REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 4, números 2 y 3; 1991.

On an Inequality of Gauss

HORST ALZER

ABSTRACT. In this note we prove a new extension and a converse of an inequality due to Gauss.

The following theorem which is important in Statistics (see [1, p. 183 and p. 256]) is due to C.F. Gauss:

If $f: [0, \infty) \rightarrow \mathfrak{R}$ is a decreasing function, then we have for all real numbers $k > 0$:

$$k^2 \int_k^\infty f(x)\,dx \leq \frac{4}{9} \int_0^\infty x^2 f(x)\,dx. \tag{1}$$

Interesting generalizations of this result were given by V. N. Volkov [3] and, recently, by J. E. Pečarić [2].

The aim of this paper is two-fold. On the one hand we establish a new extension of inequality (1) and on the other hand we present a converse of (1) which provides (under the additional assumption that f is nonnegative) a lower bound for

$$k^2 \int_k^\infty f(x)\,dx.$$

First we prove the following proposition:

1991 Mathematics Subject Classification: 26D15.
Editorial de la Universidad Complutense. Madrid, 1991.

Theorem. *Let $g:[a, b]\to\mathfrak{R}$ be strictly increasing, convex and differentiable, and let $f: I\to\mathfrak{R}$ be decreasing. Then*

$$\int_a^b f(s(x))\, g'(x)\, dx \leq \int_{g(a)}^{g(b)} f(x)\, dx \leq \int_a^b f(t(x))\, g'(x)\, dx \tag{2}$$

where

$$s(x)=\frac{g(b)-g(a)}{b-a}(x-a)+g(a)$$

and

$$t(x)=g'(x_0)\,(x-x_0)+g(x_0)\,,\ x_0\in[a, b]\,.$$

($I\subset\mathfrak{R}$ is an interval containing a, b, $g(a)$, $g(b)$, $t(a)$ and $t(b)$.)

If either g is concave (instead of convex) or f is increasing, then the reversed inequalities hold.

Proof. Let g be convex and let f be decreasing. We denote by h the function

$$h(x)=f(g(x));$$

then h is also decreasing. Since g is convex we obtain for all $x\in[a, b]$:

$$t(x)\leq g(x)\leq s(x)\,.$$

This implies

$$g^{-1}(t(x))\leq x\leq g^{-1}(s(x))$$

and

$$h(g^{-1}(t(x)))\geq h(x)\geq h(g^{-1}(s(x)))\,,$$

where g^{-1} designates the inverse function of g.

Because of $g'\geq 0$ we conclude

$$h(g^{-1}(t(x)))\, g'(x)\geq h(x)\, g'(x)\geq h(g^{-1}(s(x)))\, g'(x)$$

and integration yields

$$\int_a^b h(g^{-1}(t(x)))\, g'(x)\, dx \geq \int_a^b h(x)\, g'(x)\, dx$$
$$\geq \int_a^b h(g^{-1}(s(x)))\, g'(x)\, dx. \tag{3}$$

Finally, from (3) and

$$\int_a^b h(x)\, g'(x)\, dx = \int_{g(a)}^{g(b)} h(g^{-1}(y))\, dy$$

(which follows immediately from the substitution $y = g(x)$) we get

$$\int_a^b f(t(x))\, g'(x)\, dx \geq \int_{g(a)}^{g(b)} f(x)\, dx \geq \int_a^b f(s(x))\, g'(x)\, dx.$$

Similarly we can verify, if either g is concave (instead of convex) or f is increasing, then in the last inequalities we have to replace "$\geq$" by "$\leq$". This completes the proof. □

An application of the Theorem leads to a new proof and to a converse of inequality (1).

Corollary. *If $f: [0, \infty) \to \mathfrak{R}$ is decreasing, then we have for all real positive real numbers k:*

$$k^2 \int_k^\infty f(x)\, dx \leq \frac{4}{9} \int_0^\infty x^2 f(x)\, dx,$$

and, under the additional assumption that f is nonnegative we obtain

$$3 \int_0^k x^2 f(x+k)\, dx \leq k^2 \int_k^\infty f(x)\, dx, \tag{4}$$

where the constant 3 cannot be replaced by a greater number.

Proof. Let $a = 0$, $b \geq x_0 = k/2^{1/3}$ and $g(x) = (1/k^2)\, x^3 + k$.

Then we have

$t(x)=(3/4^{1/3})\,x$

and from the right-hand side of (2) we conclude

$$\int_k^{k+b^3/k^2} f(x)\,dx \le \frac{3}{k^2}\int_0^b x^2 f(3x/4^{1/3})\,dx = \frac{4}{9k^2}\int_0^{3b/4^{1/3}} x^2 f(x)\,dx .$$

If b tends to ∞ we obtain inequality (1).

In order to prove (4) we set $a=0$, $b=k$ and $g(x)=(1/k^2)\,x^3+k$.

Then we get

$s(x)=x+k$

and from the first inequality of (2) we obtain

$$\frac{3}{k^2}\int_0^k x^2 f(x+k)\,dx \le \int_k^{2k} f(x)\,dx . \tag{5}$$

Since f is nonnegative we conclude

$$\int_k^{2k} f(x)\,dx \le \int_k^{\infty} f(x)\,dx \tag{6}$$

such that (5) and (6) imply inequality (4).

If we put

$$f(x)=\begin{cases} 1, & 0\le x\le 2k\,, \\ 0, & 2k<x\,, \end{cases}$$

then equality holds in (4). Therefore, the constant 3 is best possible. □

References

[1] H. CRAMÉR: *Mathematical Methods of Statistics*, Princeton Univ. Press. Princeton, 1966.

[2] J. E. PEČARIĆ: *Connections among some inequalities of Gauss, Steffensen and Ostrowski*, SEA Bull. Math. **13** (1989), 89-91.

[3] V. N. VOLKOV: *Inequalities connected with an inequality of Gauss* (Russian), Volž. Mat. Sb. **7** (1969), 11-13.

Morsbacher Str. 10
5220 Waldbröl
Germany

Recibido: 31 de julio de 1990