\mathbf{R}^n$, with smooth boundary Γ; Q is the cylinder $\Omega\times]0,\infty[$ of $\mathbf{R}^{n+1}$, with lateral boundary $\Sigma=\Gamma\times[0,\infty[$; $\Delta=\sum_{i=1}^n \partial^2/\partial x_i^2$ is the Laplace operator and $|\nabla u(x,t)|^2=\sum_{i=1}^n |\partial u/\partial x_i|^2$.

The equation $(1)_1$, without damping $(-\Delta)^\alpha\frac{\partial u}{\partial t}$, has its origin in the study of vibrations of an elastic string (cf. Carrier [3]). To obtain his model, he

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admits only vertical component for the tension on the string of length L. Using a linear Hooke's law, he obtained the model:

$$\rho a \frac{\partial^2 y}{\partial t^2} = (\tau_o + \frac{aE}{2L} \int_0^L (\frac{\partial u}{\partial x})^2 dx) \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L.$$

τ_o is the initial tension on the string; E the Young's modulus of the material; ρ density; a the area of the cross section; $u(x, t)$ the vertical displacement of the point x of the string, at time t. In this work we are interested in the case $\tau_o > 0$, that is, we have an initial tension. If we consider a Hooke's law of the type:

$$\tau - \tau_o = \sigma \left(\frac{S-L}{L} \right),$$

σ a function not necessarily linear, $\tau - \tau_o$ the variation of the tension, $S-L$ the variation of the lenght of the string, then we obtain a nonlinear model of type $(1)_1$, without damping, with

$$M(s) = m_o + \theta(s),$$

$m_o > 0$, θ non linear. Therefore the so called Carrier-Narashimham model is:

$$(*) \qquad \frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u(x, t)|^2 dx \right) \Delta u = f.$$

When u_o, u_1 are chosen in a regular class of functions, Pohozhaev [17], Lions [7], Arosio-Spagnolo [1] proved that (*), $(1)_2$, $(1)_3$ has regular solution in $x \in \Omega$, global in t, that is, $0 \leq t < \infty$. For the case $n = 1$, cf. Bernstein [2], Dickey [4]. Therefore, if we restrict $u_o \in H_o^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_o^1(\Omega)$, $f \in L^1(0, T; H_o^1(\Omega))$, no global solutions has been proved to (*), $(1)_2$, $(1)_3$ and no blow up studied. With this choice for u_o, u_1, f we can prove that there exists a certain $T_o > 0$ and a solution for (*), $(1)_2$, $(1)_3$ defined only on $\Omega \times [0, T_o]$, cf. Ebihara-Medeiros-Milla Miranda [5].

However, with the perturbation $-\Delta \frac{\partial u}{\partial t}$, i.e., the case $\alpha = 1$ in $(1)_1$, there exist results on global solutions in t and on decay, when (u_o, u_1, f) belongs to $H_o^1(\Omega) \cap H^2(\Omega) \times H_o^1(\Omega) \times L^1(0, T; H_o^1(\Omega))$, cf. Nishihara [14], Yamada [18]. It is interesting to look Tsutsumi [17].

In the present work, we obtain global solutions of the mixed problem (1), that is, mixed problem for (*) with the damping $(-\Delta)^\alpha \frac{\partial u}{\partial t}$, if $0 < \alpha \leq 1$, when $u_o \in H_o^1(\Omega) \cap D((-\Delta)^\alpha)$, $u_1 \in L^2(\Omega)$, $f \in L^1(0, T; L^2(\Omega))$. Our motivation was to have information when $\alpha \to 0$, but no result we know up today. If we suppose $M \in C^1([0,\infty[;\mathbf{R})$ we prove that when $\frac{1}{2} \leq \alpha \leq 1$ we have uniqueness for solutions, cf. Theorem 1.2.

In Section 2 we study the exponential decay for solutions obtained in Section 1, $0 < \alpha \leq 1$, with $f = 0$ to avoid technalities, following the ideas of Haraux-Zuazua [6], cf. also Nakao [13], [14], Zuazua [22], Muñoz Rivera [12], Strauss [18]. In Matos-Pereira [10] they obtained algebraic decay for the energy, in the damping case $\alpha = 1$, with $M(s) = s$.

Another question, proposed by Lions [7] for (*), $(1)_2$, $(1)_3$, is to study the case (u_o, u_1, f) in $H_o^1(\Omega) \times L^2(\Omega) \times L^1(0, T; L^2(\Omega))$. An answer to this question was done by Medeiros-Milla Miranda [11]. They proved that the best choice is not the above but the following:

$$(u_o, u_1, f) \in D((-\Delta)^{3/4}) \times D((-\Delta)^{1/4}) \times L^1(0, T; D((-\Delta)^{1/4})).$$

With $D((-\Delta)^{\beta})$ one represents the domain of the operator $(-\Delta)^{\beta}$. With this choice the authors proved, in [11], that (*), $(1)_1$, $(1)_2$ has at least one weak solution, when Ω is a bounded open set. For the unbounded case cf. Matos [9], as an application of the diagonalization theorem by Von Neumann-Diximier, cf. Lions-Magenes [8].

The present work is dedicated to Jacques Louis Lions on the occasion of his 60th anniversary, as an acknowledgement of our deep admiration for his scientific work and reconnaissance for his permanent support to our research work.

1. GLOBAL SOLUTIONS: EXISTENCE AND UNIQUENESS

Let V and H be real Hilbert spaces with Hilbert structure given, respectively, by $((\cdot,\cdot))$, $||\cdot||$, $(\cdot,\cdot)$, $|\cdot|$. We suppose $V \subset H$ continuously and V dense in H. Let A be the operator defined by the triplet $\{V, H, ((\cdot,\cdot))\}$. As it is known, A is a positive self adjoint operator of H with domain $D(A) \subset H$, dense and $D(A^{\frac{1}{2}}) = V$. We also know that $((u, u)) = (Au, v)$ for all $u \in D(A)$, $v \in V$ which implies $((u, v)) = |A^{\frac{1}{2}} u|^2$. If we suppose $V \subset H$ compact, then the spectral problem $((w, v)) = \lambda (w, v)$ for all $v \in V$, has for solution a sequence of vectors $(w_\upsilon)_{\upsilon \in \mathbf{N}}$, $w_\nu \in H$, for all $\upsilon \in \mathbf{N}$, called eigenvectors of A and a sequence $(\lambda_\upsilon)_{\upsilon \in \mathbf{N}}$ of real numbers λ_υ called eigenvalues of A, such that $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_\upsilon \leq \ldots$ diverges to $+\infty$, and $((w_\upsilon, v)) = \lambda_\upsilon (w_\upsilon, v)$ for all $v \in V$, $\upsilon \in \mathbf{N}$. The sequence $(w_\upsilon)_{\upsilon \in \mathbf{N}}$ is dense in V and we suppose it is orthonormalized in H.

With this framework, the mixed problem (1) can be written:

$$u'' + M\left(\left|A^{\frac{1}{2}}u\right|^2\right) Au + A^{\alpha} u' = f \text{ in }]0, T[, \; 0<\alpha\leq 1$$
$$u(0) = u_o, \quad u'(0) = u_1 \tag{2}$$

In (2), u' denotes the derivative of u, with respect to t, in the sense of the vector distributions on $]0, T[$. We assume on M the natural condition:

$$M \in C^o([0,\infty[; \mathbf{R}), \; M(s) \geq m_o > 0 \text{ for all } s \geq 0. \tag{3}$$

Theorem 1.1: *Suppose $M(s)$ satisfies (3), $0<\alpha\leq 1$,*

$$u_o \in V \cap D(A^{\alpha}), \quad u_l \in H, \quad f \in L^1(0, T; H). \tag{4}$$

Then, there exists one function $u:[0, T] \to H, 0<T<\infty$, satisfying the conditions:

$$u \in L^{\infty}(0, T; V \cap D(A^{\alpha})) \cap L^2(0, T; D(A^{\frac{\alpha+1}{2}}))$$
$$u' \in L^{\infty}(0, T; H) \cap L^2(0, T; D(A^{\frac{\alpha}{2}})) \tag{5}$$

$$\frac{d}{dt}(u'(t), v) + M\left(\left|A^{\frac{1}{2}}u(t)\right|^2\right)\left(A^{\frac{1}{2}}u(t), A^{\frac{1}{2}}v\right) + (A^{\frac{\alpha}{2}}u'(t), A^{\frac{\alpha}{2}}v) = (f(t), v) \tag{6}$$

in $D'(0, T)$ *for all $v \in V$. We say u is weak solution of (2).*

$$u(0) = u_o, \quad u'(0) = u_l \tag{7}$$

Proof: Denote by V_m the subspace of $V \cap D(A^{\alpha})$ generated by the first m eigenvectors $\omega_1, \omega_2, \ldots, \omega_m$ of A. Let $u_m(t) \in V_m$ defined by:

$$(u''_m(t), v) + M\left(\left|A^{\frac{1}{2}}u_m(t)\right|^2\right)(Au_m(t), v) + (A^{\alpha}u'_m(t), v) = (f(t), v) \tag{8}$$

for all $v \in V_m$.

$$u_m(0) = u_{om} \to u_o \text{ strongly in } V \cap D(A^{\alpha}) \tag{9}$$

$$u'_m(0) = u_{1m} \to u_1 \text{ strongly in } H. \tag{10}$$

The system (8) plus initial conditions (9), (10) has solution on $[0, t_m[$. Its extension to the interval $[0, T]$ is a consequence of a priori estimates. The crucial point in the proof of Theorem 1.1 is to obtain strong convergence of solutions $(u_m)_{m\in\mathbf{N}}$ in the space $L^2(0, T; D(A^{\frac{1}{2}}))$, in order to obtain the limit of the nonlinear term $M\left(\left|A^{\frac{1}{2}}u_m(t)\right|^2\right)$. We need two a priori estimates. In the computation we use u in place of u_m.

FIRST A PRIORI ESTIMATE: Let us represent by $\hat{M}(\lambda)$ a primitive of $M(s)$, i.e.,

$$\hat{M}(\lambda)=\int_0^\lambda M(s)\,ds.$$

Taking $v=2u'$ in (8), integrating from 0 to t and applying Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}&|u'(t)|^2+\hat{M}\left(\left|A^{\frac{1}{2}}u(t)\right|^2\right)+2\int_0^t|A^{\frac{\alpha}{2}}u'(s)|^2\,ds\\ &\leq 2\int_0^t|f(s)|\,|u'(s)|\,ds+|u_{1m}|^2+\hat{M}\left(\left|A^{\frac{1}{2}}u_{om}\right|^2\right)\end{aligned}$$

Whence, by the assumption (3) on $M(s)$,

$$\begin{aligned}&|u'(t)|^2+m_o\left|A^{\frac{1}{2}}u(t)\right|^2+2\int_0^t|A^{\frac{\alpha}{2}}u'(s)|^2\,ds\\ &\leq 2\int_0^t|f(s)|\,|u'(s)|\,ds+|u_1|^2+\hat{M}\left(\left|A^{\frac{1}{2}}u_o\right|^2\right).\end{aligned}\tag{11}$$

Observe that $\hat{M}(\lambda)\geq m_o\lambda$. From (11) it follows:

$$|u'(t)|^2\leq|u_1|^2+\hat{M}\left(\left|A^{\frac{1}{2}}u_o\right|^2\right)+2\int_0^t|f(s)|\,|u'(s)|\,ds.$$

Since $|f(s)|\in L^1(0,T)$, this inequality implies, by Gronwall's lemma:

$$\int_0^t|f(s)|\,|u'(s)|\,ds$$

is bounded for all t in $[0,T]$. Therefore, from this estimate and (11), we obtain:

$$|u_m'(t)|^2\leq C;\ \left|A^{\frac{1}{2}}u_m(t)\right|^2\leq C;\ \int_0^t\left|A^{\frac{\alpha}{2}}u_m'(s)\right|^2ds\leq C\tag{12}$$

for all $t\in[0,T]$, where $C>0$ is a constant independent of m.

SECOND A PRIORI ESTIMATE: We take $v=A^\alpha u$ (8); as above u in place of u_m. We get:

$$\frac{d}{dt}(u',A^\alpha u)-|A^{\frac{\alpha}{2}}u'|^2+M\left(\left|A^{\frac{1}{2}}u\right|^2\right)\left|A^{\frac{\alpha+1}{2}}u\right|^2+\frac{1}{2}\frac{d}{dt}|A^\alpha u|^2=(f,A^\alpha u).$$

Integrating from 0 to t, we find:

$$\int_0^t M\left(\left|A^{\frac{1}{2}}u\right|^2\right)\left|A^{\frac{\alpha+1}{2}}u\right|^2 ds+\frac{1}{2}|A^\alpha u|^2=-(u',A^\alpha u) \\ +(u_{1m}, A^\alpha u_{om})+\int_0^t |A^{\frac{\alpha}{2}}u'|^2 ds+\frac{1}{2}|A^\alpha u_{om}|^2+\int_0^t (f, A^\alpha u)ds. \tag{13}$$

By the conditions (9), (10) on u_{om}, u_{1m}; by the first estimate; by the assumption (3) on M and by the inequality $2pq\leq\frac{p^2}{\beta}+\beta q^2$, $\beta>0$, we obtain, from (13):

$$m_o\int_0^t \left|A^{\frac{\alpha+1}{2}}u\right|^2 ds+\left(\frac{1}{2}-\beta\right)|A^\alpha u|^2\leq K+\int_0^t |f(s)||A^\alpha u(s)|\,ds \tag{14}$$

for $0<\beta<\frac{1}{2}$, $K>0$ constant.

We obtain from (14),

$$|A^\alpha u(t)|^2\leq a+b\int_0^t |f(s)||A^\alpha u(s)|\,ds,$$

with $|f(s)|\in L^1(0,T)$. By Gronwall's lemma, the last inequality and (14), it follows:

$$|A^\alpha u_m(t)|^2\leq C_1,\quad \int_0^t \left|A^{\frac{\alpha+1}{2}}u_m(s)\right|^2 ds\leq C_1 \tag{15}$$

with C_1 independent of m, for all t in $[0,T]$.

We have:

$$D(A^{\frac{\alpha+1}{2}})\subset D(A^{\frac{1}{2}})\subset H,\quad 0<\alpha\leq 1 \tag{16}$$

with continuous injections. We have compact embedding of $D(A^{\frac{\alpha+1}{2}})$ into $D(A^{\frac{1}{2}})$.

From (12), (15) we have:

$$(u_m)_{m\in\mathbf{N}} \text{ bounded in } L^2(0,T;D(A^{\frac{\alpha+1}{2}})) \\ (u'_m)_{m\in\mathbf{N}} \text{ bounded in } L^2(0,T;H) \tag{17}$$

By (16), (17) and Aubin-Lions compactness theorem, we can extract, from $(u_m)_{m\in\mathbf{N}}$, a subsequence $(u_\mu)_{\mu\in\mathbf{N}}$, such that:

$$(u_\mu)_{\mu\in\mathbf{N}} \text{ converges to } u \text{ strongly in } L^2(0,T;D(A^{\frac{1}{2}})). \tag{18}$$

The vector u obtained from (12), (15), (18), is such that:

$$
\begin{aligned}
&(u_\mu)_{\mu\in\mathbf{N}} \text{ converges to } u \text{ weak star in } L^\infty(0,T; V\cap D(A^\alpha)) \\
&\qquad \text{and weakly in } L^2(0,T; D(A^{\frac{\alpha+1}{2}})) \\
&(u'_\mu)_{\mu\in\mathbf{N}} \text{ converges to } u' \text{ weak star in } L^\infty(0,T;H) \\
&\qquad \text{and weakly in } L^2(0,T; D(A^{\frac{\alpha}{2}})).
\end{aligned}
\tag{19}
$$

The convergences (18), (19) imply that we can take the limit, when $\mu\to\infty$, in the approximated system (8), to obtain (6). We also prove that u satisfies the initial condition (7) and Theorem 1.1 is proved. Q.E.D.

Corollary 1.1: *Under the hypothesis of Theorem 1.1, with $f=0$, there exists a function u: $[0,\infty[\to H$, satisfying the conditions:*

$$u\in L^\infty(0,\infty; V\cap D(A^\alpha))\cap L^2(0,\infty; D(A^{\frac{\alpha+1}{2}}))$$

$$u'\in L^\infty(0,\infty; H)\cap L^2(0,\infty; D(A^{\frac{\alpha}{2}}))$$

$$u''+M\left(\left|A^{\frac{1}{2}}u\right|^2\right)Au+A^\alpha u'=0 \textit{ weakly as (6)}$$

$$u(0)=u_o, \quad u'(0)=u_1.$$

The next step of our work is dedicated to analyse the uniqueness. In order to be clear we fixe notations and prove two lemmas.

Let $\beta>0$ be a real number. We consider $D(A^\beta)$ with the norm $||u||_\beta=|A^\beta u|$. Then,

$$A^\beta : D(A^\beta)\to H$$

is linear and continuous. It follows that the adjoint $A^{\beta*}$,

$$A^{\beta*}: H'\to (D(A^\beta))',$$

is also linear and continuous, with

$$\langle A^{\beta*}f, u\rangle_{\beta'\beta}=\langle f, A^\beta u\rangle_{H'\times H} \tag{20}$$

where $\langle,\rangle_{\beta'\beta}$ is the duality between $(D(A^\beta))'$ and $D(A^\beta)$. Note that E' represents the dual of E.

Lemma 1.1: *If we identify H with H' by Riesz isomorphism, then $A^{\beta*}$ is an extension of A^β.*

Proof: If $f \in H^1$, then $\langle f, v\rangle = \langle Ju, v\rangle = (u, v)$, where J is Riesz's isomorphism. We have

$$D(A^\beta) \subset H = H' \subset (D(A^\beta))'$$

continuous and dense. Let $v \in D(A^\beta)$, and $v \in H$ and $Jv \in H'$. It follows, from (20), that:

$$\langle A^{\beta *} Jv, u\rangle_{\beta' \beta} = \langle Jv, A^\beta u\rangle_{H' \times H} = (v, A^\beta u)$$
$$= (A^\beta v, u) = \langle J A^\beta v, u\rangle_{\beta' \beta},$$

what implies $A^{\beta *} Jv = J A^\beta v$. Q.E.D.

Let us consider $\frac{1}{2} \leq \alpha \leq 1$. Then

$$D\left(A^{\frac{\alpha}{2}}\right) \subset D\left(A^{\frac{1-\alpha}{2}}\right) \text{ and } \left(D\left(A^{\frac{1-\alpha}{2}}\right)\right)' \subset \left(D\left(A^{\frac{\alpha}{2}}\right)\right)' \tag{21}$$

dense and continuous.

Lemma 1.2: *If* $\frac{1}{2} \leq \alpha \leq 1$ *and*

$$u \in L^2\left(0, T; D\left(A^{\frac{\alpha+1}{2}}\right)\right), \quad u' \in L^2\left(0, T; D\left(A^{\frac{\alpha}{2}}\right)\right),$$

then

$$\frac{d}{dt}\left(A^{\frac{\alpha+1}{2}} u, A^{\frac{1-\alpha}{2}} u\right) = 2\left(A^{\frac{\alpha+1}{2}} u, A^{\frac{1-\alpha}{2}} u'\right).$$

Proof: Let $W(0, T)$ the space

$$\left\{v;\ v \in L^2\left(0, T; D\left(A^{\frac{\alpha+1}{2}}\right)\right),\ v' \in L^2\left(0, T; D\left(A^{\frac{\alpha}{2}}\right)\right)\right\}$$

normed by

$$||v||^2_{W(0,T)} = ||v||^2_{L^2(0,T;D(A^{\frac{\alpha+1}{2}}))} + ||v'||^2_{L^2(0,T;D(A^{\frac{\alpha}{2}}))}.$$

We know that $D([0, T]; D(A^{\frac{\alpha+1}{2}}))$ is dense in $W(0, T)$, cf. Lions-Magenes [8], p. 13. Let $\varphi \in D([0, T]; D(A^{\frac{\alpha+1}{2}}))$. Then,

$$\frac{d}{dt}\left(A^{\frac{\alpha+1}{2}}\varphi,\ A^{\frac{1-\alpha}{2}}\varphi\right)=\left(A^{\frac{\alpha+1}{2}}\varphi',\ A^{\frac{1-\alpha}{2}}\varphi\right)+\left(A^{\frac{\alpha+1}{2}}\varphi,\ A^{\frac{1-\alpha}{2}}\varphi'\right) \tag{22}$$
$$=\left(A^{\alpha}\left(A^{\frac{1-\alpha}{2}}\varphi'\right),\ A^{\frac{1-\alpha}{2}}\varphi\right)+\left(A^{\frac{\alpha+1}{2}}\varphi,\ A^{\frac{1-\alpha}{2}}\varphi'\right)$$
$$=\left(A^{\frac{1-\alpha}{2}}\varphi',\ A^{\alpha}\left(A^{\frac{1-\alpha}{2}}\varphi\right)\right)+\left(A^{\frac{\alpha+1}{2}}\varphi,\ A^{\frac{1-\alpha}{2}}\varphi'\right),$$

since $A^{\frac{1-\alpha}{2}}\varphi$, $A^{\frac{1-\alpha}{2}}\varphi'$ belong to $D(A^{\alpha})$.

From (22), we have:

$$\frac{d}{dt}\left(A^{\frac{\alpha+1}{2}}\varphi,\ A^{\frac{1-\alpha}{2}}\varphi\right)=2\left(A^{\frac{\alpha+1}{2}}\varphi,\ A^{\frac{1-\alpha}{2}}\varphi'\right). \tag{23}$$

If u satisfies the condition in Lemma 2.1, then there exists a sequence $(\varphi_v)_{v\in\mathbf{N}}$ of $D([0,T];\ D\left(A^{\frac{\alpha+1}{2}}\right))$ such that

$$\begin{aligned}&\lim \varphi_v=u \text{ in } L^2(O,T;D\left(A^{\frac{\alpha+1}{2}}\right))\\&\lim \varphi_v'=u' \text{ in } L^2(O,T;D\left(A^{\frac{1-\alpha}{2}}\right))\end{aligned} \tag{24}$$

From (24) we obtain:

$$\lim\left(A^{\frac{\alpha+1}{2}}\varphi_v,\ A^{\frac{1-\alpha}{2}}\varphi_v\right)=\left(A^{\frac{\alpha+1}{2}}u,\ A^{\frac{1-\alpha}{2}}u\right) \text{ in } L^1(0,T)$$
$$\lim\left(A^{\frac{\alpha+1}{2}}\varphi_v,\ A^{\frac{1-\alpha}{2}}\varphi_v'\right)=\left(A^{\frac{\alpha+1}{2}}u,\ A^{\frac{1-\alpha}{2}}u'\right) \text{ in } L^1(0,T)$$

Considering these convergences in the space $\mathbf{D}'(0,T)$ and taking the limits of (23) with $\varphi=\varphi_v$, we prove Lemma 1.2. Q.E.D.

ESTIMATE FOR u'':

Let u the solution of Theorem 1.1 with $f=0$. Then, u satisfies:

$$-\int_0^T (u',\theta' v)\,dt+\int_0^T M\left(\left|A^{\frac{1}{2}}u\right|^2\right)\left(A^{\frac{\alpha+1}{2}}u,\ \theta A^{\frac{1-\alpha}{2}}v\right)dt \tag{25}$$
$$+\int_0^T\left(A^{\frac{\alpha}{2}}u',\ \theta A^{\frac{\alpha}{2}}v\right)dt=0$$

for $\theta\in D(0,T)$ and v eigenvector of A. Observe that $M\left(\left|A^{\frac{1}{2}}u\right|^2\right)\in L^\infty(0,T)$ because $(u_m)_{m\in\mathbf{N}}$ is bounded in $L^\infty(0,T;\ D(A^{\frac{1}{2}}))$. The eigenvectors of A are dense in $D\left(A^{\frac{1-\alpha}{2}}\right)\cap D\left(A^{\frac{\alpha}{2}}\right)$ which implies that (25) is true for all $v\in D\left(A^{\frac{1-\alpha}{2}}\right)\cap D\left(A^{\frac{\alpha}{2}}\right)$. From (25) with such vectors v, we obtain:

$$-\int_0^T (u', \theta' v)\,dt+\int_0^T M\left(\left|A^{\frac{1}{2}}u\right|^2\right)\left\langle\left(A^{\frac{1-\alpha}{2}}\right)^*\left(A^{\frac{\alpha+1}{2}}u\right), \theta v\right\rangle_{\left(\frac{1-\alpha}{2}\right)'\left(\frac{1+\alpha}{2}\right)} dt$$
$$+\int_0^T\left\langle\left(A^{\frac{\alpha}{2}}u\right)^*\left(A^{\frac{\alpha}{2}}u\right), \theta v\right\rangle_{\left(\frac{\alpha}{2}\right)'\left(\frac{\alpha}{2}\right)} dt=0.$$

This implies:

$$u''+M\left(\left|A^{\frac{1}{2}}u\right|\right)\left(A^{\frac{1-\alpha}{2}}\right)^*\left(A^{\frac{1-\alpha}{2}}u\right)u+\left(A^{\frac{\alpha}{2}}\right)^* A^{\frac{\alpha}{2}}u'=0 \qquad (26)$$

in the sense of $D'\left(0, T; \left(D\left(A^{\frac{1-\alpha}{2}}\right)\right)'+\left(D\left(A^{\frac{\alpha}{2}}\right)\right)'\right)$.

Since,

$$M\left(\left|A^{\frac{1}{2}}u\right|^2\right)\left(A^{\frac{1-\alpha}{2}}\right)^*\left(A^{\frac{\alpha+1}{2}}u\right)\in L^2\left(0, T; \left(D\left(A^{\frac{1-\alpha}{2}}\right)\right)'\right)$$

and

$$\left(A^{\frac{\alpha}{2}}\right)^* A^{\frac{\alpha}{2}}u\in L^2\left(0, T; \left(D\left(A^{\frac{\alpha}{2}}\right)\right)'\right)$$

from (26):

$$u''\in L^2\left(0, T; \left(D\left(A^{\frac{1-\alpha}{2}}\right)\cap D\left(A^{\frac{\alpha}{2}}\right)\right)'\right).$$

Note that $u'\in D\left(A^{\frac{\alpha}{2}}\right)$. Then to make sense the duality (u'', u') we must have:

$$D\left(A^{\frac{\alpha}{2}}\right)\subset D\left(A^{\frac{1-\alpha}{2}}\right)\cap D\left(A^{\frac{\alpha}{2}}\right),$$

what implies

$$\frac{\alpha}{2}\geq\frac{1-\alpha}{2} \text{ or } \alpha\geq\frac{1}{2}.$$

It follows that for $\alpha\geq\frac{1}{2}$, we have:

$$u''\in L^2\left(0, T; \left(D\left(A^{\frac{\alpha}{2}}\right)\right)'\right). \qquad (27)$$

Remark 1.1: From

$$u' \in L^2(0, T; D(A^{\frac{\alpha}{2}})) \text{ and } u'' \in L^2(0, T; (D(A^{\frac{\alpha}{2}}))'),$$

it follows:

$$u' \in C^o([0, T]; H)$$

and

$$\frac{d}{dt}|u'|^2 = 2(u'', u')_{(\frac{\alpha}{2})'\frac{\alpha}{2}}.$$

Theorem 1.2: *(Uniqueness). Suppose*

$$M \in C^1([0, \infty[; \mathbf{R}), \quad M(s) \geq m_o > 0, \ \frac{1}{2} \leq \alpha \leq 1.$$

If u, v are two solutions of Theorem 1.1, then u = v.

Proof: If $\omega = u - v$, from (26) we have:

$$\omega'' + M(|A^{\frac{1}{2}}u|^2)(A^{\frac{1-\alpha}{2}})^*(A^{\frac{\alpha+1}{2}}u) - M(|A^{\frac{1}{2}}v|^2)(A^{\frac{1-\alpha}{2}})^*(A^{\frac{\alpha+1}{2}}v) \quad (28)$$
$$+(A^{\frac{\alpha}{2}})^* A^{\frac{\alpha}{2}}\omega' = 0 \text{ in } L^2(0, T; (D(A^{\frac{\alpha}{2}})'))$$
$$\omega(0) = \omega'(0) = 0.$$

Since $\omega' \in L^2(0, T; D(A^{\frac{\alpha}{2}}))$, from (28) we obtain:

$$\int_0^t \langle\omega'', \omega\rangle_{(\frac{\alpha}{2})'(\frac{\alpha}{2})} ds + \int_0^t M(|A^{\frac{1}{2}}u|^2)(A^{\frac{\alpha+1}{2}}\omega, A^{\frac{1-\alpha}{2}}\omega')ds \quad (29)$$
$$+\int_0^t [M(|A^{\frac{1}{2}}u|^2) - M(|A^{\frac{1}{2}}v|^2)](A^{\frac{\alpha+1}{2}}v, A^{\frac{1-\alpha}{2}}\omega')ds$$
$$+\int_0^t |A^{\frac{\alpha}{2}}\omega'|^2 ds = 0.$$

Remark 1.2: We have $A^{\frac{1-\alpha}{2}}u, A^{\frac{1}{2}}u \in D(A^{\frac{\alpha}{2}})$. Then

$$(A^{\frac{1}{2}}u, A^{\frac{1}{2}}u) = (A^{\frac{1}{2}}u, A^{\frac{\alpha}{2}}A^{\frac{1-\alpha}{2}}u) = (A^{\frac{\alpha+1}{2}}u, A^{\frac{1-\alpha}{2}}u).$$

From this equality and Lemma 1.2, we obtain:

$$\frac{d}{dt}\left|A^{\frac{1}{2}}u\right|^2=2\left(A^{\frac{\alpha+1}{2}}u,\ A^{\frac{1-\alpha}{2}}u'\right).$$

By Remark 1.1 and Remark 1.2, we get from (29):

$$\frac{1}{2}|\omega'(t)|^2+\int_0^t\frac{1}{2}\frac{d}{ds}\left[M\left(\left|A^{\frac{1}{2}}u\right|^2\right)||\omega||^2\right]ds+\int_0^t\left|A^{\frac{\alpha}{2}}\omega'\right|^2 ds \tag{30}$$

$$=\int_0^t \dot{M}'\left(\left|A^{\frac{1}{2}}u\right|^2\right)A^{\frac{\alpha+1}{2}}u,\ A^{\frac{1-\alpha}{2}}u'\Big)\,||\omega||^2\,ds$$

$$+\int_0^t M'(\xi)(||u||+||v||)\,(||v||-||u||)\left(A^{\frac{\alpha+1}{2}}v,\ A^{\frac{1-\alpha}{2}}\omega'\right)ds.$$

We have, observing that $\frac{\alpha}{2}\geq\frac{1-\alpha}{2}$,

$$\begin{aligned}&\left|M'\left(\left|A^{\frac{1}{2}}u\right|^2\right)\left(A^{\frac{\alpha+1}{2}}u,\ A^{\frac{1-\alpha}{2}}u'\right)||\omega||^2\right|\\&\leq c\left|A^{\frac{\alpha+1}{2}}u\right|^2||\omega||^2+c\left|A^{\frac{\alpha}{2}}u'\right|^2||\omega||^2.\end{aligned}\tag{31}$$

Also we have:

$$\begin{aligned}&|M'(\xi)|(||u||+||v||)(||v||-||u||)\left(A^{\frac{\alpha+1}{2}}v,\ A^{\frac{1-\alpha}{2}}\omega'\right)\\&\leq c_1||\omega||\left|A^{\frac{\alpha+1}{2}}\omega\right|\left|A^{\frac{\alpha}{2}}\omega'\right|\leq c_2\left|A^{\frac{\alpha+1}{2}}v\right|^2||\omega||^2+\eta\left|A^{\frac{\alpha}{2}}\omega'\right|^2\end{aligned}\tag{32}$$

$0<\eta<1.$

By (30), (31) and (32), we obtain:

$$\frac{1}{2}m_o||\omega||^2+(1-\eta)\int_0^t\left|A^{\frac{\alpha}{2}}\omega'\right|^2\leq c\int_0^t\left|A^{\frac{\alpha+1}{2}}u\right|^2||\omega||^2\,ds$$

$$+c\int_0^t\left|A^{\frac{\alpha}{2}}u'\right|^2||\omega||^2\,ds+c_2\int_0^t\left|A^{\frac{\alpha+1}{2}}v\right|^2||\omega||^2\,ds.$$

This inequality implies:

$$||\omega(t)||^2\leq\int_0^t g(s)\,||\omega(s)||^2\,ds,\qquad g\in L^1(0,T),$$

that is, $\omega(t)=0$, $0\leq t\leq T$. Thus we have uniqueness. Q.E.D.

2. ASYMPTOTIC BEHAVIOR

On this section we obtain informations on the behavior of the energy associated to (6), Section 1, when t goes to ∞.

Theorem 2.1: *If u is the solution of Corollary 1.1, then*

$$E(t) \leq 4E(0)e^{-\gamma t}, \text{ for all } t \geq 0, \tag{33}$$

where

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2}\hat{M}\left|A^{\frac{1}{2}}u(t)\right|^2), \quad t \geq 0, \tag{34}$$

and

$$2\gamma = \left(\frac{m_o}{\tau}\right) \min\left(\frac{m_o^2 \lambda_1}{2m_o^{3/2} \lambda_1^{1/2} + \tau\lambda_1 + \tau}, \frac{2\tau \lambda_1^{\alpha}}{2\tau + m_o}\right) \tag{35}$$

$$\tau = \max\left\{M(s); 0 \leq s \leq \frac{2E(0)}{m_o}\right\}, \tag{36}$$

λ_1 the first eigenvalue of A.

We use, in the proof, the method of Haraux-Zuazua [6], cf. also Zuazua [22]. It is sufficient to obtain (33) for the approximated solutions u_m of (8). The convergences obtained in the proof of Theorem 1.1 imply the inequality (33) for the limit u. By this reason we write u in place of u_m and $E(t)$ or E in place of $E_m(t)$.

For $\epsilon > 0$ we define the perturbed energy $E_\epsilon(t)$ by:

$$E_\epsilon(t) = (1 + \epsilon C)\, E(t) + \epsilon(u', u), \tag{37}$$

where C is a constant to be determined later.

The method depends on an inequality relating $E(t)$ and $E_\epsilon(t)$, which is isolated in the following lemma:

Lemma 2.1: *We have*

$$\frac{1}{2}E_\epsilon(t) \leq E(t) \leq 2\, E_\epsilon(t)$$

for all $t \geq 0$ *and* $0 < \epsilon \leq \epsilon_o$, *where*

$$\epsilon_o = \frac{(m_o \lambda_1)^{1/2}}{2(1 + c(m_o \lambda_1)^{1/2})}. \tag{38}$$

Proof: Let us consider $\psi(t) = (u'(t), u(t))$. We have:

$$|\psi(t)| \leq \frac{\mu}{2} |u(t)|^2 + \frac{1}{2\mu} |u'(t)|^2, \quad \mu > 0, \tag{39}$$

and we know that $\left|A^{\frac{1}{2}} u(t)\right|^2 \geq \lambda_1 |u(t)|^2$, then

$$|u(t)| \leq \frac{1}{m_o \lambda_1} \hat{M}\left(\left|A^{\frac{1}{2}} u(t)\right|^2\right). \tag{40}$$

From (39), (40) if follows:

$$|\psi(t)| \leq \frac{\mu}{2m_o \lambda_1} \hat{M}\left(\left|A^{\frac{1}{2}} u(t)\right|^2\right) + \frac{1}{2\mu} |u'(t)|^2.$$

Choosing $\mu = (m_o \lambda_1)^{1\ 2}$, we get:

$$|\psi(t)| \leq (m_o \lambda_1)^{-\frac{1}{2}} E(t), \tag{41}$$

and we have:

$$E_\epsilon(t) \leq (1 + \epsilon C) E(t) + \epsilon (m_o \lambda_1)^{-\frac{1}{2}} E(t).$$

This inequality implies:

$$E_\epsilon(t) \leq 2E(t), \tag{42}$$

for $0 < \epsilon \leq \epsilon_o$.

From (41) and the definition of E, we get:

$$E_\epsilon(t) \geq E(t) - \epsilon |\psi(t)| \geq E(t) - \epsilon_o (m_o \lambda_1)^{-\frac{1}{2}} E(t) \geq \frac{1}{2} E(t),$$

which is the other side of the inequality in Lemma 2.1. Q.E.D.

Proof of Theorem 2.1: The idea of the proof is to obtain $E'_\epsilon(t) \leq -\eta E(t)$, η a positive constant. This inequality, with Lemma 2.1, permit us to obtain the exponential decay.

We have from the approximated equation:

$$E'(t) = -\left|A^{\frac{\alpha}{2}} u'(t)\right|^2, \tag{43}$$

hence,

$$E'(t) \leq -\lambda_1^{\alpha} |u'|^2. \tag{44}$$

Also from the approximated equation (8) we obtain:

$$\frac{d}{dt}(u', u) = |u'|^2 - M\left(\left|A^{\frac{1}{2}} u\right|^2\right) |A^{\frac{1}{2}} u|^2 - \left(A^{\frac{\alpha}{2}} u', A^{\frac{\alpha}{2}} u\right). \tag{45}$$

Thus, from (44), (45) we get:

$$E'(t) + \epsilon \frac{d}{dt}(u', u) \leq -\lambda_1^{\alpha} |u'|^2 + \epsilon |u'|^2 - \epsilon M\left(\left|A^{\frac{1}{2}} u\right|^2\right) \left|A^{\frac{1}{2}} u\right|^2 - \epsilon\left(A^{\frac{\alpha}{2}} u', A^{\frac{\alpha}{2}} u\right), \quad \epsilon > 0. \tag{46}$$

We have:

$$\left|\left(A^{\frac{\alpha}{2}} u', A^{\frac{\alpha}{2}} u\right)\right| \leq \frac{\delta}{2} |A^{\frac{\alpha}{2}} u'|^2 + \frac{1}{2\delta} \left|A^{\frac{\alpha}{2}} u\right|^2. \tag{47}$$

Since $0 < \alpha \leq 1$, we get:

$$\left|A^{\frac{\alpha}{2}} u\right|^2 = \sum_{0 < \lambda_\nu \leq 1} \lambda_\nu^{\alpha} \, |(u, \omega_\nu)|^2 + \sum_{\lambda_\nu \geq 1} \lambda_\nu^{\alpha} \, |(u, \omega_\nu)|^2$$

$$\leq |u|^2 + \left|A^{\frac{1}{2}} u\right|^2 \leq \left[\frac{1 + \lambda_1}{m_o \lambda_1}\right] \hat{M}\left(\left|A^{\frac{1}{2}} u\right|^2\right).$$

By (43) and this last inequality, we obtain, from (47):

$$\left|\left(A^{\frac{\alpha}{2}} u', A^{\frac{\alpha}{2}} u\right)\right| \leq -\frac{\delta}{2} E'(t) + \frac{k}{2\delta} \hat{M}\left(\left|A^{\frac{1}{2}} u\right|^2\right) \tag{48}$$

for

$$k = \frac{1 + \lambda_1}{m_o \lambda_1}.$$

The inequality (48) permits us to write (46) in the form:

$$E'(t)+\epsilon\frac{\delta}{2}E'(t)+\epsilon\frac{d}{dt}(u',u)\leq-(\lambda_1^\alpha-\epsilon)\,|u'|^2 \tag{49}$$

$$-\epsilon M\left(\left|A^{\frac{1}{2}}u\right|^2\right)\left|A^{\frac{1}{2}}u\right|^2+\frac{\epsilon k}{2\delta}\hat{M}\left(\left|A^{\frac{1}{2}}u\right|^2\right).$$

We need to compare the last two terms of (49). For this, we know from (11) in Section 1, since $f=0$:

$$\left|A^{\frac{1}{2}}u(t)\right|^2\leq\frac{2\,E(0)}{m_o},\ \text{for all } t\geq 0, \tag{50}$$

which implies:

$$M\left(\left|A^{\frac{1}{2}}u(t)\right|^2\right)\leq\tau \text{ for all } t\geq 0,$$

with τ defined by (36).

Define

$$a=\frac{2\,E(0)}{m_o}\quad\text{and } \delta=\frac{k\tau}{m_o}.$$

By the definition of $\hat{M}(\lambda)$, we have for $0\leq\lambda\leq a$:

$$\hat{M}(\lambda)\leq\lambda\max_{0\leq\xi\leq a}M(\xi)=\frac{\lambda\,\delta}{k}\,m_o\leq\frac{\lambda\,\delta}{k}\,M(\lambda).$$

Therefore,

$$\epsilon\lambda M(\lambda)-\frac{\epsilon k}{2\delta}\hat{M}(\lambda)\geq\frac{\epsilon k}{2\delta}\hat{M}(\lambda)\text{ for all } 0\leq\lambda\leq a. \tag{51}$$

By (50), (51) and observing that $\frac{k}{\tau}=\frac{m_o}{\tau}$, we have from (49)

$$E'(t)+\epsilon\frac{\delta}{2}E'(t)+\epsilon\frac{d}{dt}(u',u) \tag{52}$$

$$\leq-(\lambda_1^\alpha-\epsilon)\,|u'|^2-\frac{\epsilon m_o}{2\tau}\hat{M}\left(\left|A^{\frac{1}{2}}u\right|^2\right).$$

For $0<\epsilon<\epsilon_1=\dfrac{2\lambda_1^\alpha\tau}{2\tau+m_o}$, we obtain:

$$\lambda_1^\alpha-\epsilon>0 \text{ and } \frac{\epsilon m_o}{2\tau}<\lambda_1^\alpha-\epsilon.$$

Thus, from (52):

$$E'(t)+\frac{\epsilon\delta}{2}E'(t)+\epsilon\frac{d}{dt}(u',u)\leq-\frac{\epsilon m_o}{\tau}E(t), 0<\epsilon<\epsilon_1. \tag{53}$$

By Lemma 2.1, with $C=\dfrac{\delta}{2}$, it follows, from (53):

$$E_\epsilon'(t)\leq-\frac{\epsilon m_o}{2\tau}E_\epsilon(t),\quad 0<\epsilon<\epsilon_2=\min(\epsilon_o,\epsilon_1),$$

which implies:

$$E_\epsilon(t)\leq E_\epsilon(0)e^{-\frac{\epsilon m_o}{2\tau}t}. \tag{54}$$

If we apply Lemma 2.1, with $0<\epsilon<\epsilon_2$, to (54) we obtain Theorem 2.1. Q.E.D.

Remark 1.3: In Y. Yamada [21], he observed that if

$$M\in C^1([0,\infty);\mathbf{R}),\ \frac{1}{2}\leq\alpha\leq 1, \tag{55}$$

with $(u_o, u_1)\in V\cap D(A^\alpha)\times H, f=0$ to avoid technalities, then he obtained further regularity for the solution u given by Theorem 1.1. In fact, he proves that if we have (55), then u satisfy:

$$\sqrt{t}\,u\in L^\infty(0,T;D(A^{\frac{\alpha+1}{2}}))$$
$$\sqrt{t}\,u'\in L^\infty(0,T:D(A^{\frac{\alpha}{2}})\cap L^2(0,T:D(A^\alpha))$$
$$\sqrt{t}\,u''\in L^2(0,T;H)$$

for all $t\geq 0$.

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