REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 5, números 2 y 3; 1992.

Holomorphic Functions on Strict Inductive Limits of Banach Spaces

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ABSTRACT. In this article we show that a number of apparently different properties coincide on the set of holomorphic functions on a strict inductive limit (all inductive limits are assumed to be countable and proper) of Banach spaces and that they are all satisfied only in the trivial case of a strict inductive limit of finite dimensional spaces. Thus the linear properties of a strict inductive limit of Banach spaces rarely translate themselves into holomorphic properties.

§0

The main ingredients in our proof are the extension to strict inductive limits of a property of the τ_δ semi-norms which was previously known for direct sums of Banach spaces [4], a characterization of Schwartz spaces using limited sets due to Lindström [8] and a close examination of a special hypoanalytic function on a strict inductive limit of Banach spaces constructed in [7].

We refer to [6] for details concerning holomorphic functions on locally convex spaces. Information regarding holomorphic functions on direct sums of Banach spaces, $\mathcal{DF}$ spaces and on inductive limits of Fréchet spaces may be found in [1, 3, 4, 6, 7, 9]. We thank J. M. Ansemil for some helpful comments.

§1

In this section we recall some definitions and prove a number of results that we shall use later.

1991 Mathematics Subject Classification: 46G20.
Editorial Complutense. Madrid, 1992.

If E is a locally convex space over $\mathbf{C}$ we let $\mathscr{H}_G(E)$ denote the set of $\mathbf{C}$ valued functions on E whose restriction to each finite dimensional subspace of E is a holomorphic function (of several complex variables). If $f \in \mathscr{H}_G(E)$ is continuous on compact sets we say that f is hypoanalytic and the space of all such functions is denoted by $\mathscr{H}_{HY}(E)$. If $f \in \mathscr{H}_G(E)$ is continuous then f is called holomorphic and we let $\mathscr{H}(E)$ denote the space of all holomorphic functions on E. We let $\mathscr{P}(^n E)$ denote the space of all continuous n-homogeneous polynomials on E.

The compact open topology on $\mathscr{H}(E)$ and $\mathscr{H}_{HY}(E)$ is denoted by τ_0. A semi-norm p on $\mathscr{H}(E)$ is said to be τ_ω continuous if there exists a compact subset K of E such that for each open subset V of E, $K \subset V$, there exists $C(V) > 0$ such that

$$p(f) \leq C(V) \sup_{x \in V} |f(x)|$$

for all $f \in \mathscr{H}(E)$.

We introduce the τ_δ topology (see [6, proposition 3.27]) in a somewhat unusual fashion but one which is more suitable for the purposes of this article.

The τ_δ topology on $\mathscr{H}(E)$ is the topology generated by all semi-norms which satisfy the following two conditions

$$p(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) \tag{1.1}$$

for every f in $\mathscr{H}(E)$,

$$p|\mathscr{P}(^n E) \tag{1.2}$$

is τ_ω continuous for each positive integer n.

If τ is a locally convex topology on $\mathscr{H}(E)$, then $(\mathscr{H}(E), \tau)$ is T.S. τ-complete if for any sequence $(P_n)_{n=0}^{\infty}$, $P_n \in \mathscr{P}(^n E)$ all n, such that $\sum_{n=0}^{\infty} p(P_n) < \infty$ for every τ-continuous seminorm p on $\mathscr{H}(E)$ we have $\sum_{n=0}^{\infty} P_n \in \mathscr{H}(E)$ ([6, definition 3.32]).

A subset B of E is said to be *bounding* if $\|f\|_B := \sup_{x \in B} |f(x)| < \infty$ for every $f \in \mathscr{H}(E)$ and $L \subset E$ is said to be *limited* if for every equicontinuous weak* null sequence in E', $(\phi_n)_n$, we have $\|\phi_n\|_L \to 0$ as $n \to \infty$.

Proposition 1. *([8]) A locally convex space E is a Schwartz space if and only if it is quasi-normable and the bounded subsets of E are limited.*

Proposition 2. *If E is a barrelled locally convex space then every weak* null sequence in E′ converges uniformly to zero on the bounding subsets of E. In particular the bounding subsets of E are limited.*

Proof. Let B denote a bounding subset of E and let $(\phi_n)_{n=1}^{\infty}$ denote a weak* null sequence in E'. Let $f=\sum_{n=1}^{\infty}\phi_n^n$. Since the sequence $(\phi_n)_n$ is weak* null the function f is defined on all of E and hence belongs to $\mathscr{H}_G(E)$. Since the sequence $(\phi_n)_n$ is pointwise bounded in the barrelled space E, it is equicontinuous. Thus f belongs to $\mathscr{H}(E)$ by [2, proposition 1]. By corollary 4.19(b) of [6] this implies

$$\lim_{n\to\infty}(\|\phi_n^n\|_B)^{\frac{1}{n}}=\lim_{n\to\infty}\|\phi_n\|_B=0$$

and hence B is a limited subset of E. This completes the proof.

The following result is a reformulation of [7, proposition 3]. We sketch the proof for the readers' convenience

Proposition 3. *Let $E=\lim\limits_{\vec{n}} E_n$ denote a strict inductive limit of Banach spaces and suppose dimension $(E_n)=\infty$ for some integer n. Then there exists a g in $\mathscr{H}_{HY}(E)$ with the following properties;*

$$g\notin\mathscr{H}(E) \tag{1.3}$$

$$\text{if}\quad Z_m=\left\{n;\frac{\hat{d}^n g(0)}{n!}\bigg|_{E_m}\neq 0\right\}$$

$$\text{then}\sum_{n\in Z_m}\frac{\hat{d}^n g(0)}{n!}\in\mathscr{H}(E) \tag{1.4}$$

for every positive integer m.

Proof. Since $\dim(E_n)=\infty$ for some n, E is not a Schwartz space and hence proposition 1 implies that E contains a bounded subset which is not limited. By proposition 2 the bounding subsets and the limited subsets of E coincide and hence E contains a bounded set, which we may suppose is the unit ball B_1 of E_1, which is not bounding. Hence there exists $f\in\mathscr{H}(E)$,

$f=\sum_{n=1}^{\infty} P_n$, such that $\|P_{n_j}\|_{B_1}\geq 1$ for all j where $(n_j)_j$ is a strictly increasing sequence of positive integers satisfying $n_{j+1}>n_j+j$ for all j.

Let $\phi_n\in E'$, $n\geq 2$, satisfy $\phi_n|E_{n-1}=0$ and $\phi_n(x_n)=1$ for some $x_n\in E_n$ with $\|x_n\|=1$ and let $\theta_k(z)=kz$ for all $z\in E$.

We consider the function

$$g=\sum_{k=2}^{\infty}\left\{\left(\sum_{j=k}^{\infty} P_{n_j}\right)\circ\theta_k\right\}\phi_k^k.$$

Since $j\geq k$ the condition $n_{j+1}>n_j+j$ implies that $\sum_{k=2,\, j\geq k}^{\infty}(P_{nj}\circ\theta_k)\,\phi_k^k$ is the Taylor series expansion of g.

Since

$$g|E_n=\sum_{k=2}^{m+1}\left\{\sum_{j=k}^{\infty} P_{nj}\circ\theta_k\right\}\phi_k^k\Big|_{E_n}$$

it follows that $g\in\mathscr{H}_{HY}(E)$ and, moreover,

$$\sum_{n\in Z_m}\frac{\hat{d}^n g(0)}{n!}\in\mathscr{H}(E)\quad\text{for all } m.$$

We refer to [7] for the details which show that $g\notin\mathscr{H}(E)$.

Our next result was proved for countable direct sums of Banach spaces in [4] (see also [6, proposition 4.40]).

Proposition 4. *If* $E=\lim\limits_{\overrightarrow{n}} E_n$ *is a strict inductive limit of Banach spaces and* p *is a* τ_δ *continuous seminorm on* $\mathscr{H}(E)$ *then there exists a positive integer* m *such that* $f\in\mathscr{H}(E)$ *and* $f|E_m=0$ *imply* $p(f)=0$.

Proof. We may suppose without loss of generality that the semi-norm p satisfies (1.1) and (1.2). If the result is not true then for every positive integer n there exists a continuous homogeneous polynomial P_n such that $P_n|E_n=0$ and $p(P_n)\neq 0$. We now show that the sequence $(P_n)_{n=1}^{\infty}$ is locally bounded.

Let K denote a compact subset of E. Then K is contained and compact in some E_k. For each positive integer j let B_j denote the unit ball in E_j.

Now choose $M>0$ and $\lambda_1,\dots,\lambda_k$ positive numbers such that

$$\|P_j\|_{K+\sum_{r=1}^{k}\lambda_r B_r}\leq M\quad\text{for } j=1,\dots,k$$

Using a binomial expansion and the fact that $P_{k+1}|E_{k+1}=0$ we can find $\lambda_{k+1}>0$ such that

$$\|P_j\|_{K+\sum_{r=1}^{k+1}\lambda_r B_r}\leq M+\frac{1}{2^{k+1}} \text{ for } j=1 \ldots, k+1.$$

By induction we can find a sequence of positive numbers $(\lambda_r)_{r=1}^{\infty}$ such that

$$\|P_j\|_{K+\sum_{r=1}^{\infty}\lambda_r B_r}\leq M+1 \quad \text{for all } j.$$

Since E is bornological it follows that $\sum_{r=1}^{\infty}\lambda_r B_r$ is a neighbourhood of zero in E. Hence $\{P_j\}_j$ is locally bounded.

If $Q_j=\dfrac{j P_j}{p(P_j)}$ then $Q_{j+1}|_{E_j}=0$ and the above argument shows that $\{Q_j\}_j$ is also a locally bounded and hence a τ_δ bounded sequence in $\mathscr{H}(E)$ (see [6, lemma 2.43]).

Since $p(Q_j)=j$ this is impossible and proves our result.

Proposition 5. ([*6, example 1.24*]) *If* $E=\varinjlim_n E_n$ *is a inductive limit of Banach spaces then* **C**-*valued homogeneous hypocontinuous polynomials on* E *are continuous.*

We refer to [1], [3, proposition 4.1] and [6, example 1.38] for information regarding the next proposition.

Proposition 6. *If* $E=\varinjlim_n E_n$ *is an inductive limit of Banach spaces then, for each positive integer* n, τ_ω *on* $\mathscr{P}(^n E)$ *is the topology of uniform convergence on the bounded subsets of* E *and* $(\mathscr{P}(^n E), \tau_0)$ *and* $(\mathscr{P}(^n E), \tau_\omega)$ *are complete locally convex spaces.*

§2

The following theorem is our main result.

Theorem 7. *If* $E=\varinjlim_n E_n$ *is a strict inductive limit of Banach spaces then the following are equivalent:*

(a) $\mathscr{H}(E)=\mathscr{H}_{HY}(E)$,

(b) the bounded subsets of E *are bounding,*

(c) *the bounded subsets of E are limited,*

(d) $E \approx \mathbf{C}^{(N)}$,

(e) $(\mathscr{H}(E), \tau_0)$ *is a Fréchet space,*

(f) $E = \varinjlim_{n} E_n$ *in the category of locally convex spaces and holomorphic mappings,*

(g) $(\mathscr{H}(E), \tau_0)$ *is complete (resp. quasicomplete, sequentially complete, T. S. τ_0 complete),*

(h) *the τ_0 bounded subsets of $\mathscr{H}(E)$ are locally bounded,*

(i) $(\mathscr{H}(E), \tau_\delta)$ *is complete (resp. quasicomplete, sequentially complete, T. S. τ_δ complete),*

(j) *τ_δ bounded subsets of $\mathscr{H}(E)$ are locally bounded.*

Proof. The implication (a)$\Rightarrow$(b) is proved in [7] (and also the implication (b)$\Rightarrow$(a) if E is separable). By proposition 2 we have (b)$\Rightarrow$(c).

Since E is a $\mathscr{DF}$ space it is quasinormable and hence if (c) is satisfied then proposition 1 implies that E is a Schwartz space. Hence each E_n is finite dimensional and (c)$\Rightarrow$(d). By [5] condition (d) implies all the other conditions except (f). Since $\mathbf{C}^{(N)}$ is a Schwartz space (d)$\Rightarrow$(f). Hence conditions (a),..., (d) are all equivalent. By proposition 6 and [6, proposition 3.36] all the conditions in (g) are equivalent, all the conditions in (i) are equivalent and (e)$\Rightarrow$(g)$\Rightarrow$(i). If the τ bounded subsets of $\mathscr{H}(E)$ are locally bounded, $\tau = \tau_0$ or τ_δ, then we have $\mathscr{H}(E)$ T. S. τ-complete and hence (h)$\Rightarrow$(g) and (j)$\Rightarrow$(i). Hence, to complete the proof, it suffices to show (f)$\Rightarrow$(a) and $\sim$(d)$\Rightarrow\sim$(i). Suppose (f) is satisfied. Let $f \in \mathscr{H}_{HY}(E)$. For each positive integer n, $f|_{E_n} \in \mathscr{H}_{HY}(E_n) = \mathscr{H}(E_n)$, since E_n is a Banach space and the topology induced by E on E_n is its original Banach space topology. By (f) we have $f \in \mathscr{H}(E)$ and (f)$\Rightarrow$(a).

Now suppose that (d) is not satisfied. By proposition 3 there exists an $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}$ in $\mathscr{H}_{HY}(E)$ satisfying (1.3) and (1.4).

Let p denote a τ_δ continuous semi-norm on $\mathscr{H}(E)$ satisfying (1.1) and (1.2).

Let m denote an integer associated with p as in proposition 4 and let Z_m have the meaning given to it in (1.4).

Since $\sum_{n \in Z_m} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(E)$ we have $\sum_{n \in Z_m} p\left(\frac{\hat{d}^n f(0)}{n!}\right) < \infty$.

If $n \notin Z_m$ then $\left.\frac{\hat{d}^n f(0)}{n!}\right|_{E_m} = 0$ and $p\left(\frac{\hat{d}^n f(0)}{n!}\right) = 0$ by proposition 4.

Hence $\sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) < \infty$ and $\left\{\sum_{n=0}^{m} \frac{\hat{d}^n f(0)}{n!}\right\}_{m=1}^{\infty}$ is a Cauchy sequence in $(\mathscr{H}(E), \tau_\delta)$.

Since $\sum_{n=0}^{m} \frac{\hat{d}^n f(0)}{n!} \notin \mathscr{H}(E)$ this Cauchy sequence does not converge and (i) is not satisfied. This completes the proof.

Luiza A. Moraes would like to thank University College Dublin and CNPq (Brazil) for financial support to visit Dublin to undertake the research for this paper.

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Recibido: 2 de diciembre de 1991
Revisado: 22 de mayo de 1992