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*The Tensor Product of Triples as Multilinear Product**

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ABSTRACT. In this paper we introduce a notion of multilinear product for triples in Set, which if it is given by a distributive law then coincides with the one given by Bunge. We also demonstrate that the tensor product of two triples, if there exist, is an initial object in a suitable category of multilinear products.

INTRODUCTION

In "Producto de triples" ([8]) the definition that is given of the product of triples generalizes the notion of distributive law, according to Beck ([2]). The tensor product is studied by E. Manes in various articles ([10], [11], [12]), for triples in the category Set, of sets and maps.

M. Bunge, in ([3]), studies the relationship between composition triple and tensor product of triples, for triples in Set. In this paper, it is given the definition of distributive multilinear law, which is the

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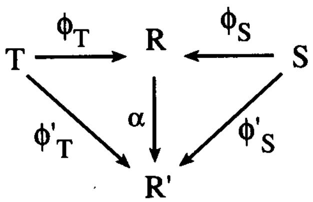
distributive law in which each algebra over the composition triple is also a bialgebra.

The aim of the present paper is to introduce a notion of multilinear product for triples in Set, which if it is given by a distributive law then coincides with the one given by Bunge, and to demonstrate that the tensor product of two triples **T** and **S**, if there exist, is an initial object in the category whose objects are multilinear products **R** = $(\mathbf{TS})_r$, and whose morphisms α: **R** $\longrightarrow$ **R'** are morphisms of triples that make the following diagram commutative

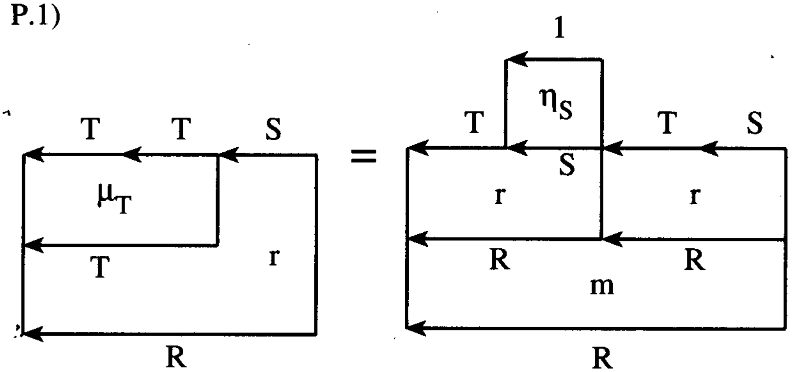


ϕ_T and ϕ_S being the morphisms of triples asociated to every product ([8]).

1. PRODUCT OF TRIPLES AND DISTRIBUTIVE LAWS

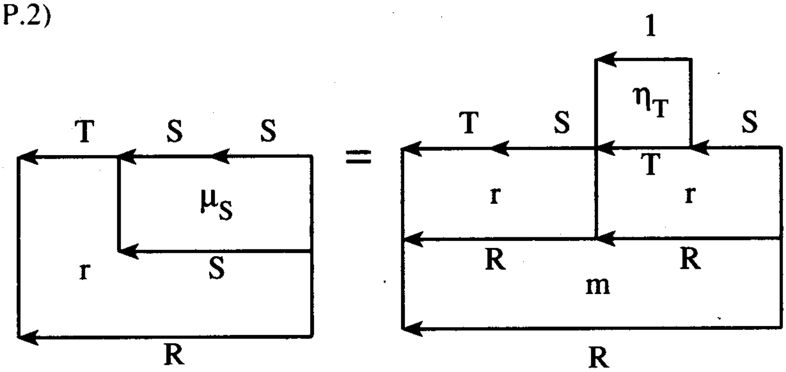
1.1 If **T** = (T, η_T, μ_T) and **S** = (S, η_S, μ_S) are triples in a category **A**, a *product* **R** = $(\mathbf{TS})_r$ is a triple **R** = (R, η, m), where $\eta = r \circ (\eta_T \star \eta_S)$ and where the natural transformation r: TS $\Longrightarrow$ R verifies the following axioms:

P.1)



i.e.: $r \circ (\mu_T \star S) = m \circ (r \star r) \circ (T \star \eta_S \star TS)$

P.2)



i.e.: $r \circ (T * \mu_S) = m \circ (r * r) \circ (TS * \eta_T * S)$ ([8], 1.1).

1.2 If $\mathbf{R} = \mathbf{(TS)}_r$ is a product, then $\phi_T := r \circ (T * \eta_S): \mathbf{T} \Longrightarrow \mathbf{R}$ and $\phi_S := r \circ (\eta_T * S): \mathbf{S} \Longrightarrow \mathbf{R}$ are morphisms of triples.

Conversely, if $\phi_T: \mathbf{T} \Longrightarrow \mathbf{R}$ and $\phi_S: \mathbf{S} \Longrightarrow \mathbf{R}$ are morphisms of triples, with $\mathbf{R} = (R, \eta_R, m)$ then $\mathbf{R} = \mathbf{(TS)}_r$ with $r := m \circ (\phi_T * \phi_S): TS \Longrightarrow R$.

Moreover, if $\mathbf{R} = \mathbf{(TS)}_r$ is a product, then $\mathbf{R} = \mathbf{(ST)}_{r'}$ is also a product, where $r' = m \circ (\phi_S * \phi_T)$ ([8], 1.2, 1.3, 1.5).

1.3 If $\mathbf{T} = (T, \eta_T, \mu_T)$ and $\mathbf{S} = (S, \eta_S, \mu_S)$ are triples in **A**, a *distributive law* of **T** over **S** is a natural transformation $\tau: TS \Longrightarrow ST$ which verifies:

D.L. 1) $\tau \circ (\eta_T * S) = S * \eta_T$
D.L. 2) $\tau \circ (T * \eta_S) = \eta_S * T$
D.L. 3) $(S * \mu_T) \circ (\tau * T) \circ (T * \tau) = \tau \circ (\mu_T * S)$
D.L. 4) $(\mu_S * T) \circ (S * \tau) \circ (\tau * S) = \tau \circ (T * \mu_S)$
([2], 1).

1.4 A distributive law τ of **T** over **S** makes a product $\mathbf{R} = \mathbf{(TS)}_r$ with $r = \tau$, $\mathbf{R} = (ST, \eta_S * \eta_T, (\mu_S * \mu_T) \circ (S * \tau * T)$ and $r' = 1_{ST}$ ("half unitary law") ([8], 2.2).

1.5 Conversely if $\mathbf{R} = (\mathbf{TS})_r$ is a product with R = ST and verifies the half unitary law, $r' = 1_{ST}$, then r is a distributive law of **T** over **S** ([8], 2.3).

1.6 Taking one of the examples given in [2], we obtain a product $(\mathbf{TS})_r$, in which r is not a distributive law. In fact, if T and S are graduated rings, R = S⊗T is a ring with the product operation:

$$(s_1 \otimes t_1)(s_2 \otimes t_2) = (-1)^{\partial s_1 \partial t_2} s_1 s_2 \otimes t_1 t_2$$

(∂ indicates the degree), being 1⊗1 the unity element. Moreover, the maps

$$\phi_T: T \longrightarrow T\otimes S,\ \phi_T(t) = 1\otimes t$$
$$\phi_S: S \longrightarrow S\otimes T,\ \phi_S(s) = s\otimes 1$$

are homomorphisms of rings.

The rings T, S and R give the triples $\mathbf{T} = (-\otimes T, \eta_T, \mu_T)$, $\mathbf{S} = (-\otimes S, \eta_S, \mu_S)$ and $\mathbf{R} = (-\otimes R, \eta_R, \mu_R)$ in the category **A** of abelians groups (the natural transformations η and μ are the ones induced by the unities and the multiplications of the rings). The homomorphisms ϕ_T and ϕ_S induce morphisms of triples

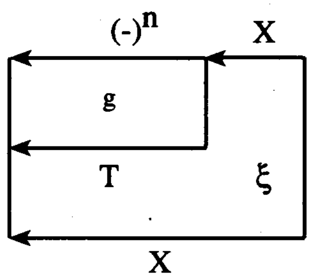
$$\phi_T: \mathbf{T} \Longrightarrow \mathbf{R} \text{ and } \phi_S: \mathbf{S} \Longrightarrow \mathbf{R}$$

$\mathbf{R} = (\mathbf{TS})_r$, being $r = \mu_R \circ (\phi_T \star \phi_S)$ (1.2). However, in this case r is not a distributive law, because the half unitary law is not verified, that is, $r' \neq 1$.

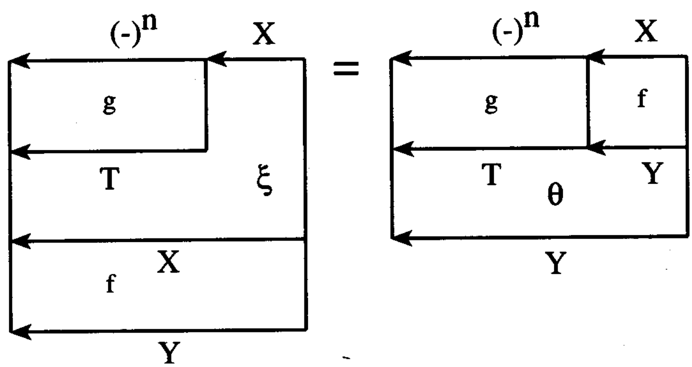
2. TENSOR PRODUCT OF TRIPLES

2.1 Let $n \in |Set|$ and $(-)^n$: Set $\longrightarrow$ Set be the functor Hom(n,-). If $\mathbf{T} = (T, \eta_T, \mu_T)$ is a triple in Set, a *n-ary operation* over **T** is a natural transformation $(-)^n \Longrightarrow T$.

If (X,ξ) is a **T**-algebra, each n-ary operation g over **T** induces an operation, $\xi^g = \xi \circ (g\ X): X^n \longrightarrow X$, over the set X

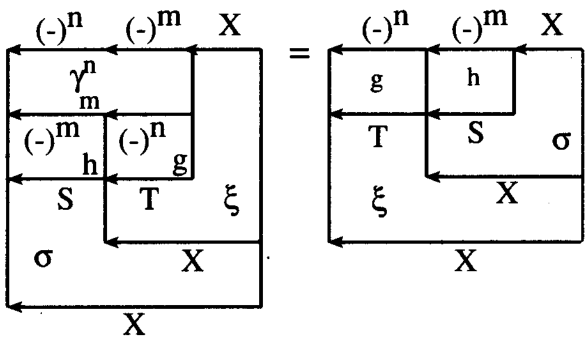


Moreover, a **T**-morphism f: (X,ξ) ⟶ (Y,θ) is a map f: X ⟶ Y which is a morphism in the classic sense, commuting with each operation, that is, for each g: $(-)^n \Rightarrow T$



i.e.: $f \circ \xi^g = f \circ \xi \circ (g\ X) = \theta \circ (g \ast f) = \theta \circ (g\ Y) \circ f^n = \theta^g \circ f^n$
([3], 1).

2.2 If **T** and **S** are triples in Set, a **S**-**T**-*bialgebra* is a 3-triple (X,σ,ξ), with (X,σ) and **S**-algebra and (X,ξ) a **T**-algebra such that for all n,m ∈ |Set|, g: $(-)^n \Rightarrow T$ and h: $(-)^m \Rightarrow S$ the following holds true:



i.e.: $\sigma \circ (S_*\xi) \circ ((h_*g)\ X) \circ (\gamma^n_m\ X) = \xi \circ (T_*\sigma) \circ ((g_*h)\ X)$, where

$$\gamma^n_m: (-)^n(-)^m \Longrightarrow (-)^m(-)^n$$

is the canonical isomorphism.

This is equivalent to, for every g: $(-)^n \Longrightarrow T, \xi^g$ is an **S**-morphism, or equivalently, for every h: $(-)^m \Longrightarrow S$, σ^h is a **T**-morphism ([3], 1).

This defines the category $Set^{[S,T]}$ of **S**-**T**-bialgebras as a full subcategory of the category $Set^{(S,T)}$ whose objects are triples (X,σ,ξ) with (X,σ) an **S**-algebra and (X,ξ) a **T**-algebra, and whose morphisms
f: $(X, \sigma, \xi) \longrightarrow (Y, \tau, \theta)$ are maps f: $X \longrightarrow Y$, being f an **S**-morphism and **T**-morphism.

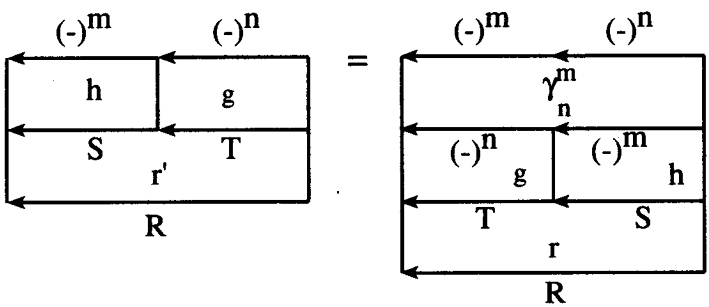
If the forgetful functor $U^{[S,T]}: Set^{[S,T]} \longrightarrow Set$ is tripleable, it makes a triple **S**⊗**T** that is called tensor product (symmetrically, **T**⊗**S**) ([3], [10], [11], [12]).

The existence of tensor product of triples is, in general, an open question.

3. MULTILINEAR PRODUCTS

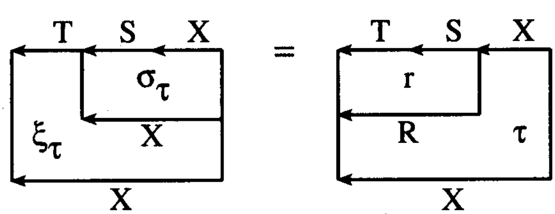
3.1 Let **T** = (T, η_T, μ_T) and **S** = (T, η_S, μ_S) be triples in Set and **R** = $(\mathbf{TS})_r$ a product, **R** = $(R, r \circ (\eta_{T*}\eta_S), m)$.

We will say that **R** is a *multilinear product* if for whatever $g:(-)^n \Rightarrow T$ and $h:(-)^m \Rightarrow S$ it are verified

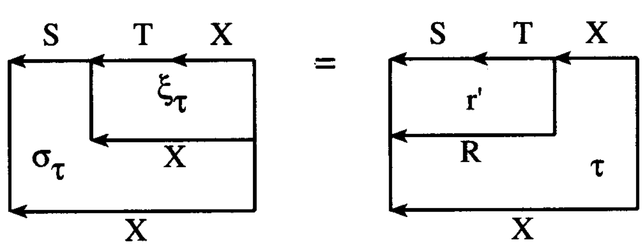


i.e.: $r' \circ (h_* g) = r \circ (g_* h) \circ \gamma^m_n$.

3.2 If $\mathbf{R} = (\mathbf{TS})_r$ is a product, the morphisms of triples ϕ_S: **S** $\Rightarrow$ **R** and ϕ_T: **T** $\Rightarrow$ **R** (1.2) give functors (change of triple) $Set^{\phi_S}: Set^{\mathbf{R}} \longrightarrow Set^{\mathbf{S}}$ and $Set^{\phi_T}: Set^{\mathbf{R}} \longrightarrow Set^{\mathbf{T}}$, respectively, that commute with the forgetful functors to Set. As a result, each **R**-algebra (X,τ) gives an **S**-algebra $(X,\sigma_\tau) = (X,\tau \circ (\phi_S X))$ and a **T**-algebra $(X,\xi_\tau) = (X,\tau \circ (\phi_T X))$. Moreover, it is verified

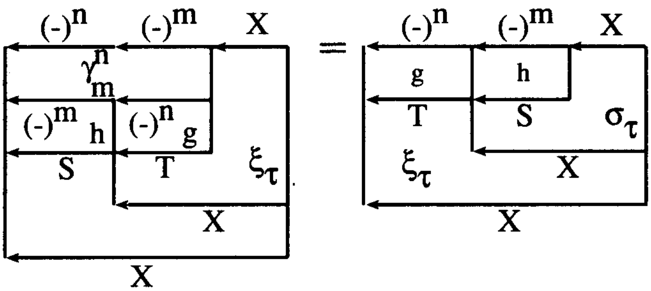


i.e.: $\xi_\tau \circ (T_* \sigma_\tau) = \tau_* (r\ X)$, and



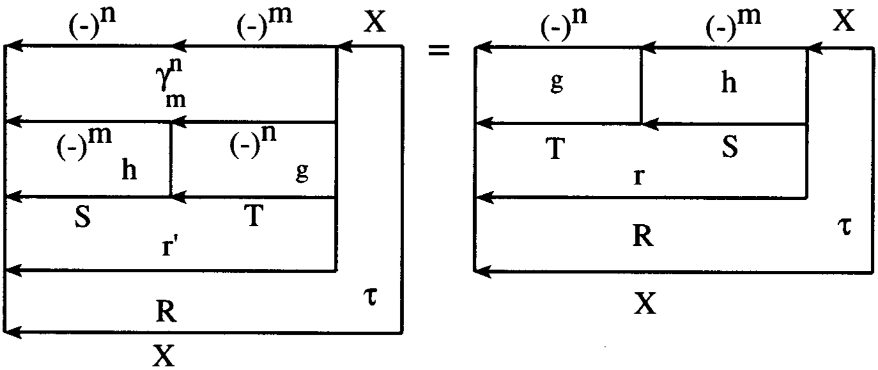
i.e.: $\sigma_\tau \circ (S_* \xi_\tau) = \tau \circ (r'\ X)$ [12], proposition 2.9, page 210).

We will say that an **R**-algebra (X,τ) is a *bialgebra* if the **S**-algebra (X,σ_τ) and the **T**-algebra (X,ξ_τ) make an **S**-**T**-bialgebra $(X, \sigma_\tau, \xi_\tau)$, i.e.:



i.e.: $\sigma_\tau \circ (S_* \xi_\tau) \circ ((h_* g)X) \circ (\gamma^n_m X) = \xi_\tau \circ (T_* \sigma_\tau) \circ ((g_* h)\ X)$, for all operations $g\colon (-)^n \Longrightarrow T$ and $h\colon (-)^m \Longrightarrow S$.

It is immediately proved that the last equality is equivalent to:

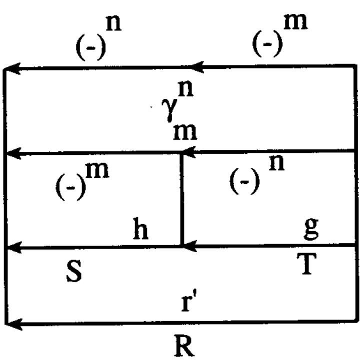


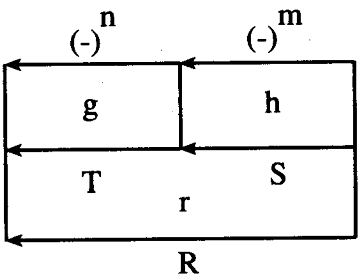
i.e.: $\tau \circ (r'\ X) \circ ((h_* g)\ X) \circ (\gamma^n_m\ X) = \tau \circ (r\ X) \circ ((g_* h)\ X)$.

3.3 From former definitions we can conclude, trivially, that for a multilinear product **R**, every **R**-algebra is a bialgebra. Its reciprocal result is also true. If **R** is a product and every **R**-algebra is a bialgebra, then **R** is multilinear as a consequence of the following result:

If **T** is a triple in Set and $\alpha,\beta\colon (-)^k \Longrightarrow T$ are k-ary operations over **T**, then $\alpha = \beta$ if and only if $\tau^\alpha = \tau^\beta$ for every **T**-algebra (X,τ) ([3], lemma 2.5, page 145).

To obtain our result, it is enough to take

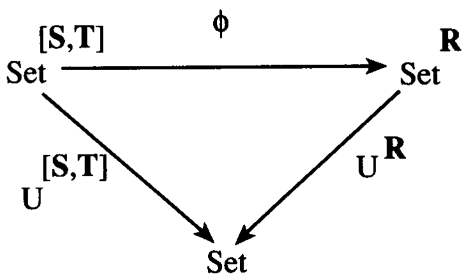
$\alpha =$ 

and $\beta =$ 

i.e.: $\alpha = r' \circ (h_* g) \circ \gamma^n_m$ and $\beta = r \circ (g_* h)$.

4. THE TENSOR PRODUCT AS MULTILINEAR PRODUCT

4.1 Let $\mathbf{T} = (T, \eta_T, \mu_T)$ and $\mathbf{S} = (T, \eta_S, \mu_S)$ be triples in Set. Let us suppose that the tensor product $\mathbf{S}\otimes\mathbf{T} = \mathbf{R} = (R, \eta_R, m)$ exist, i.e., the following diagram

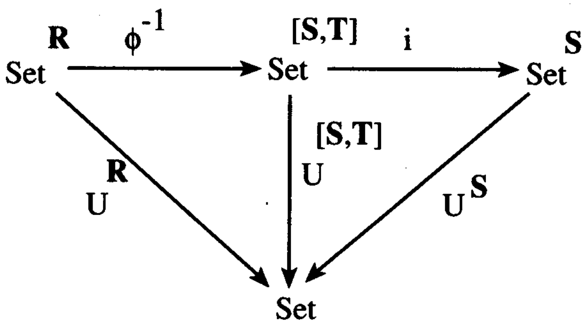


is commutative, ϕ being an isomorphism.

For the inclusion functor i: $\mathrm{Set}^{[S,T]} \longrightarrow \mathrm{Set}^{S}$,

$$i(f: (X, \varepsilon, \delta) \longrightarrow (X', \varepsilon', \delta')) = f: (X,\varepsilon) \longrightarrow (X',\varepsilon'))$$

the following diagram of functors is commutative:



Since $i \circ \phi^{-1}$ commutes with the forgetful functors, a natural transformation exists σ: SR $\Longrightarrow$ R such that $\phi_S = \sigma \circ (S_* \eta_R)$: **S** $\Longrightarrow$ **R** is a morphism of triples ($i \circ \phi^{-1} = Set^{\phi_s}$ is the change of triple functor). For $X \in |Set|$, and (RX,mX) being the free **R**-algebra over X, $(i \circ \phi^{-1})$ (RX,mX) = (RX,σX). For any **R**-algebra (X,τ), $(i \circ \phi^{-1})$ (X,τ) = (X,$\tau \circ (\phi_S$ X)).

In the same way, for the triple **T**, a natural transformation exists ξ: TR $\Longrightarrow$ R, such that $\phi_T = \xi \circ (T_* \eta_R)$: **T** $\Longrightarrow$ **R** is a morphism of triples.

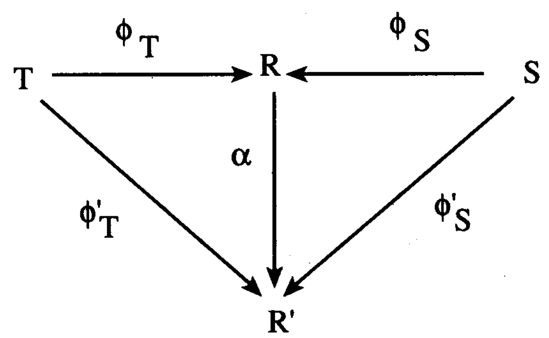
If (X,τ) is an **R**-algebra, ϕ^{-1}(X,τ) = (X,$\tau \circ (\phi_S$ X), $\tau \circ (\phi_T$ X)). In particular, ϕ^{-1}(RX, mX) = (RX, σX, ξX).

4.2 From all this and from (1.2) it follows that **R** is a product, **R**=$(\mathbf{TS})_r$ with $r = m \circ (\phi_{T*}\phi_S) = \xi \circ (T_*\sigma) \circ (TS_*\eta_R)$ (this last equality is true since mX: (RRX, σRX, ξRX) $\longrightarrow$ (RX, σX, ξX) is a morphism of **S**-algebras and of **T**-algebras).

If (X,τ) is an **R**-algebra, ϕ^{-1}(X,τ) = (X,$\tau \circ (\phi_S$ X), $\tau \circ (\phi_T$ X)) = (X, σ_τ, ξ_τ) is an **S**-**T**-algebra and, by (3.2) and (3.3), **R** is a multilinear product.

5. THE CATEGORY OF MULTILINEAR PRODUCTS

5.1 Let **T** and **S** be triples in Set. Taking as objects the multilinear products $\mathbf{R} = (\mathbf{TS})_r$ and as morphisms, α: $\mathbf{R} = (\mathbf{TS})_r \longrightarrow \mathbf{R'} = (\mathbf{TS})_{r'}$, those morphisms of triples α: $\mathbf{R} \longrightarrow \mathbf{R'}$ that make the diagram commutative

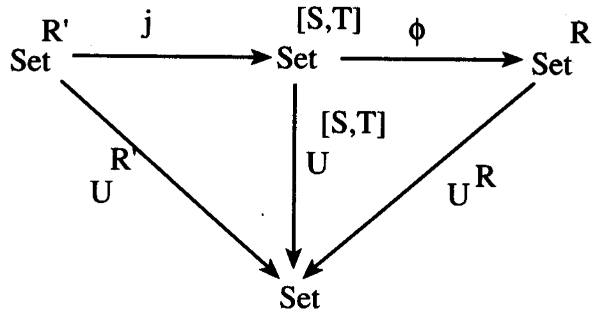


we obtain the category of multilinear products of **T** and **S**.

5.2 In the category of multilinear products of **T** and **S**, **S⊗T** is an initial object. Let us write $\mathbf{S\otimes T} = \mathbf{R} = (R, \eta_R, m) = (\mathbf{TS})_r$.

Let $\mathbf{R'} = (R', \eta_{R'}, m') = (\mathbf{TS})_{r'}$ be an arbitrary multilinear product and ϕ'_S: $\mathbf{S} \Longrightarrow \mathbf{R'}$, ϕ'_T: $\mathbf{T} \Longrightarrow \mathbf{R'}$ the corresponding morphisms of triples. If (X,τ') is an **R'**-algebra, $(X, \sigma_{\tau'}, \xi_{\tau'}) = (X, \tau' \circ (\phi'_S X), \tau' \circ (\phi'_T X))$ is an **S**-**T**-bialgebra.

If j is the functor j: $Set^{\mathbf{R'}} \longrightarrow Set^{[S,T]}$ such that $j(X,\tau') = (X,\sigma_{\tau'},\xi_{\tau'})$, then the following diagram of functors is commutative:



Thus, a natural transformation exists λ: RR' $\Longrightarrow$ R', so that $\alpha = \lambda \circ (R_* \eta_{R'})$: **R** $\Longrightarrow$ **R'** is a morphism of triples, being $\phi \circ j = Set^{\alpha}$ the change of triple functor. Moreover, $(\phi \circ j)(X,\tau') = (X,\tau' \circ (\alpha X))$ and, in particular, $(\phi \circ j)(R'X, m'X) = (R'X, \lambda X) = (R'X,(m'X) \circ (\alpha R'X))$. Then,

$$(R'X,(m'X) \circ (\phi'_S R'X),(m'X) \circ (\phi'_T R'X)) = \phi^{-1} ((\phi \circ j)(R'X,m'X) =$$

$= \phi^{-1}(R'X,(m'X) \circ (\alpha R'X)) = (R'X, (m'X) \circ (\alpha R'X) \circ (\phi_S R'X), (m'X) \circ (\alpha R'X) \circ (\phi_T R'X))$ from which we can obtain

$$m' \circ (\phi'_{S*}R') = m' \circ ((\alpha \circ \phi_S)_*R') \Longleftrightarrow \phi'_S = \alpha \circ \phi_S$$
$$m' \circ (\phi'_{T*}R') = m' \circ ((\alpha \circ \phi_T)_*R') \Longleftrightarrow \phi'_T = \alpha \circ \phi_T$$

that is, there is a morphism in the category of multilinear products α: **R** $\longrightarrow$ **R'**.

5.3 Let us see the uniqueness of α. Let β: **R** $\longrightarrow$ **R'** be a morphism of triples such that $\beta \circ \phi_T = \phi'_T$ and $\beta \circ \phi_S = \phi'_S$. Set^{β}: $Set^{\mathbf{R'}} \longrightarrow Set^{\mathbf{R}}$ is the change of triple functor corresponding to β (note that $\phi \circ j = Set^{\alpha}$). Then

$\alpha = \beta \Longleftrightarrow Set^{\alpha} = Set^{\beta} \Longleftrightarrow Set^{\alpha}(R'X, m'X) = Set^{\beta}(R'X, m'X)$ for every free **R'**-algebra (R'X, m'X) ([12], prop. 2.9, page 210). Also

$$\beta\circ\phi_T = \phi'_T = \alpha\circ\phi_T \Leftrightarrow m'\circ(\beta_* R')\circ(\phi_{T*}R') = m'\circ(\alpha_* R')\circ(\phi_{T*}R')$$
$$\beta\circ\phi_S = \phi'_S = \alpha\circ\phi_S \Leftrightarrow m'\circ(\beta_* R')\circ(\phi_{S*}R') = m'\circ(\alpha_* R')\circ(\phi_{S*}R')$$

But, $j(R'X, m'X) = (R'X, (m'X) \circ (\alpha R'X) \circ (\phi_S R'X), (m'X) \circ (\alpha R'X) \circ (\phi_T R'X)) = \phi^{-1}Set^{\beta}(R'X,m'X)$, so $Set^{\alpha}(R'X,m'X) = Set^{\beta}(R'X,m'X)$,

References

[1] BARR, M. *Coequalizers and free triples.* Math. Z. 116 (1970), 307-322.

[2] BECK, J. *Distributive laws.* Lect. Notes in Math. 80. Springer (1969), 119-140.

[3] BUNGE, M. *On the relationship between composite and tensor product triples.* Journal of Pure and Applied Algebra 13 (1978), 139-156.

[4] CARUNCHO CASTRO, J.R. *Teoría de triples.* Alxebra 5. Dpto. Álgebra y Fundamentos. Santiago (1971).

[5] FREIRE NISTAL, J.L. *Propiedades universales de triples de grado superior.* Tesis. Alxebra 11. Dpto. Algebra y Fundamentos. Santiago (1971).

[6] FREYD, P. *Algebra-valued functors in general and tensor products in particular.* Colloq. Mathem. XIV (1966), 89-106.

[7] LOPEZ LOPEZ, M.A. *Algebras de Hopf respecto a un cotriple.* Tesis. Alxebra 17, Dpto. Álgebra y Fundamentos. Santiago (1976).

[8] LOPEZ LOPEZ, M.A. y VILLANUEVA NOVOA, E. *Producto de triples.* Rev. Mat. Hisp.-Amer. T. 38 (1978), 61-70.

[9] LOPEZ LOPEZ, M.A. y BARJA PEREZ, J.M. *El triple Hom(M,-) en conjuntos.* Alxebra 26. Dpto. Álgebra y Fundamentos. Santiago (1980), 51-61.

[10] MANES, E. *A triple miscellany.* Thesis. Wesleyan University (1967).

[11] MANES, E. *A triple theoretic construction of compact algebras.* Lect. Not. in Math. 80. Springer (1969), 91-118.

[12] MANES, E. *Algebraic Theories.* Springer (1976).

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