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On Multilinear Mappings of Nuclear Type

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"Dedicate to the memory of L. NACHBIN"

ABSTRACT. The space of multilinear mappings of nuclear type $(s;r_1,...,r_n)$ between Banach spaces is considered, some of its properties are described (including the relationship with tensor products) and its topological dual is characterized as a Banach space of absolutely summing mappings.

1. INTRODUCTION

In [15] A. Pietsch remarked that the attempt to classify the nonlinear operators between Banach spaces should have its starting point in the study of the different classes of multilinear mappings. See also [4] and [19]. On the other side the research in the Theory of Infinite Dimensional Holomorphy has dealt with several topological vector spaces of n-linear mappings. See [1], [3], [5], [6], [7], [8], [10], [11], [12], [13] and [14] among other articles. Motivated by these facts we introduce in this paper the class of n-linear mappings of nuclear type $(s;r_1,...,r_n)$. We may think them as the multilinear counterpart of the $(n,+\infty,q)$-nuclear linear operators of Pietsch (see [16], Chapter 18) as well as the natural generalization of the n-linear mappings of nuclear type considered by C. Gupta in [10]. Naturally we cannot forgive that the nuclear linear

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operators were introduced independently by A. Grothendieck in [9] and A.F. Ruston in [17] and [18]. The spaces we are studying will lead to the consideration of new examples of the holomorphy types conceptually introduced by L. Nachbin in [13] and S. Dineen in [7]. In section 2, after defining the vector space of n-linear mappings of nuclear type $(s;r_1,...,r_n)$, we consider a natural complete metrizable topology on it determined by a (quasi-)norm. In the presence of the bounded approximation property, we show that this topology can be generated by a simpler (quasi-)norm when restricted to a convenient dense subspace (that of the finite type n-linear mappings). In section 3 we consider certain (quasi-)norms on tensor products and their connections with the spaces of absolutely summing mappings. These results are a preparation for the proofs and results of section 4, which characterize the topological duals of the spaces of multilinear mappings of nuclear type as spaces of absolutely summing mappings. These characterizations might be important for a further study of convolution equations on certain spaces of entire functions on Banach spaces. See remark at the end of this paper.

I would like to thank professors K. Floret and J. Mujica for some remarks and suggestions made during my seminar exposition of these results at UNICAMP.

Now we fix the notations used in this paper. For Banach spaces $E_1,...,E_n$ and F over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) we denote by $\mathcal{L}(E_1,...,E_n;F)$ the Banach space of all continuous n-linear mappings from $E_1 \times ... \times E_n$ into F, under the natural norm

$$\|T\| = \sup_{\substack{x_k \in B_{E_k} \\ k=1,...,n}} \|T(x_1,...,x_n)\|.$$

Here B_{E_k} denotes the closed unit ball of E_k centered at 0. If φ_k is in the topological dual E_k' of E_k, $k = 1,...,n$ and $b \in F$ we denote by $\varphi_1 \times ... \times \varphi_n b$ the element of $\mathcal{L}(E_1,...,E_n;F)$ defined as being $\varphi_1(x_1)...\varphi_n(x_n)b$ at the point $(x_1,...,x_n)$. These mappings generate the vector subspace $\mathcal{L}_f(E_1,...E_n;F)$ of the finite type n-linear mappings.

If $s \in (0,+\infty)$ we denote by $\ell_s(F)$ (or ℓ_s, if $F = \mathbb{K}$) the vector space of all sequences $(y_j)_{j=1}^{\infty}$ of elements of F such that

$$\|(y_j)_{j=1}^{\infty}\|_s = [\sum_{j=1}^{\infty} \|y_j\|^s]^{\frac{1}{s}} < +\infty$$

If $s \geq 1$, $\|\cdot\|_s$ is a norm and, for $s < 1$, it is an s-norm. In any case we have a complete metrizable topological vector space. We denote by $\ell_s^w(F)$ the vector space of all sequences $(y_j)_{j=1}^{\infty}$ of elements of F such that

$$\|(y_j)_{j=1}^{\infty}\|_{w,s} = \sup_{\varphi \in B_{F'}} \|(\varphi(y_j))_{j=1}^{\infty}\|_s < +\infty$$

$(\ell_s^w(F), \|\cdot\|_{w,s})$ is a complete metrizable topological vector space. If $s=+\infty$ we consider $\ell_\infty(F) = \ell_\infty^w(F)$ as being the Banach space of all bounded sequences $(y_j)_{j=1}^{\infty}$ of elements of F under the norm

$$\|(y_j)_{j=1}^{\infty}\|_\infty = \|(y_j)_{j=1}^{\infty}\|_{w,\infty} = \sup_{j \in \mathbb{N}} \|y_j\|$$

2. MULTILINEAR MAPPINGS OF NUCLEAR TYPE $(s;r_1,\ldots,r_n)$

We consider $s \in (0,+\infty]$, $r_k \in [1,+\infty]$, $k = 1,\ldots,n$ such that

$$1 \leq \frac{1}{s} + \frac{1}{r_1'} + \ldots + \frac{1}{r_n'}$$

2.1. Definition. *A mapping $T \in \mathcal{L}(E_1,\ldots,E_n;F)$ is said to be nuclear of type $(s;r_1,\ldots,r_n)$ if there are $(\lambda_j)_{j=1}^{\infty} \in \ell_s (\in c_0$ if $s = +\infty)$, $(y_j)_{j=1}^{\infty} \in \ell_\infty(F)$ and $(\varphi_{kj})_{j=1}^{\infty} \in \ell_{r_k'}^w(E_k')$,, $k = 1,\ldots,n$ such that*

$$T(x_1,\ldots,x_n) = \sum_{j=1}^{\infty} \lambda_j \varphi_{1,j}(x_1)\ldots\varphi_{n,j}(x_n) y_j \tag{1}$$

The vector space of all such mappings is denoted by $\mathcal{L}_N^{(s;r_1,...,r_n)}(E_1,...,E_n;F)$ and it is a complete metrizable topological vector space under the t_n-norm

$$\|T\|_{N,(s;r_1,...,r_n)}=\inf\|(\lambda_j)_{j=1}^{\infty}\|_s\|(y_j)_{j=1}^{\infty}\|_{\infty}\prod_{k=1}^{n}\|(\varphi_{k,j})_{j=1}^{\infty}\|_{w,r_k'}$$

where the infimum is taken over all possible representations of T as described in (1) and $t_n \in (0,1]$ is given by

$$\frac{1}{t_n}=\frac{1}{s}+\frac{1}{r_1'}+...+\frac{1}{r_n'}$$

If $r_1 =...=r_n=r$ we replace $(s;r_1,...,r_n)$ by $(s;r)$ in the preceding notations.

If $t_n = 1$, s can be writen in terms of the r_k's and we say that T is of *nuclear type* $(r_1,...,r_n)$. In this case $(s;r_1,...,r_n)$ is replaced by $(r_1,...,r_n)$ (or r, if $r_1 = ... = r_n = r$) in the above notations. We simplify these notations in the case r=1 by ommiting the letter r in the notations.

The following result can be proved easily:

2.2. Proposition. *If $T \in \mathcal{L}_N^{(s;r_1,...,r_n)}(E_1,...,E_n;F)$, A_k belongs to $\mathcal{L}(D_k;E_k)$, $k=1,...,n$ and $S \in \mathcal{L}(F;G)$, then $S \circ T \circ (A_1,...,A_n)$ is of nuclear type $(s;r_1,...,r_n)$ and*

$$\|S\circ T\circ(A_1,...,A_n)\|_{N,(s;r_1,...,r_n)}\leq\|S\|\,\|T\|_{N,(s;r_1,...,r_n)}\prod_{k=1}^{n}\|A_k\|$$

Another characterization of the n-linear mappings of nuclear type $(s;r_1,...,r_n)$ uses the following examples.

2.3. Examples.

(1) It is clear that $\mathcal{L}_f(E_1,...,E_n;F) \subset \mathcal{L}_N^{(s;r_1,...,r_n)}(E_1,...,E_n;F)$ and

$$\|\varphi_1\times\ldots\times\varphi_n b\|_{N,(s;r_1,\ldots,r_n)}=\|\varphi_1\|\ldots\|\varphi_n\|\|b\|$$

for $\varphi_k \in E_k'$, $k = 1,\ldots,n$ and $b\in F$.

(2) An easy application of Hölder's inequality shows that

$$\|T\|\leq\|T\|_{N,(s;r_1,\ldots,r_n)}$$

for every $T \in \mathcal{L}_N^{(s;r_1,\ldots,r_n)}(E_1,\ldots,E_n;F)$.

(3) We consider $(\sigma_j)_{j=1}^{\infty}$ in ℓ_s for $s \in (0,+\infty)$ and in c_0 for $s = +\infty$. Now we consider the "diagonal" mapping

$$D_{(\sigma_j)_{j=1}^{\infty}} \in \mathcal{L}(\ell_{r_1'},\ldots,\ell_{r_n'};\ell_1)$$

defined by

$$D_{(\sigma_j)_{j=1}^{\infty}}((\xi_{1,j})_{j=1}^{\infty},\ldots,(\xi_{n,j})_{j=1}^{\infty})=(\sigma_j\xi_{i,j}\ldots\xi_{n,j})_{j=1}^{\infty}$$

We note that this mapping can be represented by

$$D_{(\sigma_j)_{j=1}^{\infty}}=\sum_{j=1}^{\infty}\sigma_j(\pi_j\times\ldots\times\pi_j)e_j$$

where $\pi_j((\xi_{k,m})_{m=1}^{\infty}) = \xi_{k,j}$ for $k = 1,\ldots,n$ and $j \in \mathbb{N}$ and $e_j = (0,\ldots,0,1,0\ldots)$, with 1 in the j-th component. Since $(\pi_j)_{j=1}^{\infty} \in \ell_{r_k'}^{w}(\ell_{r_k}) \subset \ell_{r_k'}^{w}((\ell_{r_k'})')$ (stric inclusion for $r_k' = +\infty$) with $\|(\pi_j)_{j=1}^{\infty}\|_{w,r_k'}=1$, for $k = 1,\ldots,n$ and $\|e_j\| = 1$ for all $j \in \mathbb{N}$ we have $D_{(\sigma_j)_{j=1}^{\infty}}$ of nuclear type $(s;r_1,\ldots,r_n)$ and

$$\|D_{(\sigma_j)_{j=1}^{\infty}}\|_{N,(s;r_1;\ldots,r_n)}\leq\|(\sigma_j)_{j=1}^{\infty}\|_s$$

2.4. Theorem. *For $T \in \mathcal{L}(E_1,\ldots,E_n;F)$ the following conditions are equivalent:*

(a) T is of nuclear type $(s; r_1,\ldots,r_n)$.

(b) There are $A_k \in \mathcal{L}(E_k;\ell_{r_k'})$, $k = 1,\ldots,n$, $Y \in \mathcal{L}(l_1;F)$ and $(\sigma_j)_{j=1}^{\infty} \in \ell_s$ ($\in c_0$ when $s = +\infty$) such that

$$T=Y\circ D_{(\sigma_j)_{j=1}^{\infty}}\circ(A_1,\ldots,A_k)$$

In this case

$$\|T\|_{N,(s;r_1,\ldots,r_n)}=\inf\|Y\|\prod_{k=1}^{n}\|A_k\|\cdot\|(\sigma_j)_{j=1}^{\infty}\|_s$$

where the infimum is taken over all possible factorizations as decribed in (b).

Proof. By 2.2 and 2.3 (3) we have that (b) implies (a) with

$$\|T\|_{N,(s;r_1,\ldots,r_n)}\leq\|Y\|\prod_{k=1}^{n}\|A_k\|\cdot\|(\sigma_j)_{j=1}^{\infty}\|_s$$

Now we assume (a). For each $\varepsilon > 0$ we consider a representation of T as in 2.1. such that

$$\|(\lambda_j)_{j=1}^{\infty}\|_s\|(y_j)_{j=1}^{\infty}\|_{\infty}\prod_{k=1}^{n}\|(\varphi_{k,j})_{j=1}^{\infty}\|_{w,r_k'}\leq(1+\varepsilon)\|T\|_{N,(s;r_1,\ldots,r_n)}$$

If we define $A_k(x_k) = (\varphi_{k,j}(x_k))_{j=1}^{\infty}$ for $x_k \in E_k$, then $A_k \in \mathcal{L}(E_k;\ell_{r_k'})$ with $\|A_k\| \leq \|(\varphi_{kj})_{j=1}^{\infty}\|_{w,r_k'}$, $k = 1,\ldots,n$. Now we consider $Y \in \mathcal{L}(\ell_1;F)$ defined by

$$Y((\xi_j)_{j=1}^{\infty})=\sum_{j=1}^{\infty}\xi_j b_j$$

for $(\xi_j)_{j=1}^{\infty} \in \ell_1$. Thus $\|Y\| \leq \|(b_j)_{j=1}^{\infty}\|_{\infty}$. If follows that

$$T=Y\circ D_{(\lambda_j)_{j=1}^{\infty}}\circ(A_1,\ldots,A_n)$$

with

$$\|Y\| \|(\lambda_j)_{j=1}^{\infty}\|_s \prod_{k=1}^{n} \|A_k\| \leq (1+\varepsilon)\|T\|_{N,(s;r_1,\dots,r_n)}$$

Now we consider some inclusion results.

2.5. Theorem. *For $s,t \in (0,+\infty]$, $r_k,p_k \in [1,+\infty]$ such that $s \leq t$, $r_k \leq p_k$, $k = 1,\dots,n$*

$$1 \leq \frac{1}{s} + \frac{1}{r_1'} + \dots + \frac{1}{r_n'}, \quad 1 \leq \frac{1}{t} + \frac{1}{p_1'} + \dots + \frac{1}{p_n'}$$

and

$$\frac{1}{r_1} + \dots + \frac{1}{r_n} - \frac{1}{s} \leq \frac{1}{p_1} + \dots + \frac{1}{p_n} - \frac{1}{t}$$

then $\mathcal{L}_N^{(s;r_1,\dots,r_n)}(E_1,\dots,E_n;F) \subset \mathcal{L}_N^{(t;p_1,\dots,p_n)}(E_1,\dots,E_n;F)$ with

$$\|T\|_{N,(t;p_1,\dots,p_n)} \leq \|T\|_{N,(s;r_1,\dots,r_n)}$$

for every T of nuclear type $(s;r_1,\dots,r_n)$.

Proof. We consider

$$\frac{1}{q_k} = \frac{1}{r_k} - \frac{1}{p_k}, \quad k=1,\dots,n; \quad \frac{1}{u} = \frac{1}{s} - \left(\frac{1}{q_1} + \dots + \frac{1}{q_n}\right)$$

Hence $u \leq t$. For T of nuclear type $(s;r_1,\dots,r_n)$ and $\varepsilon > 0$ we choose a representation of T in the form

$$T=\sum_{j=1}^{\infty}\sigma_j\varphi_{1,j}\times\ldots\times\varphi_{n,j}y_j$$

such that $\sigma_j \geq 0$ for all $j\in\mathbb{N}$ and

$$\|(\sigma_j)_{j=1}^{\infty}\|_s\prod_{k=1}^{n}\|(\varphi_{k,j})_{j=1}^{\infty}\|_{w,r_k'}\|(y_j)_{j=1}^{\infty}\|_{\infty}\leq(1+\varepsilon)\|T\|_{N,(s;r_1,\ldots,r_n)}.$$

We can write

$$T=\sum_{j=1}^{\infty}\sigma_j^{\frac{s}{u}}\left(\sigma_j^{\frac{s}{q_1}}\varphi_{1,j}\right)\times\ldots\times\left(\sigma_j^{\frac{s}{q_n}}\varphi_{n,j}\right)y_j$$

and have

$$\|(\sigma_j^{\frac{s}{u}})_{j=1}^{\infty}\|_t\leq\|(\sigma_j^{\frac{s}{u}})_{j=1}^{\infty}\|_u=\left[\|(\sigma_j)_{j=1}^{\infty}\|_s\right]^{\frac{s}{u}}$$

$$\|(\sigma_j^{\frac{s}{q_k}}(\varphi_{k,j})_{j=1}^{\infty}\|_{w,p_k'}\leq\|(\sigma_j^{\frac{s}{q_k}})_{j=1}^{\infty}\|_{q_k}\|(\varphi_{k,j})_{j=1}^{\infty}\|_{w,r_k'}$$

$$=\left[(\sigma_j)_{j=1}^{\infty}\|_s\right]^{\frac{s}{q_k}}\|(\varphi_{k,j})_{j=1}^{\infty}\|_{w,r_k'}$$

for $k = 1,\ldots,n$. Thus T is of nuclear type $(t;p_1,\ldots,p_n)$ and

$$\|T\|_{N,(t;p_1,\ldots,p_n)}\leq(1+\varepsilon)\|T\|_{N,(s;r_1,\ldots,r_n)}$$

for each $\varepsilon > 0$.

2.6. Corollary. *(1) If $r_k \leq p_k$, $k = 1,\ldots,n$ every T of nuclear type $(r_1,\ldots,r_n)$ is of nuclear type $(p_1,\ldots,p_n)$ and*

$$\|T\|_{N,(p_1,\dots,p_n)} \leq \|T\|_{N,(r_1,\dots,r_n)}$$

(2) If $s \leq t$, every T of nuclear type $(s; r_1,\dots,r_n)$ is of nuclear type $(t;r_1,\dots,r_n)$ and

$$\|T\|_{N,(t;r_1,\dots,r_n)} \leq \|T\|_{N,(s;r_1,\dots,r_n)}$$

(3) If $r_k \geq p_k$, $k = 1,\dots,n$ every T of nuclear type $(s;r_1,\dots,r_n)$ is of nuclear type $(s;p_1;\dots,p_n)$ and

$$\|T\|_{s,(t;p_1,\dots,p_n)} \leq \|T\|_{N,(s;r_1,\dots,r_n)}$$

2.7. Remark. It follows from the definition that $\mathfrak{L}_f(E_1,\dots,E_n;F)$ is dense in $\mathfrak{L}_N^{(s;r_1,\dots,r_n)}(E_1,\dots,E_n;F)$. Since every T of $\mathfrak{L}_f(E_1,\dots,E_n;F)$ has a *finite* representation of the form

$$T=\sum_{j=1}^{m} \sigma_j \varphi_{1,j} \times \dots \times \varphi_{n,j} b_j \tag{2}$$

with $\sigma_j \in \mathbb{K}$, $\varphi_{k,j} \in E_k'$, $k = 1,\dots,n$, $b_j \in F$, $j = 1,\dots,m$, it is natural to ask when it is possible to have

$$\|T\|_{N,(s;r_1;\dots,r_n)} = \|T\|_{N_f(s;r_1,\dots,r_n)}$$

where

$$\|T\|_{N_f(s;r_1,\dots,r_n)} = \inf \|(\sigma_j)_{j=1}^m\|_s \prod_{k=1}^{n} \|(\varphi_{k,j})_{j=1}^m\|_{w,r_k'} \|(b_j)_{j=1}^m\|_\infty$$

with the infimum taken for all representations of T as in (2). It is clear that we always have

$$\|T\|_{N,(s;r_1;\dots,r_n)} \leq \|T\|_{N_f(s;r_1,\dots,r_n)}$$

It is natural to hope that equality might take place when E_k is finite dimensional for $k = 1,...,n$. This is indead a fact as we show in the following result.

2.8. Proposition. *If $E_1,...,E_n$ are finite dimensional and $T \in \mathcal{L}(E_1,...,E_n;F)$, then*

$$\|T\|_{N_f(s;r_1,...,r_n)} = \|T\|_{N,(s;r_1,...,r_n)}$$

Proof. In this case $\mathcal{L}(E_1,...,E_n;F) = \mathcal{L}_f(E_1,...,E_n;F)$ and this is a complete space for both t_n-norms $\|\cdot\|_{N,(s;r_1,...,r_n)}$ and $\|\cdot\|_{N_f,(s;r_1,...,r_n)}$. By the open mapping theorem these t_n-norms are equivalent. Hence there is $c \geq 0$ such that

$$\|T\|_{N_f(s;r_1;...,r_n)} \leq c\|T\|_{N,(s;r_1,...,r_n)}$$

for every T of finite type. For $\varepsilon > 0$ we choose a representation

$$T = \sum_{j=1}^{\infty} \sigma_j \varphi_{1,j} \times ... \times \varphi_{n,j} y_j$$

such that

$$\|(\sigma_j)_{j=1}^{\infty}\|_s \|(y_j)_{j=1}^{\infty}\|_{\infty} \prod_{k=1}^{n} \|(\varphi_{k,j})_{j=1}^{\infty}\|_{w,r_k'} \leq (1+\varepsilon)\|T\|_{N,(s;r_1,...,r_n)}$$

There is $m \in N$ such that

$$c\|\sum_{j>m} \sigma_j \varphi_{1,j} \times ... \times \varphi_{n,j} y_j\|_{N,(s;r_1,...,r_n)} \leq \varepsilon \|T\|_{N,(s;r_1,...,r_n)}$$

We have

$$(\|T\|_{N_\rho(s;r_1,\dots,r_n)})^{t_n} \leq (\|\sum_{j=1}^{m} \sigma_j\varphi_{1,j} \times \dots \times \varphi_{n,j}y_j\|_{N_\rho(s;r_1,\dots,r_n)})^{t_n}$$

$$+(\|\sum_{j>m} \sigma_j\varphi_{1,j} \times \dots \times \varphi_{n,j}y_j\|_{N_\rho(s;r_1,\dots,r_n)})^{t_n}$$

$$\leq (1+\varepsilon)^{t_n}(\|T\|_{N,(s;r_1,\dots,r_n)})^{t_n}$$

$$+c^{t_n}(\|\sum_{j>m} \sigma_j\varphi_{1,j} \times \dots \times \varphi_{n,j}y_j\|_{N_\rho(s;r_1,\dots,r_n)})^{t_n}$$

$$\leq \left[(1+\varepsilon)^{t_n}+\varepsilon^{t_n}\right](\|T\|_{N,(s;r_1,\dots,r_n)})^{t_n}$$

2.9. Proposition. *If* $T \in \mathcal{L}_N^{\rho(s;r_1,\dots,r_n)}(E_1,\dots,E_n;F)$ *and* S_k *is in* $\mathcal{L}_f(D_k;E_k)$, $k=1,\dots,n$, *then*

$$\|T\circ(S_1,\dots,S_n)\|_{N_\rho(s;r_1,\dots,r_n)} \leq \|T\|_{N,(s;r_1,\dots,r_n)} \prod_{k=1}^{n} \|S_k\|$$

Proof. If J_k is the natural injection from $S_k(D_k)$ into E_k, we can write $S_k = J_k \circ \tilde{S}_k$ with $\|\tilde{S}_k\| = \|S_k\|$, $k = 1,\dots,n$. Hence $T \circ (J_1,\dots,J_n)$ is in $\mathcal{L}_f(S_1(D_1),\dots,S_n(D_n);F)$. Now we apply 2.8. and 2.2. in order to have the result.

2.10. Theorem. *If* $E_1',\dots,E_n'$ *have the bounded approximation property, then*

$$\|T\|_{N_f(s;r_1,\dots,r_n)}=\|T\|_{N,(s;r_1,\dots,r_n)}$$

for every $T \in \mathcal{L}_f(E_1,\dots,E_n;F)$.

Proof. First we consider $n = 2$. The proof for other n's is analogous as one can note easily. Since $T_1 \in \mathcal{L}_f(E_1;\mathcal{L}(E_2;F))$ for $T_1(x_1)(x_2) = T(x_1,x_2)$, $x_k \in E_k$, $k = 1,2$, for each $\varepsilon > 0$ there is $S_1 \in \mathcal{L}_f(E_1;E_1)$ such that $T_1 \circ S_1 = T_1$ and $\|S_1\| \leq (1+\varepsilon)\lambda_1$ (because E_1' has the λ_1-approximation property for some $\lambda_1 > 0$). Hence

$$T(S_1(x_1),x_2)=T(x_1,x_2) \qquad (\forall x_k\in E_k,\ k=1,2)$$

By the same reasoning $T_2 \in \mathcal{L}_f(E_2;\mathcal{L}(E_1;F))$, given by $T_2(x_2)(x_1) = T(x_1,x_2)$, with $x_k \in E_k$, $k = 1,2$, is such that there is $S_2 \in \mathcal{L}_f(E_2;E_2)$ satisfying $T_2 \circ S_2 = T_2$ and $\|S_2\| \leq (1+\varepsilon)\lambda_2$. We have

$$T(x_1,S_2(x_2))=T(x_1,x_2) \qquad (\forall x_k\in E_k,\ k=1,2)$$

Thus $T = T \circ (S_1,S_2)$ and, by 2.9,

$$\|T\|_{N_f(s;r_1,r_2)}=\|T\circ(S_1,S_2)\|_{N_f(s;r_1,r_2)}$$

$$\leq\|T\|_{N,(s;r_1,r_2)}\|S_1\|\|S_2\|\leq(1+\varepsilon)^2\lambda_1\lambda_2\|T\|_{N,(s;r_1,r_2)}$$

Therefore

$$\|T\|_{N_f(s;r_1,r_2)}\leq\lambda_1\lambda_2\|T\|_{N,(s;r_1,r_2)}$$

Now with the same argument used in the proof of 2.8 we have

$$\|T\|_{N_f(s;r_1,r_2)}\leq\|T\|_{N,(s;r_1,r_2)}$$

3. ABSOLUTELY SUMMING MULTILINEAR MAPPINGS AND TENSOR PRODUCTS

We recall the concept of absolutely $(s;r_1,...,r_n)$-summing n-linear mapping from $E_1 \times ... \times E_n$ into F (See Pietsch [14] for scalar valued mappings).

In this section, s, $r_k \in (0,+\infty]$, $k = 1,...,n$ are such that

$$\frac{1}{s} \leq \frac{1}{r_1} + ... + \frac{1}{r_n}$$

3.1. Definition. *A mapping $T \in \mathcal{L}(E_1,...,E_n;F)$ is said to be absolutely $(s;r_1,...,r_n)$-summing if there is $c \geq 0$ such that*

$$\|(T(x_{1,j},...,x_{n,j}))_{j=1}^m\|_s \leq c \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^m\|_{w,r_k} \tag{3}$$

for $m \in \mathbb{N}$, $x_{k_j} \in E_k$, $k = 1,...,n$ and $j = 1,...,m$.

We denote by $\mathcal{L}_{as}^{(s;r_1,...,r_n)}(E_1,...,E_n;F)$ the vector space of all these mappings. The smallest possible value of c in (3) is denoted by $\|T\|_{as,(s;r_1,...,r_n)}$. This gives a s-norm (for $s \in (0,1)$) and a norm (for $s \geq 1$) making $\mathcal{L}_{as}^{(s;r_1,...,r_n)}(E_1,...,E_n;F)$ a complete metrizable topological vector space. When $r_1 = ... = r_n = r$ we replace $(s;r_1,...,r_n)$ by $(s;r)$ in the preceding notations. In the last case for $s=r$ we replace $(s;r)$ by r in the above notations. When $r = 1$ we do not write it.

An interesting special case of absolutely $(s;r_1,...,r_n)$-summing mappings is obtained when we have

$$\frac{1}{s} = \frac{1}{r_1} + ... + \frac{1}{r_n}$$

In this situation the mappings are called $(r_1,...,r_n)$-*dominated* and we denote the corresponding vector space and s-norm (or norm) by

$\mathcal{L}_d^{(r_1,\ldots,r_n)}(E_1,\ldots,E_n;F)$ and $\|\cdot\|_{d,(r_1,\ldots,r_n)}$ respectively. If $r_1 = \ldots = r$ we replace $(r_1,\ldots,r_n)$ by r in the preceding notations. When $r = 1$ we do not write it. The use of the word "dominated" is justified by the following result mentioned by Pietsch in [14], for scalar valued mappings.

3.2. Theorem. *For $T \in \mathcal{L}(E_1,\ldots,E_n;F)$ and $s, r_k \in (0,+\infty]$, $k = 1,\ldots,n$ such that*

$$\frac{1}{s}=\frac{1}{r_1}+\ldots+\frac{1}{r_n}$$

the following conditions are equivalent

(a) *T is $(r_1,\ldots,r_n)$-dominated*

(b) *There are $c \geq 0$ and regular probability measures $\mu_k \in W(B_{E_k'})$, $k=1,\ldots,n$ such that*

$$\|T(x_1,\ldots,x_n)\|\leq c\prod_{k=1}^{n}\left[\int_{B_{E_k'}}|\varphi(x_k)|^{r_k}d\mu_k(\varphi)\right]^{\frac{1}{r_k}} \tag{4}$$

for ever $x_k \in E_k$, $k = 1,\ldots,n$.

In this case

$$\inf_{(4)} c=\min_{(4)} c=\|T\|_{d,(r_1,\ldots,r_n)}$$

Here $W(B_{E_k'})$ denotes the set of all regular probability measures defined on the Borel σ-algebra of $B_{E_k'}$, with the weak-star topology.

Now we are going to introduce some special t_n-norms on $E_1 \otimes \ldots \otimes E_n \otimes F$ in such way that the topological dual of this metrizable topological vector space is isometric to a space of F'-valued absolutely summing mappings on $E_1 \times \ldots \times E_n$.

3.3. Definition. *For $r_k \in (0,+\infty]$, $k = 1,\ldots,n$ and $s \in [1,+\infty]$ such that*

$$\frac{1}{s} \leq \frac{1}{r_1} + \ldots + \frac{1}{r_n} \tag{5}$$

we consider $t_n \in (0,1]$ such that

$$\frac{1}{t_n} = \frac{1}{r_1} + \ldots + \frac{1}{r_n} + \frac{1}{s'} \geq 1$$

and define

$$\delta_{(s;r_1,\ldots,r_n,+\infty)}(u) = \inf \|(\lambda_j)_{j=1}^m\|_{s'} \prod_{k=1}^n \|(x_{k,j})_{j=1}^m\|_{w,r_k} \|(b_j)_{j=1}^m\|_\infty$$

where $u \in E_1 \otimes \ldots \otimes E_n \otimes F$ and the infimum is taken over all representations of u of the form

$$u = \sum_{j=1}^m \lambda_j x_{1,j} \otimes \ldots \otimes x_{n,j} \otimes b_j$$

with $m \in \mathbb{N}$, $x_{k,j} \in E_k$, $k = 1,\ldots,n$, $\lambda_j \in \mathbb{K}$, $b_j \in F$, $j = 1,\ldots,m$.

When we have equality in (5), $t_n = 1$. In this case we have a norm denoted by $\delta_{(r_1,\ldots,r_n,+\infty)}$.

3.4. Proposition. *$\delta_{(s;r_1,\ldots,r_n,+\infty)}$ is a t_n-norm on $E_1 \otimes \ldots \otimes E_n \otimes F$ and $\varepsilon \leq \delta_{(s;r_1,\ldots,r_n,+\infty)}$, where ε denotes the injective norm on $E_1 \otimes \ldots \otimes E_n \otimes F$.*

3.5. Proposition. *If $s \leq t$, $r_k \leq p_k$, $k = 1,\ldots,n$ and*

$$\frac{1}{r_1} + \ldots + \frac{1}{r_n} - \frac{1}{s} \leq \frac{1}{p_1} + \ldots + \frac{1}{p_n} - \frac{1}{t}$$

then

$$\delta_{(s;r_1,\dots,r_n,+\infty)} \le \delta_{(t;p_1,\dots,p_n,+\infty)}$$

Proof. It is an adaptation to this case of the proof of 2.5.

3.6. Remark. It is a consequence of 3.5. and the fact that $\delta_{(+\infty,\dots,+\infty,+\infty)} = \Pi$ (the projective norm) that all norms $\delta_{(r_1,\dots,r_n,+\infty)}$ are reasonable cross-norms on $E_1 \otimes \dots \otimes E_n \otimes F$.

3.7. Theorem. *The topological dual of* $(E_1 \otimes \dots \otimes E_n \otimes F, \delta_{(s;r_1,\dots,r_n,+\infty)})$ *is isometric to* $\mathcal{L}_{as}^{(s;r_1,\dots,r_n)}(E_1 \otimes \dots \otimes E_n;F')$.

Proof. (1) If $T \in \mathcal{L}_{as}^{(s;r_1,\dots,r_n)}(E_1 \otimes \dots \otimes E_n;F')$ we consider the linear functional f_T on $E_1 \otimes \dots \otimes E_n \otimes F$ defined by

$$f_T(u)=\sum_{j=1}^{m} \lambda_j T(x_{1,j},\dots,x_{n,j})(b_j)$$

for u with a representation of the form

$$u=\sum_{j=1}^{m} \lambda_j x_{1,j} \otimes \dots, x_{n,j} \otimes b_j$$

where $m \in \mathbb{N}$, $\lambda_j \in \mathbb{K}$, $x_{k,j} \in E_k$, $k = 1,\dots,n$, $b_j \in F$, $j = 1,\dots,m$. Hence

$$|f_T(u)| \le \|(\lambda_j)_{j=1}^m\|_{s'} \|(T(x_{1,j},\dots,x_{n,j}))_{j=1}^m\|_s \|(b_j)_{j=1}^m\|_\infty$$

$$\le \|(\lambda_j)_{j=1}^m\|_{s'} \|T\|_{as,(s;r_1,\dots,r_n)} \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^m\|_{w,r_k} \|(b_j)_{j=1}^m\|_\infty$$

Thus f_T is $\delta_{(s;r_1,\dots,r_n,+\infty)}$-continuous and

$$\|f_T\| \leq \|T\|_{as;(s;r_1,\ldots,r_n)}$$

(2) Now we consider a continuous linear functional f on $(E_1 \otimes \ldots \otimes E_n \otimes F, \delta_{(s;r_1,\ldots,r_n,+\infty)})$ and define the n-linear mapping T_f from $E_1 \times \ldots \times E_n$ into F' by

$$T_f(x_1,\ldots,x_n)(b) = f(x_1 \otimes \ldots \otimes x_n \otimes b)$$

for $x_k \in E_k$, $k = 1,\ldots,n$ and $b \in F$.

We consider $m \in \mathbb{N}$, $x_{k,j} \in E_k$, $k = 1,\ldots,n$ and $j = 1,\ldots,m$. For each $\varepsilon>0$ there are $b_j \in F$, $\|b_j\| = 1$, $j = 1,\ldots,m$ such that

$$\sum_{j=1}^{m} \|T_f(x_{1,j},\ldots,x_{n,j})\|^s \leq \varepsilon + \sum_{j=1}^{m} |T_f(x_{1,j},\ldots,x_{n,j})(b_j)|^s = \otimes$$

For conveniente $\lambda_j \in \mathbb{K}$, $|\lambda_j| = 1$, $j = 1,\ldots,m$ we may write

$$\otimes = \varepsilon + |\sum_{j=1}^{m} \lambda_j f(|f(x_{1,j} \otimes \ldots \otimes x_{n,j} \otimes b_j)|^{s-1} x_{1,j} \otimes \ldots \otimes x_{n,j} \otimes b_j)|$$

$$\leq \varepsilon + \|f\| \cdot \delta_{(s;r_1,\ldots,r_n,+\infty)}(\sum_{j=1}^{m} \lambda_j |f(x_{1,j} \otimes \ldots \otimes x_{n,j} \otimes b_j)|^{s-1} x_{1,j} \otimes \ldots \otimes x_{n,j} \otimes b_j)$$

$$\leq \varepsilon + \|f\| [\sum_{j=1}^{m} |f(x_{1,j} \otimes \ldots \otimes x_{n,j} \otimes b_j)|^{(s-1)s'}]^{\frac{1}{s'}} \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^{m}\|_{w,r_k} \|(b_j)_{j=1}^{m}\|_{\infty}$$

$$\leq \varepsilon + \|f\| \left[\|T_f(x_{1,j},\ldots,x_{n,j}))_{j=1}^{m}\|_s \right]^{\frac{1}{s'}} \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^{m}\|_{w,r_k}$$

Since $\varepsilon > 0$ is arbitrary we have

$$\|(T_f(x_{1,j},\ldots,x_{n,j}))_{j=1}^m\|_s \leq \|f\| \prod_{k=1}^{n} \|(x_{k,j})_{j=1}^m\|_{w,r_k}$$

Hence T_f is absolutely $(s;r_1,\ldots,r_n)$-summing and

$$\|T_f\|_{as,(s;r_1,\ldots,r_n)} \leq \|f\|$$

4. CHARACTERIZATION OF THE TOPOLOGICAL DUALS OF THE SPACES OF MAPPINGS OF NUCLEAR TYPE

In this section, $s, r_k \in [1,+\infty]$, $k = 1,\ldots,n$ and

$$1 \leq \frac{1}{s} + \frac{1}{r_1'} + \ldots + \frac{1}{r_k'} \tag{6}$$

If we consider $E_1' \otimes \ldots \otimes E_n' \otimes F$ with $\delta_{(s';r'_1,\ldots,r'_n,+\infty)}$ it is clear that it is isometric to $\mathcal{L}_f(E_1,\ldots,E_n;F)$ with $\|\cdot\|_{N(s;r_1,\ldots,r_n)}$ through the mapping taking

$$u = \sum_{j=1}^{m} \lambda_j \varphi_{1,j} \otimes \ldots \otimes \varphi_{n,j} \otimes b_j$$

into

$$T_u = \sum_{j=1}^{m} \lambda_j \varphi_{1,j} \times \ldots \times \varphi_{n,j} b_j$$

This fact, Theorem 3.7. and Theorem 2.10 give the following characterization result.

4.1. Theorem. *If $E_1',\ldots,E_n'$ have the bounded approximation property, the topological dual of $\mathcal{L}_N^{(s;r_1,\ldots,r_n)}(E_1,\ldots,E_n;F)$ is isometric to $\mathcal{L}_{as}^{(s';r_1,\ldots,r_n)}(E_1',\ldots,E_n';F')$ through the mapping*

$$B(\psi)(\varphi_1,\ldots,\varphi_n)(b)=\psi(\varphi_1 \times \ldots \times \varphi_n b)$$

for all $b \in F$, $\varphi_k \in E_k'$, $k = 1,\ldots, n$ *and* ψ *in the required dual.*

In particular the topological dual of $\mathcal{L}_N^{(r_1,\ldots,r_n)}(E_1,\ldots,E_n;F)$ *is isometric to* $\mathcal{L}_d^{(r'_1,\ldots,r'_n)}(E_1',\ldots,E_n';F')$ [*under condition (5) with* $s \geq 1$].

A result of A. Defant and J. Voigt (see [2] for a proof) state that $\mathcal{L}_{as}(E_1,\ldots,E_n;\mathbb{K})$ is identically isometric to $\mathcal{L}(E_1,\ldots,E_n;\mathbb{K})$. If $n \geq 2$ it is proved in [2] that $\mathcal{L}_{as}(E_1,\ldots,E_n;F)$ is identically homeomorphic to $\mathcal{L}(E_1,\ldots,E_n;F)$ when E_k has the Orlicz property, $k = 1,\ldots,n$. These results and Theorem 4.1 allow us to state the following results.

4.2. Theorem. *If* $E_1',\ldots,E_n'$ *have the bounded approximation property, the topological dual of the space of scalar valued n-linear mappings of nuclear type* $(+\infty;+\infty)$ *on* $E_1 \times \ldots \times E_n$ *is isometric to* $\mathcal{L}(E_1',\ldots,E_n';\mathbb{K})$.

4.3. Theorem. *If* E_k *is either* c_0 *or* ℓ_p, $p \in [2,+\infty)$, *for* $k = 1,\ldots,n$ *and* $n \geq 2$, *then the topological dual of* $\mathcal{L}_N^{(+\infty;+\infty)}(E_1,\ldots,E_n;F)$ *is homeomorphic to* $\mathcal{L}(E_1',\ldots,E_n';F')$.

Remark. When $s \leq r$, $s \in (0,+\infty]$ and $r \in [1,+\infty]$ it is possible to consider the n-homogeneous polynomials from E into F of nuclear type $(s;r)$ canonically associated to the symmetric elements of $\mathcal{L}_N^{(s;r)}(E\ldots,E;F)$. Analogous results as described in this paper are still valid for the spaces of polynomials. Now the use of the idea of homomorphy types (see [13] and [7]) allows us to consider holomorphic mappings of nuclear type $(s;r)$ from open subsets of E into F. See [10] for the case $s = r = 1$. A careful lecture of the result of [10] shows us that we can define a complete metrizable space of entire functions on E of bounded nuclear type $(s;r)$, with topological dual isomorphic to a natural subspace of "absolutely (s',r')-summing" entire functions on E'. Again following the routine of [10] we can show that convolution equations on the space of entire functions of bounded nuclear type $(+\infty,+\infty)$ have always solutions and a

Malgrange type approximation result holds for solutions of the homogeneous equations. Theorem 4.2. is a key lemma for the proof of these facts. A hard open problem is to know if results of this type for convolution equations can be proved when $(s;r)$ is different from $(1;1)$ and $(+\infty;+\infty)$. Another problem is to check if the results of Dinnen's paper [7] are true in this case.

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