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Dunford-Pettis-like Properties of Projective and Natural Tensor Product Spaces

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ABSTRACT. Several properties of weakly p-summable sequences and of the scale of *p-converging* operators (i.e., operators transforming weakly p-summable sequences into convergent sequences) in projective and natural tensor products with an l_p space are considered. The last section studies the *Dunford-Pettis property of order p* (i.e., every weakly compact operator is p-convergent) in those spaces.

0. INTRODUCTION

In this paper several properties of the scale of p-converging operators in projective and natural tensor products with an l_p space are considered. This scale, introduced in [2] and [3], is intermediate between the ideals of *unconditionally converging* operators and the ideal of *completely continuous* or *Dunford-Pettis* operators. Since p-converging operators are characterized by the property of sending *weakly-p-summable* sequences into convergent ones, a part of the study is devoted to a special class of subsets of vector sequence spaces, termed *almost compact* sets, nontrivial examples of which are, in certain spaces, precisely the weakly-p-summable sequences, $1\leq p<+\infty$. Section 2 characterizes the compact sets of $l_p\widehat{\otimes}_\pi X$ and $l_p\widehat{\otimes}_{\Delta_p}$ X, extending results of Leonard [8] and Bombal [1]. Section 3 considers the *Dunford-Pettis properties of order p* in projective and natural tensor product spaces of l_p and a Banach space X. For l_p-sums of sequences of Banach spaces, generalizations of results of Bombal [1] are obtained. Those properties were introduced in [3].

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1. BACKGROUND

Throughout the paper p^* denotes the conjugate number of p. We base our approach to the properties of natural and projective tensor products on the use of the representations of those spaces as sequence spaces. A sequence (x_n) in a Banach space X is said to be *weakly-p-summable* $(p \geq 1)$ if for every $x^* \in X^*$ the sequence $(x^*(x_n))$ is in l_p; equivalently (see [7] 19.4), if there is a constant $C>0$ such that, for each (ξ_n) in l_{p^*}, $w_p((x_n)) = \sup_k \{\|\Sigma_{n=1}^k \xi_n x_n\| : \|(\xi_n)\|_{l_{p^*}} \leq 1\} \leq C < +\infty$. (Here, if $p=1$, c_0 plays the role of l_∞.) It is said to be *absolutely-p-summable,* when $p \geq 1$, if $s_p((x_n)) = (\Sigma_{n=1}^{+\infty} \|x_n\|^p)^{\frac{1}{p}} < +\infty$. (If $p=+\infty$, the l_p norm has to be replaced by the *sup* norm.) It is said to be *strongly-p-summable* for $p \geq 1$ if $\sigma_p((x_n)) = \sup\{|\Sigma_{n=1}^{+\infty} f_n(x_n)| : w_{p^*}(\{f_n\}) \leq 1, (f_n) \in X^*\} < +\infty$. Following [7], we shall denote by $l_p[X]$, $l_p\{X\}$ and $l_p<X>$ respectively the spaces of weakly-p-summable, absolutely-p-summable and strongly-p-summable sequences of X, endowed with their natural topologies: those induced by the norms w_p, s_p and σ_p, respectively. The following isometries are well-known (see [7] 19.4.3): $l_p[X] = L(l_{p^*}, X)$, for $1<p<+\infty$, and $l_1[X] = L(c_0, X)$. The symbols π and ε shall denote the projective and injective norms on the space $l_p \otimes X$: they are, respectively, the strongest and coarsest *crossnorms* (i.e. norms satisfying $\|x \otimes y\| = \|x\| \, \|y\|$) which is possible to define on that space. The symbol Δ_p denotes the norm induced by s_p over $l_p \otimes X$; the topology induced by s_p is termed the *natural* topology. We shall denote by $l_p \hat{\otimes}_\varepsilon X$, $l_p \hat{\otimes}_\pi X$ *and* $l_p \hat{\otimes}_{\Delta_p} X = l_p\{X\}$ the completion of $l_p \otimes X$ with respect to ε, π, and Δ_p, respectively. The space $l_p \hat{\otimes}_\pi X$ also admits a representation as a vector sequence space: it is the closed subspace of the space $l_p<X>$ formed by those sequences which are the limit of their finite sections; this can be deduced without difficulty from [5], where it is proved that the norm σ_p induces π over $l_p \otimes X$.

Let E be any of the spaces $l_p \hat{\otimes}_\pi X$ or $l_p \hat{\otimes}_{\Delta_p} X$, p_k be the continuous projection onto the k^{th} coordinate, and i_k the canonical inclusion of X into the k^{th} coordinate. If $T: E \to Y$ is a continuous operator, then a sequence of operators $T_k \in L(X, Y)$ exists such that $T = \Sigma_k T_k p_k$: explicitely, $T_k = Ti_k$. We shall say that (T_k) is the representing sequence of T. If (X_n) is a sequence of Banach spaces, and T is an operator from the Banach space $(\Sigma_n \oplus X_n)_p = \{x = (x_n) \in \Pi_n X_n : \|x\|_p = (\Sigma_n \|x_n\|^p)^{1/p} < +\infty\}$ into Y, then the sequence (T_k) defined by $T_k = Ti_k$ is again called the representing sequence of T (cf. [1]).

We shall consider the following operator ideals: The ideal L of all continuous operators; the ideal W of *weakly compact* operators; the ideal U of *unconditionally converging* operators, i.e., those sending weakly-1-summable sequences into unconditionally summable sequences; the ideal K of compact operators; and the ideal DP of *completely continuous or Dunford-Pettis operators*, i.e., those sending weakly convergent sequences into convergent ones.

Definition. *We say that an operator $T \in L(X, Y)$ is p-converging, $1 \leq p < +\infty$, if it transforms weakly-p-summable sequences of X into norm null sequences of Y. We shall use C_p to denote the ideal of p-converging operators.*

The classes C_p form injective, non-surjective closed operator ideals. It is not difficult to see that $C_1 = U$ and, with the convention that the weakly-∞-summable are the weakly null sequences, that $C_\infty = DP$. A characterization of p-converging operators is contained in the following proposition (see [3]):

Proposition 0. *Let X be a Banach space, and $1 \leq p < +\infty$. If $p > 1$ the operator Id(X) belongs to C_p if and only if all operators from l_{p^*} into X are compact. If $p = 1$, Id(X) belongs to C_1 if and only if all operators from c_0 into X are compact.*

2. COMPACT SETS

We shall study in Section 3 the relation between the membership of an operator T in a class C_p, and the membership of the operators forming its representing sequence in that same class. To this end, we shall introduce a class of subsets which have something of the flavour of compact sets.

Lemma 1. *Let $1 \leq p < +\infty$. Let X and Y be Banach spaces. Consider a set $A \subset l_p \hat{\otimes}_\pi X$ (resp. $A \subset l_p \hat{\otimes}_{\Delta_p} X$). The following are equivalent:*

1. *For each continuous operator $T \in L(l_p \hat{\otimes}_\pi X, Y)$ (resp. $T \in L(l_p \hat{\otimes}_{\Delta_p} X, Y)$), the representing sequence of T converges to T uniformly over A.*

2. $\lim\limits_{N\to+\infty} \sup\limits_{x\in A} \pi\,[(x_k)_{k\geq N}]=0$, (resp., $\lim\limits_{N\to+\infty} \sup\limits_{x\in A} \Delta_p\,[(x_k)_{k\geq N}]=0$).

Proof. That $1\Rightarrow 2$ is obvious. Let us show that $2\Rightarrow 1$ for the case of the projective tensor product. Let (x_n) be any sequence in A. Then

$$\|T((x_n))-(T_1(x_1),\, T_2(x_2),\, \ldots,\, T_N(x_N),\, 0,\, 0,\ldots)\|_Y =$$
$$=\|T(0,\, 0,\ldots,\, 0,\, x_{N+1},\, x_{N+2},\ldots)\|_Y \leq \|T\|\pi[(0,\, 0,\ldots,\, 0,\, x_{N+1},\, x_{N+2},\ldots)]$$

and this converges uniformly on A by Condition 2.

The computations for the natural product are very similar. ■

Definition. *Let $p<+\infty$. A set $A\subseteq l_p \hat{\otimes}_\pi X$, (resp. $A\subset l_p \hat{\otimes}_{\Delta_p} X$), is said to be almost-compact if it satisfies either of the equivalent conditions of Lemma 1.*

Proposition 2. *Let $p<+\infty$. A subset $A\subseteq l_p \hat{\otimes}_\pi X$, (resp. $A\subset l_p \hat{\otimes}_{\Delta_p} X$), is relatively compact if and only if it is bounded, almost-compact, and its continuous projections $p_k(A)$ are relatively compact in X for all $k\in\mathbb{N}$.*

Proof. It is easy to see that all the conditions are necessary. They are also sufficient: Let (x^n) be a sequence contained in $A\subseteq l_p\hat{\otimes}_\pi X$. Condition 1 of Lemma 1 and a diagonal argument show that a certain sub-sequence, again denoted (x^n), exists having pointwise convergence to an element x. To verify that the convergence occurs in the projective norm, it is only necessary to take, in the following expression, the supremum over all elements x^* in the unit ball of $l_{p^*}[X^*]$:

$$\sum_{k=1}^{+\infty}|\langle x_k^n - x_k, x_k^*\rangle| \leq \sum_{k=1}^{k=N}|\langle x_k^n - x_k\,,\, x_k^*\rangle| +$$

$$+\sum_{k=N+1}^{+\infty}|\langle x_k^n, x_k^*\rangle| + \sum_{k=N+1}^{+\infty}|\langle x_k\,,\, x_k^*\rangle|,$$

and observe that the first summand can be made, for large N, less than ε; since A is almost compact, the second and third summands tend to zero when N tends to infinity.

The proof for the natural product is analogous. ■

Remark. Lemma 1 and Proposition 2 have been proved in [1] and [8] for $l_p\{X\}$. The referee has informed to us that this proposition is a particular case of an old theorem due to Mazur, who stated it for the case of a Banach space having a basis, and that a more general result has been established by Goes and Welland as follows:

Theorem ([6] Thm. 1.) *Let X be a complete locally convex topological vector space. Let A be a bounded subset of X and $\{P_\beta\}_{\beta\in I}$ a net in L(X, X). Then A is relatively compact if $\{P_\beta\}_{\beta\in I}$ converges uniformly to the identity on A and $P_\beta(A)$ is relatively compact for each $\beta\in I$.*

Proposition 2 follows taking $P_N((x_n)_n)=(x_1, x_2,\ldots, x_N, 0, 0,\ldots)$ for $N\in\mathbb{N}$. We have left the proof of Proposition 2 for the sake of completeness.

Nontrivial examples of almost compact sets in natural and projective tensor products are provided by the next proposition.

Proposition 3. *Assume that X is a Banach space and that $1\leq p,\ r<+\infty$. If $r<p^*$, then a weakly-r-summable sequence of $l_p\widehat{\otimes}_\pi X$ or $l_p\widehat{\otimes}_{\Delta_p} X$ is an almost compact set. For $p=1$, a weakly null sequence of $l_1\{X\}$ is an almost compact set.*

Proof. We first show the proof for the projective product. Let (a^n) be a weakly-r-summable sequence in $l_p\widehat{\otimes}_\pi X$. Assume that $A=\{a^n : n\in\mathbb{N}\}$ is not an almost compact set. In that case, an $\varepsilon>0$ and two sequences (n_i) and (N_i) of naturals exist such that if I_i denotes the set $\{N_i+1, \ldots, N_{i+1}\}$ and $P_i : l_p\widehat{\otimes}_\pi X \to l_p\widehat{\otimes}_\pi X$ denotes the projection over the indices of I_i then

$$\pi_p(P_i(a^{n_i}))>\varepsilon.$$

Elements $z_i\in(l_p\widehat{\otimes}_\pi X)^*=L(l_p, X^*)$ with $\|z_i\|\leq 1$ can be chosen such that $|<P_i(a^{n_i}), z_i>|>\varepsilon$. The proof of [4, Thm. 1] shows that if $Q_i : l_p\to l_p$ denotes the projection over the indices of I_i, then $|<P_i(a^{n_i}), z_iQ_i>|>\varepsilon$. Once more, the proof of [4, Thm. 1] shows that the operator $B : l_p\widehat{\otimes}_\pi X\to l_p$

defined by $B(Y)=(<P_i y, z_i Q_i>)$ is continuous. By [3, Prop. 1.6.], it transforms (a^n) into a norm-null sequence of l_p, which is a contradiction.

The proof for the natural product is essentially the same. We shall give it for the sake of completeness: If $A=\{a^n : n\in\mathbb{N}\}$ is not almost compact, then an $\varepsilon>0$ and two sequences (n_i) and (N_i) of naturals exist such that

$$\sum_{k=N_i}^{k=N_{i+1}} \|a_k^{n_i}\|^p > \varepsilon .$$

Normalized elements $x_i^*(k) \in X^*$ can be chosen such that: $<x_i^*(k), a_k^{n_i}> = \|a_k^{n_i}\|$. If $y_k^* = x_i^*(k)$, for $N_i \leq k < N_{i+1}$, then (y_k^*) is a bounded sequence of X^* which defines an element of $L(l_p\{X\}, l_p)$. This operator transforms (a^n) into a weakly-r-summable sequence of l_p, which must be norm-null (see [3, Prop. 1.6.]). Thus one has

$$\lim_{N\to+\infty} \sup_{n\in\mathbb{N}} \left[\sum_{k=N}^{k=+\infty} | <y_k^*, a_k^n> |^p\right]^{\frac{1}{p}} = 0,$$

which is a contradiction.

The proof for the case $p=1$ follows closely that of the natural product, and it is only necessary to recall that l_1 has the Schur property, i.e.: weakly null sequences are norm null. That yields the proof for the projective tensor product since $l_1 \hat{\otimes}_\pi X = l_1\{X\}$. In other words: the statement holds for $p=1$ and $r=\infty$. ■

Remark. Let X_n be a sequence of Banach spaces, and $1\leq p<+\infty$. A set $A\subseteq(\Sigma_n \oplus X_n)_p$ is said to be *almost compact* if Conditions 1 or 2 of Lemma 1, with suitable modifications, are satisfied. In this form, Propositions 2 and 3 can be translated to l_p-sums of sequences of Banach spaces.

3. DUNFORD-PETTIS PROPERTIES

A Banach space X is said to have the *Dunford-Pettis property* (*DPP*) if weakly compact operators defined on X are completely continuous, that is, if for any Banach space $Y: W(X, Y)\subseteq DP(X, Y)$. Typical examples of Banach spaces having *DPP* are L_∞ and L_1 spaces. No reflexive Banach space can have *DPP*. Weakened versions of the Dunford-Pettis property were introduced in [3]. A Banach space X is said to have the Dunford-Pettis property of order $p \geq 1$, if $W(X, Y) \subseteq C_p(X, Y)$ for all Banach spaces Y. We shall call this property DPP_p. Notice that $DPP_\infty = DPP$. Every Banach space has DPP_1. Other examples are (see [3] for details): l_p has DPP_r for all $r < p^*$; $L_p[0, 1]$ has DPP_r for $r < min\{p^*, 2\}$; Tsirelson's space has DPP_r for all $r < +\infty$, but not DPP since it is reflexive; if $id(X) \in C_p$ then $C(K, X)$ has DPP_p.

Lemma 4. *Let* $1 \leq p < +\infty$. *Let* (X_n) *be a sequence of Banach spaces. Assume that E represents any of the spaces* $(\Sigma \oplus X_n)_p$ *or* $l_p \hat{\otimes}_\pi X$, *and that T is a continuous operator from E into a Banach space Y, having* (T_k) *as a representing sequence. If* $r < p^*$ *(or* $p = 1$ *and* $r = \infty$*), then T is r-converging if and only if each* T_k *is r-converging.*

Proof. Let (a^n) be a weakly-r-summable sequence of E. Since (a^n) is an almost compact set, the convergence of (T_k) to T is uniform over the set $\{a^n\}$. Furthermore, $T_k(\{a^n\})$ is relatively compact in Y since T_k is r-converging. The relationship

$$T(\{a^n\}) \subseteq \sum_{k=1}^{k=N(\varepsilon)} T_k(\{a^n\}) + \varepsilon B_Y$$

implies that $T(\{a^n\})$ is relatively compact, and therefore (Ta^n) must be norm-null. ■

Remark. When $p^* \leq r < \infty$ the result is clearly false: simply consider the example $l_p\{l_1\}$ and $T = id$.

Proposition 5. *Let A denote an operator ideal and* $r < p^*$ *(or* $p = 1$ *and* $r = \infty$*). With the same notation as in Lemma 4,* $A((\Sigma \oplus X_n)_p, Y) \subseteq C_r((\Sigma \oplus X_n)_p, Y)$ *if and only if, for all n,* $A(X_n, Y) \subseteq C_r(X_n, Y)$. *Moreover* $A(l_p \hat{\otimes}_\pi X) \subseteq C_r(l_p \hat{\otimes}_\pi X)$ *if and only if* $A(X, Y) \subseteq C_r(X, Y)$.

Remark. Recalling that $C_1 = U$ and that $C_\infty = DP$, one sees that these results include and generalize the following results of Bombal [1]: Theorem1.5, part a) for the unconditionally converging operators ($p = 1$,

$r=1$ in Lemma 4) and Dunford-Pettis operators ($p=1$, $r=\infty$ in Lemma 4; Corollary 1.6. part a) for the unconditionally converging operators ($p=1$, $r=1$ in Proposition 5) and Dunford-Pettis operators ($p=1$, $r=\infty$ in Proposition 5) and Proposition 2.6. a) ($p=1$, $r=\infty$, $A=L$), and c) ($p=1$, $r=1$, $A=L$); this last case also appears in [4].

Theorem 6. *Let $1 \leq p < +\infty$. Assume that $r<p^*$ (or $p=1$ and $r=\infty$): these are the cases when l_p has DPP_r. Assume that X also has DPP_r. Then $l_p \hat{\otimes}_\pi X$ and $l_p \hat{\otimes}_{\Delta_p} X$ also have DPP_r.*

Proof. Let E denote any of those spaces, and let $T:E \to Y$ be a weakly compact operator. Since X has DPP_r the operators (T_k) in the representing sequence of T, which necessarily are weakly compact, are p-converging. By Lemma 4, T must also be p-converging. ∎

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