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Complemented subspaces of sums and products of copies of $L^1[0,1]$.

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Abstract

We prove that the direct sum and the product of countably many copies of $L^1[0,1]$ are primary locally convex spaces. We also give some related results.

For a while it was an open problem whether a complemented subspace of a countable product of Banach spaces can be written as a product of Banach spaces. This question has been solved in negative by M. I. Ostrovskii [12], but it is still open for X^N where X is a classical Banach space. The only countable products of classical Banach spaces whose complemented subspaces have been fully described are: $\omega; (l^p)^N, 1 \le p \le \infty$, and $(c_0)^N$ ([5], [9]) and for these the answer is positive. Moreover, in [1] it was shown that, for $1 < p < \infty$, $(L^p[0,1])^N$ is *primary*, i.e. if $(L^p[0,1])^N = F \bigoplus G$, then either F or G is isomorphic to $(L^p[0,1])^N$; it follows, by reflexivity, that also the direct sum of countably many copies of $L^p[0,1]$ is primary. The purpose of this note is to extend these results to the case $L^1[0,1]$, i.e. we will prove that the direct sum and the product of countably many copies of $L^1[0,1]$ are also primary spaces. However it remains an open problem whether both the complements F and G of a direct decomposition of $(L^p[0,1])^N$, with $1 \le p < \infty$, are isomorphic to a product of Banach spaces. Note that $(L^p[0,1])^N$ is isomorphic to $L^p_{loc}(\mathbf{R}), 1 \le p \le \infty$.

Our proof is completely different from the one in [1]: the technique of that proof cannot be applied to the case when $p = 1$, as it based

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on some special features of the spaces $L^p[0,1], 1 < p < \infty$, and on the fact that the Haar-system is an unconditional basis in such spaces (the Haar-system is only a basis of $L^1[0,1]$; there is no unconditional basis in $L^1[0,1]$!). Actually, in order to obtain our results we will use some known facts about a special class of operators on $L^1[0,1]$, the so-called $E-$ operators (see [6]), together with a method given in [9].

For other examples of primary non-Banach Fréchet spaces, we refer the reader to [1], [2], [4], [5], [9] and [10].

We will use standard terminology (like e.g. [7], [8] and [9]). In particular, for two locally convex spaces E and F, we write $E \simeq F$ and $E < F$ to mean respectively that E is topologically isomorphic to F or to a complemented subspace of F. Finally, we put $L^1 = L^1[0,1]$.

1 Preliminaries

We recall some definitions and facts which will be used later on.

Definition 1 ([6]). *A bush is a sequence* $(E_i^n), i = 1, \cdots, M_n, n = 0, 1, \cdots,$ *of Lebesgue measurable subsets of [0,1] such that*

(a) $M_0 = 1$ and $| E_1^0 | > 0$,

(b) for each n $\cup_{i=1}^{M_n} E_i^n = E_1^0$,

(c) for each n $E_i^n \cap E_j^n = \emptyset$ if $i \neq j$,

(d) for each n and each $j, 1 \leq j \leq M_{n+1}$, there exists an $i, 1 \leq i \leq M_n$, with $E_j^{n+1} \subset E_i^n$,

(e) $\max_{1 \leq i \leq M_n} | E_i^n | \to 0$ as $n \to \infty$.
Here $| E |$ *denotes the Lebesgue measure of a mesurable subset* $E \subset$ *[0,1].*

Definition 2 ([6]). *Let* $T : L^1 \to L^1$ *be a bounded linear operator.* T *is called an* E- **operator** *if there exist* $\delta > 0$ *and a bush* (E_i^n) *with*

$$\frac{1}{| E_1^0 |} \int_0^1 \max_{1 \leq i \leq M_n} \left| T\left(\chi_{E_i^n}\right)\right| dx > \delta$$

for each n*, where* χ_E *denotes the characteristic function of a measurable subset* $E \subset$ *[0,1].*

Enflo and Starbid [6] proved the following useful fact:

Theorem 0. *Let $T : L^1 \to L^1$ be a bounded linear operator. T is an E-operator if and only if there exists a subspace Y of L^1 with Y isomorphic to L^1, with $T_{|Y}$ an isomorphism onto, and with TY complemented in L^1.*

Remark. (1) If $T_1 + T_2$ is an E-operator, then either T_1 or T_2 must be an E-operator. (2) Obviously, the identity map of L^1 is an E-operator.

For more about such operators the reader is referred to [6].

2 Complemented subspaces of $(L^1)^N$

We denote by $(L^1)^N$ the product of countably many copies of L^1. In particular, the space $(L^1)^N$ can be represented as the projective limit of the Banach spaces $\prod_{i=1}^{n} L^1$ with respect to the linking maps

$$p_{n-1,n} : \prod_{i=1}^{n} L^1 \to \prod_{i=1}^{n-1} L^1, \ (x_1, \cdots, x_{n-1}, x_n) \to (x_1, \cdots, x_{n-1}),$$

which are surjective. It is clear that, for each increasing sequence $(k(n)) \subset \mathbf{N}$, we have $(L^1)^N = proj_n \left(\prod_{i=1}^{k(n)} L^1, p_{k(n-1),k(n)}\right)$, where $p_{k(n-1),k(n)} = p_{k(n-1),k(n-1)+1} \cdots p_{k(n)-1,k(n)}$.

Now, let $p_n : (L^1)^N \to \prod_{i=1}^{n} L^1$ be the canonical projection $(x_i)_i \to (x_1, \cdots, x_n)$. Then $p_{n,n+1} p_{n+1} = p_n$.

Now we are ready to prove

Theorem 1. *The space $(L^1)^N$ is primary.*

Proof. Suppose that $(L^1)^N = F \bigoplus G$ with P projection from $(L^1)^N$ onto F and $ker P = G$. Put $Q = I - P$.

Because F and G are closed subspaces of $(L^1)^N$, by Lemma 1.1 of [9], we may write $F = proj_n(F_n, p_{n-1,n})$ and $G = proj_n(G_n, p_{n-1,n})$, where F_n (resp. G_n) denotes the closure of $p_n(F)$ (resp. $p_n(G)$) in $\prod_{i=1}^{n} L^1$ and $p_{n-1,n}$ also denotes the restriction of $p_{n-1,n}$ to F_n (resp. G_n). Moreover, since F_n (resp. G_n) is Banach every map $p_n P$ (resp. $p_n Q$) factors canonically through $\prod_{i=1}^{k(n)} L^1$. Therefore, we can find two sequence $(k(n))_n$ and $(h(n))_n$ of integer numbers with $1 = h(1) < k(1) <$

$h(2) < \cdots < h(n) < k(n) < h(n+1) < \cdots$ such that the diagrams

$$\begin{array}{ccc} (L^1)^N & \stackrel{p_{h(n)}P}{\to} & F_{h(n)} \\ p_{k(n)} \downarrow & \nearrow r_n & \\ \prod_{i=1}^{k(n)} L^1 & & \end{array} \quad \text{and} \quad \begin{array}{ccc} (L^1)^N & \stackrel{p_{h(n)}Q}{\to} & G_{h(n)} \\ p_{k(n)} \downarrow & \nearrow s_n & \\ \prod_{i=1}^{k(n)} L^1 & & \end{array} \qquad (1)$$

commute, where r_n (resp. s_n) denotes the map associated with $p_{h(n)}P$ (resp. $p_{h(n)}Q$).

Put $E_{0,1} = \prod_{i=1}^{k(1)} L^1, E_{n-1,n} = \{0\}^{k(n-1)} \times \prod_{i=k(n-1)+1}^{k(n)} L^1$, and $p_{h(n),k(n)} = p_{h(n),h(n)+1} \cdots p_{k(n)-1,k(n)}$. Then, by (1), as it is easy to verify, we obtain that, for each $x \in E_{n-1,n}, (r_n + s_n)(x) = p_{h(n),k(n)}(x)$, i.e.

$$r_n + s_n = p_{h(n),k(n)|E_{n-1,n}} : E_{n-1,n} \to \prod_{i=1}^{h(n)} L^1$$

is the canonical projection ($\neq 0$ as $k(n-1) < h(n) < k(n)$) and hence is an E-operator as it follows from Theorem 0. This implies that, by Remark 1, either $r_{n|E_{n-1,n}}$ or $s_{n|E_{n-1,n}}$ is an E-operator for each n. Therefore, we can suppose that $r_{n|E_{n-1,n}}$ is an E-operator for infinite indices n.

Now, for the sake of simplicity, we assume that for each n $r_{n|E_{n-1,n}}$ is an E-operator and $k(n) = n+1, h(n) = n$. Thus, we have that the following diagram

$$\begin{array}{ccc} \prod_{i=1}^{n+2} L^1 & \stackrel{p_{n+1,n+2}}{\to} & \prod_{i=1}^{n+1} L^1 \\ r_{n+1} \downarrow & & \downarrow r_n \\ F_{n+1} & \stackrel{p_{n,n+1}}{\to} & F_n \end{array}$$

commutes for each n.

Because $r_{1|E_{0,1}}$ is an E-operator, by Theorem 0 there exists a closed subspace $M_1 \subset E_{0,1} = L^1$ with $r_{1|M_1}$ an isomorphism into, with $H_1 =$

$r_1(M_1) < L^1$ and, with $H_1 \simeq L^1$. Since $p_{n,n+1}r_{n+1} = r_n p_{n+1,n+2}$, it is clear that also the maps

$$p_{n,n+1} : r_{n+1}(M_1) \to r_n(M_1)$$

are isomorphism onto.

Now, note that $p_{1,2}(x_1, x_2) = x_1$ and hence $N = ker\, p_{1,2} = F_2 \cap (\{0\} \times L^1)$. Because $p_{1,2} : r_2(M_1) \to r_1(M_1) = H_1$ is an isomorphism onto, there exists a continuous linear map $A : H_1 \to \{0\} \times L^1$ with $r_2(M_1) = \{(x, Ax) : x \in H_1\}$. It follows that, if $t_1 : L^1 \to H_1$ is a projection, then the map $r : L^1 \times L^1 \to r_2(M_1)$ defined by $r(x_1, x_2) = (t_1x_1, At_1x_1)$ is a projection onto $r_2(M_1)$ with $ker\, r = \{(x_1, x_2) : t_1x_1 = 0\} = ker\, t_1 \times L^1$. Now, we observe that $r_2^{-1}(N) \subset E_{1,2}$ and $r_{2|E_{1,2}}$ is an E-operator. Then, again Theorem 0 gives that there exists a closed subspace $M_2 \subset E_{1,2}$ with $r_{2|M_2}$ an isomorphism into, with $H_2 = r_2(M_2) < \{0\} \times L^1$ and with $H_2 \simeq L^1$. As before, all the maps $p_{n,n+1} : r_{n+1}(M_2) \to r_n(M_2)$ are isomorphism onto. If I is the identity map of $L^1 \times L^1$ and $q : \{0\} \times L^1 \to H_2$ is a projection onto H_2, we consider the diagram

$$L^1 \times L^1 \stackrel{I-r}{\to} ker\, t_1 \times L^1 \stackrel{I-p_{1,2}}{\to} \{0\} \times L^1 \stackrel{q}{\to} H_2.$$

Then the map

$$s = qp_{1,2}(I - r) : L^1 \times L^1 \to H_2$$

is a projection onto H_2 and $rs = 0 = sr$. It follows that $r_2(M_1) + H_2$ is a closed subspace of F_2, hence equal to $r_2(M_1) \oplus r_2(M_2) \simeq L^1 \oplus L^1$, and the map $t_2 = r + s$ is clearly a projection from $L^1 \times L^1$, hence from F_2, onto $r_2(M_1) \oplus r_2(M_2)$ such that $p_{1,2}t_2 = t_1p_{1,2}$.

Continuing in this way, we inductively obtain that for each n there exists a closed subspace $X_n = \oplus_{i=1}^n r_n(M_i) \simeq \prod_{i=1}^n L^1$ of F_n and a projection $t_n : \prod_{i=1}^n L^1 \to X_n$ such that

$$p_{n,n+1}t_{n+1} = t_n p_{n,n+1} \tag{2}$$

so that $p_{n,n+1}(X_{n+1}) = X_n$. Now, if we form the projective limit X of the spaces X_n with respect to the restriction maps $p_{n-1,n} : X_n \to X_{n-1}$, we see that $X \subset F, X \simeq (L^1)^N$. Moreover, by using (2), we see that the map

$$t : (L^1)^N \to X \, , x = (x_n)_n \to (t_n p_n(x_n))_n$$

is a projection onto X. Therefore, we have the situation $(L^1)^N < F < (L^1)^N$ which gives, by using Pelczyinski's decomposition method, that $F \simeq (L^1)^N$ and hence the proof is complete.

Moreover

Proposition 1. *If $F < (L^1)^N$ then one of the following cases occurs: $(i)F$ is a complemented subspace of $L^1 \cdot (ii)F \simeq \omega \cdot (iii)F \simeq \omega \oplus X$ where X is a complemented subspace of $L^1 \cdot (iv)F'_\beta \simeq (l^\infty)^{(N)}$, moreover in this case F contains a complemented copy of $(l^1)^N$.*

In order to prove Proposition 1, we need the following Lemma

Lemma. *Let E be a quojection (i.e., E is a projective limit of a projective sequence $(E_n, r_{n,n+1})$ of Banach spaces E_n and surjective linking maps $r_{n,n+1} : E_{n+1} \to E_n$). If E'_β has a subspace isomorphic to $(l^\infty)^{(N)}$, then E contains a complemented copy of $(l^1)^N$.*

Proof. First, we write $E'_\beta = ind \, E'_n$, where the increasing sequence (E'_n) of Banach spaces is strict since E is a quojection.

Now, we assume that E'_β contains a copy of $(l^\infty)^{(N)}$. Put $X_n = l^\infty$ for all $n, (l^\infty)^{(N)} = \oplus X_n$. Then there is a $k(1)$ such that $X_1 \subset E'_{k(1)}$ since X_1 is Banach. By Proposition 2.e.8 of [8] it follows that $E_{k(1)}$ contains a complemented copy of l^1, i.e. there is a subspace G_1 of $E'_{k(1)}$ with $G_1 \simeq l^1$ and a projection $t_1 : E_{k(1)} \to G_1$. We denote by (e_j) the unit vectors basis of G_1: because E is a quojection there is a bounded sequence $(x_j) \subset E$ such that $r_{k(1)}x_j = e_j$ (for each n, r_n denotes the map $r_n : E \to E_n$ defined by $r_n x = x_n$). Therefore, the map $s_1 : G_1 \to E \, , \sum_{j=1}^\infty a_j e_j \to \sum_{j=1}^\infty a_j x_j$ is an isomorphism onto $\widetilde{G}_1 = [x_j]$. Actually, $s_1 = \left(r_{k(1)_{|G_1}}\right)^{-1}$. It follows that the composition map

$$\widetilde{t}_1 = s_1 t_1 r_{k(1)} : E \to E_{k(1)} \to G_1 \to \widetilde{G}_1$$

is also a projection from E onto $\widetilde{G}_1 \simeq l^1$. So, $E = \widetilde{G}_1 \oplus ker\widetilde{t}_1 \simeq l^1 \oplus ker\widetilde{t}_1$ and, hence, $E'_\beta = \widetilde{G}_1 \oplus (ker\widetilde{t}_1)'_\beta$, where $F = ker\widetilde{t}_1$ is also a quojection as a quotient of a quojection (see Proposition 3 of [3]).

In order to complete the proof, we observe that $(l^\infty)^{(N)}$ is also a complemented subspace of E'_β (it is an easy consequence of the fact that l^∞ is injective (see Proposition 2.f.2 of [8]) and that E'_β is a strict LB-space). Then, we denote by p a projection from E'_β onto $(l^\infty)^{(N)}$: because $\widetilde{G}'_1$ is a Banach subspace of E'_β there is a $k \in N$ such that $q_k p\left(\widetilde{G}'_1\right) = 0$, where q_k denotes the canonical k-th projection from $(l^\infty)^{(N)} = \oplus_n X_n$ onto $\oplus_{n>k} X_n$. By noting that $q_k p$ is a projection from E'_β onto $\oplus_{n>k} X_n$, it follows that, for $x \in \oplus_{n>k} X_n, x = \left(id_E - \widetilde{t}_1\right)' x + \widetilde{t}'_1 x$ and hence $x = q_k p x = q_k p\left(id_E - \widetilde{t}_1\right)' x + q_k p \widetilde{t}'_1 x = q_k p\left(id_E - \widetilde{t}_1\right)' x$, i.e. $q_k p\left(id_E - \widetilde{t}_1\right)'_{|\oplus_{n>k} X_n} = id_{\oplus_{n>k} X_n}$. Therefore, the composition map

$$\left(id_E - \widetilde{t}_1\right)' q_k p : F'_\beta \to \oplus_{n>k} X_n \to \left(id_E - \widetilde{t}_1\right)' (\oplus_{n>k} X_n) \subset F'_\beta$$

is a projection from F'_β onto $Y = \left(id_E - \widetilde{t}_1\right)' (\oplus_{n>k} X_n)$ and $Y \simeq (l^\infty)^{(N)} \subset F'_\beta$.

Since F'_β contains also a (complemented) copy of $(l^\infty)^{(N)}$, as before, we find a subspace $\widetilde{G}_2$ of F with $\widetilde{G}_2 \simeq l^1$ and a projection $\widetilde{t}_2 : F \to \widetilde{G}_2$ so that $E = \widetilde{G}_1 \oplus F = \widetilde{G}_1 \oplus \widetilde{G}_2 \oplus ker\widetilde{t}_2 \simeq l^1 \oplus l^1 \oplus ker\widetilde{t}_2$, where $\widetilde{t}_1 + \widetilde{t}_2(id_E - \widetilde{t}_1)$ is a projection from E onto $\widetilde{G}_1 \oplus \widetilde{G}_2$. Iterating this procedure, for each n we find a subspace $\widetilde{G}_n$ of $ker\widetilde{t}_{n-1}$ with $\widetilde{G}_n \simeq l^1$ and a projection $\widetilde{t}_n : ker\widetilde{t}_{n-1} \to \widetilde{G}_n$ so that $E = \oplus_{i=1}^n \widetilde{G}_i \oplus ker\widetilde{t}_n \simeq \oplus_{i=1}^n l^1 \oplus ker\widetilde{t}_n$. Then, if we form the projective limit G of the Banach spaces $\oplus_{i=1}^n \widetilde{G}_i$ with respect to the maps s_n, where s_n is the restriction of the map $\sum_{i=1}^n \widetilde{t}_i\left(id_E - \widetilde{t}_{i-1}\right)\cdots\left(id_E - \widetilde{t}_1\right)$ to $\oplus_{i=1}^{n+1} \widetilde{G}_i$, we obtain that $G \subset E$ and $G \simeq (l^1)^N$. Moreover, the map $s = \sum_{i=1}^\infty \widetilde{t}_i\left(id_E - \widetilde{t}_{i-1}\right)\cdots\left(id_E - \widetilde{t}_1\right)$ is a projection from E onto G. This completes the proof.

Proof of Proposition 1.

It follows from assumption that F is a quojection (because it is a quotient of $(L^1)^N$)) and $F'_\beta < (L^\infty)^{(N)} \simeq (l^\infty)^{(N)}$. Thus Theorem 2.1 of [9] implies that one of the cases $(i) \div (iv)$ must occur. In particular, when the case (iv) occurs, by the above lemma, we get that F contains a complemented copy of $(l^1)^N$.

Remark. We observe that, for a Fréchet space E, the fact the dual of E is a countable direct sum of Banach spaces (thus the bidual is a contable product of Banach spaces) does not necessarily imply that E is a countable product of Banach spaces. The second author and Metafune [11] constructed examples of quojections which are not countable products of Banach spaces but whose duals are countable direct sums of Banach spaces. Thus case (iv) need not imply that the complemented subspace F is a countable product of Banach spaces.

3 Complemented subspaces of $(L^1)^{(N)}$

We denote by $(L^1)^{(N)}$ the sum of countably many copies of L^1. In particular, the space $(L^1)^{(N)}$ can be represented as the inductive limit of the Banach spaces $\oplus_{i=1}^{n} L^1$ with respect to the linking maps

$$i_{n+1,n} : \oplus_{i=1}^{n} L^1 \to \oplus_{i=1}^{n+1} L^1 \,,\ (x_1 \cdots, x_n) \to (x_1, \cdots, x_n, 0),$$

which are isomorphism into. Clearly, if $(k(n))$ is an increasing sequence of integer numbers, we have also that $(L^1)^{(N)} = ind_n \left(\oplus_{i=1}^{k(n)} L^1, i_{k(n+1),k(n)}\right)$, where $i_{k(n+1),k(n)} = i_{k(n+1),k(n+1)-1} \cdots i_{k(n)+1,k(n)}$.

Also recall that if E a complemented subspace of $(L^1)^{(N)}, E$ is an LB-space and hence we may represent it as the strict inductive limit of the Banach spaces $E_n = E \cap (\oplus_{i=1}^{n} L^1)$.

Theorem 2. *The space $(L^1)^{(N)}$ is primary.*

Proof. We suppose that $(L^1)^{(N)} = F \oplus G$ with P projection from $(L^1)^{(N)}$ onto F and $ker P = G$. Put $Q = I - P$. Then, $F = ind_n F_n$ (resp. $G = ind_n G_n$), where $F_n = F \cap (\oplus_{i=1}^{n} L^1)$ (resp. $G_n = G \cap (\oplus_{i=1}^{n} L^1)$). Clearly $(L^1)^{(N)} = ind_n F_n \oplus G_n$.

Now, let $P_1 = P_{|L^1}$ (resp. $Q_1 = Q_{|L^1}$) be. Then there exists an $h(1) > 1$ such that the maps $P_1 : L^1 \to F_{h(1)}, Q_1 : L^1 \to G_{h(1)}$ are bounded and $F_{h(1)} \oplus G_{h(1)} \supset L^1$. Put $P_2 = P_{|\oplus_{i=1}^{h(1)+1} L^1}$ and $Q_2 = Q_{|\oplus_{i=1}^{h(1)+1} L^1}$, we also find an $h(2) > h(1) + 1$ such that the maps $P_2 : \oplus_{i=1}^{h(1)+1} L^1 \to F_{h(2)}$ and $Q_2 : \oplus_{i=1}^{n(1)+1} L^1 \to G_{h(2)}$ are bounded and $F_{h(2)} \oplus G_{h(2)} \supset \oplus_{i=1}^{h(1)+1} L^1$.

Continuing in this way, we inductively find a sequence $(h(n))$ of integer numbers with $h(n) > h(n-1)+1, h(0) = 1$, such that the maps

$$P_n = P|_{\oplus_{i=1}^{h(n-1)+1} L^1} : \oplus_{i=1}^{h(n-1)+1} L^1 \to F_{h(n)}$$

and

$$Q_n = Q|_{\oplus_{i=1}^{h(n-1)+1} L^1} : \oplus_{i=1}^{h(n-1)+1} L^1 \to G_{h(n)}$$

are bounded and $F_{h(n)} \oplus G_{h(n)} \supset \oplus_{i=1}^{h(n-1)+1} L^1$ for each $n \geq 1$.

Now, we note that the following diagram

$$\begin{array}{ccccc} \oplus_{i=1}^{h(n-1)+1} L^1 & \overset{P_n+Q_n}{\to} F_{h(n)} \oplus G_{h(n)} \overset{q_n}{\to} & \frac{F_{h(n)} \oplus G_{h(n)}}{F_{h(n-1)} \oplus G_{h(n-1)}} & \simeq \frac{F_{h(n)}}{F_{h(n-1)}} \oplus \frac{G_{h(n)}}{G_{h(n-1)}} \\ \widetilde{p}_{n-1} \downarrow & & \downarrow p_n & \\ \frac{\oplus_{i=1}^{h(n-1)+1} L^1}{\oplus_{i=1}^{h(n-1)} L^1} \simeq L^1 & \overset{j_n}{\to} & \frac{F_{h(n)} \oplus G_{h(n)}}{\oplus_{i=1}^{h(n-1)} L^1} & \end{array}$$

commutes for each $n > 1$, where q_n, p_n and $\widetilde{p}_{n-1}$ are the quotient maps and j_n is the canonical isomorphism into. Moreover, for $n = 1$,

$$P_1 + Q_1 = i_{h(1),1} : L^1 \to \oplus_{i=1}^{h(1)} L^1$$

is the canonical inclusion. By Remark 1 $i_{h(1),1}$ is an E-operator and, hence, either P_1 or Q_1 is an E-operator. Also, $j_n\widetilde{p}_{n-1}$ is an E-operator and, as follows from the above diagram, $p_n q_n (P_n + Q_n)$ is an E-operator. Then, by Remark 1 either $p_n q_n P_n$ or $p_n q_n Q_n$ is an E-operator, where, clearly,

$$\oplus_{i=1}^{h(n-1)+1} L^1 \overset{P_n}{\to} F_{h(n)} \overset{q_n}{\to} \frac{F_{h(n)}}{F_{h(n-1)}} \overset{p_n}{\to} \frac{F_{h(n)} \oplus G_{h(n)}}{\oplus_{i=1}^{h(n-1)} L^1}$$

and

$$\oplus_{i=1}^{h(n-1)+1} L^1 \overset{Q_n}{\to} G_{h(n)} \overset{q_n}{\to} \frac{G_{h(n)}}{G_{h(n-1)}} \overset{p_n}{\to} \frac{F_{h(n)} \oplus G_{h(n)}}{\oplus_{i=1}^{h(n-1)} L^1}.$$

Therefore, we can suppose that $p_n q_n P_n$ (for $n = 0$ q_0 denotes the identity map of $F_{h(1)} \oplus G_{h(1)}$) is an E-operator for infinite indices n.

For the sake of simplicity, we assume that $p_n q_n P_n$ is an E-operator for each n.

Because P_1 is an E-operator, by Theorem 0 there exists a closed subspace $M_1 \subset L^1$ with $P_{1|M_1}$ an isomorphism into, with $P_1(M_1) = H_1 \simeq L^1$ and, with $H_1 < \oplus_{i=1}^{h(1)} L^1$. Also $p_2 q_2 P_2$ is an E-operator and, hence, by Theorem 0 there exists a closed subspace $M_2 \subset \oplus_{i=1}^{h(1)+1} L^1$, with $M_2 \simeq L^1$, on which $p_2 q_2 P_2$ is an isomorphism onto a complemented subspace of $\frac{F_{h(2)} \oplus G_{h(2)}}{\oplus_{i=1}^{h(1)} L^1}$. Putting $H_2 = P_2(M_2)$, we then have $H_2 \subset F_{h(2)}$, $H_2 \simeq L^1$ and $p_2 q_2(H_2) < \frac{F_{h(2)} \oplus G_{h(2)}}{\oplus_{i=1}^{h(1)} L^1}$, $P_{2|M_2}$ an isomorphism into and $H_2 \cap F_{h(1)} = \{0\}$, $H_1 + H_2$ is closed in $F_{h(2)}$, hence equal to $H_1 \oplus H_2 \simeq L^1 \oplus L^1$.

Continuing in this way, we inductively obtain for each n a closed subspace $M_n \subset \oplus_{i=1}^{h(n-1)+1} L^1$ with $P_{n|M_n}$ an isomorphism into, $P_n(M_n) = H_n \subset F_{h(n)}$, with $H_n \simeq L^1$ and $p_n q_n(H_n) < \frac{F_{h(n)} \oplus G_{h(n)}}{\oplus_{i=1}^{h(n-1)} L^1}$ and $H_n \cap F_{h(n-1)} = \{0\}$, with $H_n + F_{h(n-1)}$ closed subspace of $F_{h(n)}$, hence $H_n + H_{n-1} = H_n \oplus H_{n-1}$ closed subspace of $F_{h(n)}$.

Clearly, if we now form the inductive limit X of the Banach spaces $X_n = \oplus_{i=1}^{n} H_i$ with respect to the canonical inclusions $X_n \to X_{n+1}$, we see that $X \subset F$ and $X \simeq (L^1)^{(N)}$.

To conclude the proof we have to show that $X < F$ and again to apply Pelczynski's decomposition method. Then we proceed as follows.

Let $r_i : \oplus_{i=1}^{h(1)} L^1 \to H_1$ be a projection. Now, recall that $p_2 q_2(H_2)$ is a complemented subspace of $\frac{F_{h(2)} \oplus G_{h(2)}}{\oplus_{i=1}^{h(1)} L^1}$ and $p_2 q_2(H_2) \simeq H_2$. Moreover, the following diagram

$$\begin{array}{ccc} \oplus_{i=1}^{h(2)} L^1 & \overset{\overline{p}_2}{\to} & \frac{\oplus_{i=1}^{h(2)} L^1}{\oplus_{i=1}^{h(1)} L^1} \\ s_{2,1} \downarrow & & \nearrow \swarrow \ t_{2,1} \\ \oplus_{i=h(1)+1}^{h(2)} L^1 & & \end{array}$$

commutes, where $s_{2,1}$ denotes the canonical inclusion, $t_{2,1}$ denotes the canonical isomorphism and $\overline{p}_2$ denotes the quotient map (we note that $\overline{p}_{2|F_{h(2)} \oplus G_{h(2)}} = p_2 q_2$). Then $s_{2,1}(H_2) \simeq H_2$ and $s_{2,1}(H_2) < \oplus_{i=h(1)+1}^{h(2)} L^1$. It follows that there exists a continuous linear map $A : s_{2,1}(H_2) \to \oplus_{i=1}^{h(1)} L^1$ with $H_2 = \{(Ay, y) : y \in s_{2,1}(H_2)\}$. Moreover, if r_2 :

$\oplus_{i=h(1)+1}^{h(2)} L^1 \to s_{2,1}(H_2)$ is a projection, then the map $\tilde{r}_2 : \oplus_{i=1}^{h(2)} L^1 \to H_2$ defined by $\tilde{r}_2(x,y) = (Ar_2y, r_2y)$ is a projection onto H_2 with $ker\,\tilde{r}_2 = \oplus_{i=1}^{h(1)} L^1 \oplus ker\, r_2$. Now, if I is the identity map of $\oplus_{i=1}^{h(2)} L^1$, we consider the diagram

$$\oplus_{i=1}^{h(1)} L^1 \oplus \oplus_{i=h(1)+1}^{h(2)} L^1 \overset{I-\tilde{r}_2}{\to} \oplus_{i=1}^{h(1)} L^1 \oplus ker\, r_2 \overset{I-s_{2,1}}{\to} \oplus_{i=1}^{h(1)} L^1 \oplus \{0\} \overset{r_1}{\to} H_1.$$

It is immediate to verify that the composition map $v_2 = r_1(I - s_{2,1}')(I - \tilde{r}_2)$ is a projection onto $H_1, v_2\tilde{r}_2 = 0 = \tilde{r}_2 v_2$ and $v_{2|\oplus_{i=1}^{h(1)} L^1} = r_1 = (v_2 + \tilde{r}_2)_{|\oplus_{i=1}^{h(1)} L^1}$. Therefore, $v_2 + \tilde{r}_2$ is a projection from $\oplus_{i=1}^{h(2)} L^1$ onto $H_1 \oplus H_2$ which extends r_1.

Also the diagram

$$\begin{array}{ccc} \oplus_{i=1}^{h(3)} L^1 & \overset{\overline{p}_3}{\to} & \dfrac{\oplus_{i=1}^{h(3)} L^1}{\oplus_{i=1}^{h(2)} L^1} \\ s_{3,2} \downarrow & \nearrow \swarrow & t_{3,2} \\ \oplus_{i=h(2)+1}^{h(3)} L^1 & & \end{array}$$

commutes, where $s_{3,2}$ denotes the canonical inclusion, $t_{3,2}$ denotes the canonical isomorphism and $\overline{p}_3$ denotes the quotient map ($\overline{p}_{3|F_{h(3)}\oplus G_{h(3)}} = p_3 q_3$). Then $s_{3,2}(H_3) \simeq H_3$ and $s_{3,2}(H_3) < \oplus_{i=h(2)+1}^{h(3)} L^1$. As before, it follows that there exists a continuous linear map (which, for simplicity, we again denotes by A) $A : s_{3,2}(H_3) \to \oplus_{i=1}^{h(2)} L^1$ with $H_3 = \{(Ay, y) : y \in s_{3,2}(H_3)\}$. Moreover, if $r_3 : \oplus_{i=h(2)+1}^{h(3)} L^1 \to s_{3,2}(H_3)$ is a projection, then the map $\tilde{r}_3 : \oplus_{i=1}^{h(3)} L^1 \to H_3$ defined by $\tilde{r}_3(x,y) = (Ar_3y, r_3y)$ is a projection onto H_3 with $ker\,\tilde{r}_3 = \oplus_{i=1}^{h(2)} L^1 \oplus ker\, r_3$. Then, again denoting by I the identity map of $\oplus_{i=1}^{h(3)} L^1$, the composition map $v_3 = (v_2 + \tilde{r}_2)(I - s_{3,2})(I - \tilde{r}_3)$ is a projection from the space $\oplus_{i=1}^{h(3)} L^1$ onto $H_1 \oplus H_2$ such that $v_3\tilde{r}_3 = 0 = \tilde{r}_3 v_3, v_{3|\oplus_{i=1}^{h(2)} L^1} = v_2 + \tilde{r}_2 = (v_3 + \tilde{r}_3)_{|\oplus_{i=1}^{h(2)} L^1}$.

Therefore, $v_3 + \tilde{r}_3$ is a projection from $\oplus_{i=1}^{h(3)} L^1$ onto $H_1 \oplus H_2 \oplus H_3$ which extends $v_2 + \tilde{r}_2$.

Continuing in this way, for each n we find a projection t_n from $\oplus_{i=1}^{h(n)} L^1$ onto X_n satisfying $t_{n|\oplus_{i=1}^{h(n-1)} L^1} = t_{n-1}$. To complete the proof it is enough to notice that the map $t : (L^1)^{(N)} \to X \simeq (L^1)^{(N)}$, defined by the sequence (t_n), is the desired projection.

Moreover

Proposition 2. ([9]). *If $F < (L^1)^{(N)}$ then one of the following cases occurs: (i) F is a complemented subspace of L^1. (ii) $F \simeq \varphi$. (iii) $F \simeq \varphi \oplus X$ where X is a complemented subspace of L^1. (iv) $F'_\beta \simeq (l^\infty)^N$, moreover in this case F contains a complemented copy of $(l^1)^{(N)}$.*

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