

On Some Weak Monomorphisms and Weak Epimorphisms of Pro- $H\text{Top}^$*

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ABSTRACT. Related to Shape Theory, in a previous paper [6] we studied weak monomorphisms and weak epimorphisms in the category of pro-groups. In this note we give some intrinsic characterizations of the weak monomorphisms and the weak epimorphisms in $\text{pro-}H\text{Top}^*$ in the case when one of the two objects of such a morphism is a rudimentary system.

1. INTRODUCTION

If \mathcal{C} is a category with zero-objects then a morphism $f : A \rightarrow B$ of \mathcal{C} is a weak monomorphism if $f \circ u = 0$ implies $u = 0$. A morphism $f : A \rightarrow B$ is called a weak epimorphism if $u \circ f = 0$ implies $u = 0$.

Weakened versions of categorical notions of monomorphism and epimorphism have proved to be of some interest in pointed homotopy theory. A study of the comparison between weak monomorphism and monomorphism in homotopy theory was carried by T.Ganea [3] who, in

particular, obtained examples of weak monomorphisms which are not monomorphisms. Examples of homotopy weak monomorphisms which are not homotopy epimorphisms have been given by J.Roitberg [7]. Certainly, the study of shape monomorphisms and epimorphisms and their weakened versions can be interesting (see, for the homotopy case, the recently papers of E.Dyer & J.Roitberg [2] and J.Dydak [1]). In [6] we characterized weak monomorphisms and weak epimorphisms in the category of pro-groups and we defined the notion of weakly exact sequence and we studied this notion in the category of pro-groups.

In this note we consider the pro-category of HTop^* , the homotopy category of pointed topological spaces, and we give some intrinsic characterizations of weak monomorphisms and weak epimorphisms $\underline{f} : \underline{X} \rightarrow \underline{Y}$ in pro-HTop^* , when \underline{X} or \underline{Y} is a rudimentary system. These results can be interesting (and maybe sufficient) so a shape morphism $F : X \rightarrow Y$ between topological spaces X and Y can be given by means of such morphisms $f : \underline{X} \rightarrow \underline{Y}$ in pro-HTop^* (approaching morphisms). The study of an arbitrary morphism $f : \underline{X} \rightarrow \underline{Y}$ of pro-HTop^* is more complicated.

The notions and properties of pro-categories which are used in this paper are those of the book of S.Mardešić and J.Segal [4].

2. WEAK MONOMORPHISMS IN THE CATEGORY PRO-HTOP^*

The category pro-HTop^* is a category with zero objects. A zero-object is a single point rudimentary system.

If $(X, *)$ is a rudimentary system in pro-HTop^* and if $\underline{Y} = ((Y_\lambda, *), q_{\lambda\lambda'}, \Lambda)$ is an arbitrary object in pro-HTop^* , then the morphisms $\underline{f} = (f_\lambda) : (X, *) \rightarrow \underline{Y}$ coincide with the morphisms in inv-HTop^* , the category of inverse systems in HTop^* [4, p.20]. This means that for each $\lambda \in \Lambda$ is given a morphism $f_\lambda : (X, *) \rightarrow (Y_\lambda, *)$ in HTop^* and for each pair $\lambda \leq \lambda'$ we have $q_{\lambda\lambda'} f_{\lambda'} = f_\lambda$.

Lemma 1. *For a morphism $\underline{f} : (X, *) \rightarrow \underline{Y} = ((Y_\lambda, *), q_{\lambda\lambda'}, \Lambda)$ in pro-HTop^* , there exist an object $\underline{P} = ((P_\lambda, *), r_{\lambda\lambda'}, \Lambda)$ and two morphisms $\underline{p} = (p_\lambda, 1_\Lambda) : \underline{P} \rightarrow \underline{Y}$, $\underline{h} = (h_\lambda) : (X, *) \rightarrow \underline{P}$ such that for each $\lambda \in \Lambda$:*

(i) $h_\lambda : (X, *) \rightarrow (P_\lambda, *)$ is a pointed homotopy equivalence,

- (ii) $p_\lambda : (P, *) \rightarrow (Y_\lambda, *)$ is a pointed fiber map,
- (iii) $f_\lambda = p_\lambda \circ h_\lambda$.

Proof. The existence for each $\lambda \in \Lambda$ of a factorization (iii), satisfying (i) and (ii), is well known [5, p.249]. For a pair $\lambda \leq \lambda'$ in Λ we define $r_{\lambda\lambda'} = h_\lambda \circ h_{\lambda'}^{-1}$, for which is immediate that $\underline{P} = ((P_\lambda, *), r_{\lambda\lambda'}, \Lambda)$ is an inverse system in HTop^* and that $\underline{h} = (h_\lambda) : (X, *) \rightarrow \underline{P}$ is a morphism in pro-HTop^* . Also, from the relations $q_{\lambda\lambda'} \circ f_{\lambda'} \approx f_\lambda, f_\lambda = p_\lambda h_\lambda, f_{\lambda'} = p_{\lambda'} \circ h_{\lambda'}$, we deduce that $q_{\lambda\lambda'} p_\lambda \approx p_{\lambda'} \circ r_{\lambda\lambda'}$, which shows that $\underline{p} = (p_\lambda, 1_\Lambda) : \underline{P} \rightarrow \underline{Y}$ a morphism of pro-HTop^* .

Remark 1. It is obvious from Lemma 1 that we can write the equality $\underline{f} = \underline{p} \circ \underline{h}$, in the category pro-HTop^* , where \underline{h} is an isomorphism. Then it is clear that \underline{f} is a weak monomorphism if and only if \underline{p} is a weak monomorphism. We will refer to the morphism $\underline{p} : \underline{P} \rightarrow \underline{Y}$ as the *fibred factor* of the morphism \underline{f} .

Remark 2. If $\underline{p} = (p_\lambda, 1_\Lambda) : \underline{P} = ((P_\lambda, *), r_{\lambda\lambda'}, \Lambda) \rightarrow \underline{Y} = ((Y_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ is a fibred factor in pro-HTop^* , we can consider the fiber of this morphism by the object $\underline{F} = ((F_\lambda, *), r'_{\lambda\lambda'}, \Lambda)$, where $F_\lambda = p_\lambda^{-1}(*)$ and $r'_{\lambda\lambda'} = r_{\lambda\lambda'} / F_{\lambda'}$, for $\lambda \leq \lambda'$. Then we can define a morphism $\underline{i} = (i_\lambda, 1_\Lambda) : \underline{F} \rightarrow \underline{P}$, where i_λ is the inclusion of $(F_\lambda, *)$ in $(P_\lambda, *)$.

Definition 1. We will say that the fiber $\underline{F} = ((F_\lambda, *), r'_{\lambda\lambda'}, \Lambda)$ of the fibred factor $\underline{p} = (p_\lambda, 1_\Lambda) : \underline{P} = ((P_\lambda, *), r_{\lambda\lambda'}, \Lambda) \rightarrow \underline{Y} = ((Y_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ is contractible in \underline{P} if for each $\lambda \in \Lambda$ there is a $\lambda' \geq \lambda$ such that $i_\lambda \circ r'_{\lambda\lambda'} \approx *$.

Theorem 1. A morphism $\underline{f} = (f_\lambda) : (X, *) \rightarrow \underline{Y}$ is a weak monomorphism in the category pro-HTop^* if and only if the fibre \underline{F} of every fibred factor $\underline{p} = \underline{P} \rightarrow \underline{Y}$ is contractible in \underline{P} .

Proof. By Remark 1 it is sufficient to prove that \underline{p} is a weak monomorphism if and only if \underline{F} is contractible in \underline{P} .

Suppose that $\underline{p} : \underline{P} \rightarrow \underline{Y}$ is a weak monomorphism in the category pro-HTop^* . For the morphism $\underline{i} : \underline{F} \rightarrow \underline{P}$ from Remark 2 we have $\underline{p} \circ \underline{i} = *$ and by hypothesis it follows $\underline{i} = *$. If $* = (*, \Phi)$ then we have an equivalence $(i_\lambda, 1_\Lambda) \sim (*, \Phi)$ [4, p.6] which implies that for each

$\lambda \in \Lambda$ there is $\lambda' \geq \lambda$ (and $\lambda' \geq \Phi(\lambda)$) such that the following diagram in HTop^* commutes

$$\begin{array}{ccc}
 F_{\Phi(\lambda)} & \xleftarrow{r'_{\Phi(\lambda)\lambda'}} & F_{\lambda'} \\
 * \downarrow & & \downarrow r_{\lambda\lambda'} \\
 P_{\lambda} & \xleftarrow{i_{\lambda}} & F_{\lambda}
 \end{array}$$

This implies that $i_{\lambda} \circ r'_{\lambda\lambda'}$ is pointed null-homotopic, i.e. $i_{\lambda} \circ r'_{\lambda\lambda'} \approx *$. Thus, \underline{F} is contractible in \underline{P} .

Conversely, suppose that the libre \underline{F} is contractible in \underline{P} and let $\underline{u} = (u_{\lambda}, \Phi) : \underline{Z} = ((Z_{\mu}, *), s_{\mu\mu'}, M) \rightarrow \underline{P} = ((P_{\lambda}, *), r_{\lambda\lambda'}, \Lambda)$ be a morphism, such that $\underline{p} \circ \underline{u} = *$. But $\underline{p} \circ \underline{u} = (p_{\lambda} \circ u_{\lambda}, \Phi)$, with the function $\Phi : \Lambda \rightarrow M$, and $p_{\lambda} \circ u_{\lambda} : Z_{\Phi(\lambda)} \rightarrow P_{\lambda} \rightarrow Y_{\lambda}$. This relation implies that each $\lambda \in \Lambda$ admits $\mu \in M$, $\mu \geq \Phi(\lambda)$ such that $p_{\lambda} \circ u_{\lambda} \circ s_{\Phi(\lambda)\mu} \approx *$, by a pointed homotopy $H_{\lambda} : Z_{\mu} \times [0, 1] \rightarrow Y_{\lambda}$. Then, by the homotopy covering property of p_{λ} , there exists a pointed homotopy $K_{\lambda} : Z_{\mu} \times [0, 1] \rightarrow P_{\lambda}$ such that $K_{\lambda}(\cdot, 0) = u_{\lambda} \circ s_{\Phi(\lambda)\mu}$ and $p_{\lambda} \circ K_{\lambda} = H_{\lambda}$. Thus we have $u_{\lambda} \circ s_{\Phi(\lambda)\mu} \approx K_{\lambda}(\cdot, 1)$ and $\text{Im } K_{\lambda} \subseteq F_{\lambda}$. By the proof of Lemma 1 and since the index sets are directed, for each $\lambda \in \Lambda$ we can choose the indices $\lambda' \in \Lambda$ and $\mu, \mu' \in M$ such that $i_{\lambda} \circ r_{\lambda\lambda'} \approx *$ and the following diagram commutes

$$\begin{array}{ccccc}
 Z_{\mu'} & \xleftarrow{s_{\mu'\mu}} & Z_{\mu} & \xrightarrow{s_{\Phi(\lambda)\mu}} & Z_{\Phi(\lambda)} \\
 K_{\lambda'}(\cdot, 1) \downarrow & & \downarrow K_{\lambda}(\cdot, 1) & & \downarrow u_{\lambda} \\
 F_{\lambda'} & \xrightarrow{r_{\lambda\lambda'}} & F_{\lambda} & \xrightarrow{i_{\lambda}} & P_{\lambda}
 \end{array}$$

This means that $u_{\lambda} s_{\Phi(\lambda)\mu} \approx i_{\lambda} r_{\lambda\lambda'} K_{\lambda'}(\cdot, 1) s_{\mu'\mu} \approx *$, i.e. $(u_{\lambda}, \Phi) \sim (*, \Phi')$ for satisfactory function $\Phi' : \Lambda \rightarrow M$. Thus we obtained $\underline{u} = *$, what finishes the proof of the theorem.

Remark 3. If $f : (X, *) \rightarrow (Y, *)$ is a pointed continuous map then f is a weak monomorphism in HTop^* if and only if it is a weak monomorphism in pro-HTop^* . Theorem 1 generalizes the usual result for pointed continuous map [7, Prop. 2.2, (ii)].

3. WEAK EPIMORPHISMS IN THE CATEGORY PRO-HTOP*

In this section we consider only morphisms of the form $f : \underline{X} \rightarrow (Y, *)$, where \underline{X} is an arbitrary inverse system in HTop^* . In fact the morphism f can be represented by a continuous map $f_\lambda : (X_\lambda, *) \rightarrow (Y, *)$, if $\underline{X} = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$, and two such maps $f_{\lambda_1}, f_{\lambda_2}$ define the same morphism f if there is $\lambda \geq \lambda_1, \lambda_2$ such that $f_\lambda p_{\lambda_1\lambda} = f_{\lambda_2} p_{\lambda_2\lambda}$ in HTop^* .

Lemma 2. *For a morphism $f : \underline{X} = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda) \rightarrow (Y, *)$ there exist an object $\underline{M} = ((M_\lambda, *), r_{\lambda\lambda'}, \Lambda)$, $\Phi : \Lambda' \hookrightarrow \Lambda$ and two morphisms $\underline{j} = (j_\lambda, \Phi) : \underline{X} \rightarrow \underline{M}$, $\underline{h} : \underline{M} \rightarrow (Y, *)$, such that for each $\lambda \in \Lambda'$:*

- (i) $h_\lambda : (M_\lambda, *) \rightarrow (Y, *)$ is a pointed homotopy equivalence,
- (ii) $j_\lambda : (X_\lambda, *) \rightarrow (M_\lambda, *)$ is a pointed cofiber inclusion map,
- (iii) $f_\lambda = h_\lambda \circ j_\lambda$.

Proof. Denote by Λ' the subset of Λ such that an index λ is in Λ' if and only if there is a map $f_\lambda : (X_\lambda, *) \rightarrow (Y, *)$ defining f .

The existence for each $\lambda \in \Lambda'$ of a factorization (iii) satisfying (i) and (ii) is well known [5, p.246]. For a pair $\lambda \leq \lambda'$ in Λ' define $r_{\lambda\lambda'} = h_{\lambda'}^{-1} \circ h_\lambda$, from which is immediate that $\underline{M} = (M_\lambda, *, r_{\lambda\lambda'}, \Lambda')$ is an inverse system in HTop^* and that all maps $h_\lambda, \lambda \in \Lambda'$ define the same morphism $\underline{h} : \underline{M} \rightarrow (Y, *)$. Finally, if $\Phi : \Lambda' \hookrightarrow \Lambda$ is the inclusion function, then $\underline{j} = (j_\lambda, \Phi) : \underline{X} \rightarrow \underline{M}$ is a morphism of pro-HTop^* .

Remark 4. It is obvious from Lemma 2 that we can write $f = \underline{h} \circ \underline{j}$, in the category pro-HTop^* , where \underline{h} is an isomorphism. Then it is clear that f is a weak epimorphism if and only if \underline{j} is a weak epimorphism. We will refer to the morphism $\underline{j} : \underline{X} \rightarrow \underline{M}$ as the *cofibred factor* of the morphism f .

Remark 5. Let $\underline{j} = (j_\lambda, \Phi) : \underline{X} = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda) \rightarrow \underline{M} = ((M_\lambda, *), r_{\lambda\lambda'}, \Lambda')$ be a cofibred factor in pro-HTop^* . Then for each $\lambda \in \Lambda'$ we can consider the pointed quotient space M_λ/X_λ with the pointed identification map $\pi_\lambda : M_\lambda \rightarrow M_\lambda/X_\lambda$. We can consider the

inverse system $\underline{M}/\underline{X} = ((M_\lambda/X_\lambda, *), \bar{r}_{\lambda\lambda'}, \Lambda')$ and the morphism $\underline{\pi} = (\pi_\lambda, 1_{\Lambda'}) : \underline{M} \rightarrow \underline{M}/\underline{X}$. For the morphism $\underline{f} : \underline{X} \rightarrow (Y, *)$ we will say that $(Y, *)$ is contractible in the cofibred factor of f if each $\lambda \in \Lambda'$ admits a $\lambda' \geq \lambda$ such that $\pi_\lambda \circ r_{\lambda\lambda'} \approx *$.

Theorem 2. *A morphism $\underline{f} : \underline{X} \rightarrow (Y, *)$ of pro- HTop^* is a weak epimorphism if and only if $(Y, *)$ is contractible in every cofibred factor.*

Proof. Suppose that \underline{f} is a weak epimorphism, what is equivalent to the fact that the morphism $\underline{j} : \underline{X} \rightarrow \underline{M}$ is a weak epimorphism. Since $\underline{\pi} \circ \underline{j} = *$, the hypothesis implies $\underline{\pi} = *$ in pro HTop^* . This means $(\pi_\lambda, 1_{\Lambda'}) \sim (*, \Phi')$, i.e. each $\lambda \in \Lambda'$ admits a $\lambda' \geq \lambda$ such that $\pi_\lambda \circ r_{\lambda\lambda'} = *$ in HTop^* . Thus, $(Y, *)$ is contractible in the cofibred factor $\underline{j} : \underline{X} \rightarrow \underline{M}$ of \underline{f} .

Conversely, suppose that $(Y, *)$ is contractible in the cofibred factor of \underline{f} . It is sufficient to prove that \underline{f} is a weak epimorphism. For this, suppose that for a morphism $\underline{u} = (u_\nu, \Psi) : \underline{M} \rightarrow \underline{Z} = ((Z_\nu, *), s_{\nu\nu'}, N)$ we have $\underline{u} \circ \underline{j} = *$. This implies that for each $\nu \in N$ there is a pointed homotopy $H_\nu : u_\nu \circ j_{\Psi(\nu)} \approx *$. Then, by the pointed homotopy extension property of the pair $(M_{\Psi(\nu)}, X_{\Psi(\nu)})$ there exists a pointed homotopy $K_\nu : M_{\Psi(\nu)} \times [0, 1] \rightarrow Z_\nu$, such that $K_\nu(\cdot, 0) = u_\nu$ and $K_\nu/K_{\Psi(\nu)} \times [0, 1] = H_\nu$. Now, if we consider the pointed map $\varphi_\nu : M_{\Psi(\nu)} \rightarrow Z_\nu, \varphi_\nu = K_\nu(\cdot, 1)$, then we have $\varphi_\nu/X_{\Psi(\nu)} = K_\nu/X_{\Psi(\nu)} \times \{1\} = H_\nu(\cdot, 1) = *$. Therefore, we can define the pointed map $\tilde{\varphi}_\nu : M_{\Psi(\nu)}/X_{\Psi(\nu)} \rightarrow Z_\nu$, such that $\tilde{\varphi}_\nu \circ \pi_{\Psi(\nu)} = \varphi_\nu$ and the pointed homotopy $\tilde{\varphi}_\nu \circ F_{\Psi(\nu)} : M_{\lambda'} \times [0, 1] \rightarrow M_{\Psi(\nu)}/X_{\Psi(\nu)} \rightarrow Z_\nu$, where $F_{\Psi(\nu)} : \pi_{\Psi(\nu)} \circ r_{\Psi(\nu)\lambda'} \approx *$, for a convenient $\lambda' \geq \Psi(\nu)$. For this we have $\tilde{\varphi}_{\Psi(\nu)} \circ F_{\Psi(\nu)} : \varphi_\nu \circ r_{\Psi(\nu)\lambda} \approx *$ in Top^* . On the other hand K_ν is a pointed homotopy, $K_\nu : u_\nu \circ r_{\Psi(\nu)\lambda} \approx \varphi_\nu \circ r_{\Psi(\nu)\lambda}$, and therefore $\tilde{\varphi}_\nu \circ F_{\Psi(\nu)} \circ K_\nu : u_\nu \circ r_{\Psi(\nu)\lambda} \approx *$. This proves the equivalence $(u_\nu, \Psi) \sim (*, \Psi)$ for every $\nu \in N$ and therefore $\underline{u} = *$, what finishes the proof of the theorem.

Remark 6. If $f : (X, *) \rightarrow (Y, *)$ is a pointed continuous map then f is a weak epimorphism in HTop^* if and only if it is a weak epimorphism in pro- HTop^* . Particularly, Theorem 2 generalizes the usual intrinsic characterization of a weak epimorphism in HTop^* [7, Prop. 2.2 (i)].

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